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Lecture Notes on Risk Management & Financial Regulation

Structure of the lecture notes

Preface

Warning

These lectures notes are currently being written and far from complete. My objective is first to provide a preliminary draft for each chapter. In a second time, I will improve the English text of the manuscript and illustrate with more examples and data. Corrections and comments are then welcome!

Three decades of financial regulation

Two decades of risk management

One decade of financial instability

About these lecture notes

These lecture notes are divided into three parts. After an introductory chapter presenting the main concepts of risk management and an overview of the financial regulation, the first part is dedicated to the risk management in the banking sector and consists of six chapters: market risk, credit risk, counterparty credit risk and collateral risk, operational risk, liquidity risk and asset/liability management risk. We begin with the market risk, because it permits to introduce naturally the concepts of risk factor and risk measure and to define the risk allocation approach. For each chapter, we present the corresponding regulation framework and the risk management tools. The second part is dedicated to non-banking financial sectors with four chapters dedicated to insurance, asset management, investors and market infrastructure (including central counterparties). This second part ends with a fifth chapter on systemic risk and shadow banking system. The third part of these lecture notes develops the mathematical and statistical tools used in risk management. It contains seven chapters: risk model and derivatives hedging, statistical inference and model estimation, copula functions, extreme value theory, Monte Carlo simulation, stress testing methods and scoring models. Each chapter of these lectures notes are extensively illustrated by numerical examples and contains also tutorial exercises. Finally, a technical appendix completes the lecture notes and contains some important elements on numerical analysis.

The writing of these lectures notes started in April 2015 and is the result of fifteen years of academic courses. When I began to teach risk management, a large part of my course was dedicated to statistical tools. Over the years, financial regulation became however increasingly important. This is why risk management is now mainly driven by the regulation, not by the progress with the mathematical models. The preparation of this book has benefited from the existing materials of my French book called "La Gestion des Risques Financiers". Nevertheless, the structure of the two books is different, because my previous book only concerned risk management in the banking sector and before Basel III. Three years ago, I decided to extend the course to other financial sectors, especially insurance, asset management and market infrastructure. It appears that even if the quantitative tools of risk management are the same across the different financial areas, each sector presents some particular aspects. The knowledge of the different regulations is especially not an easy task for students. However, it is necessary if one would like to understand what is the role of risk management in financial institutions in the present-day world. Moreover, reducing the practice of risk management to the assimilation of the regulation rules is not sufficient. The sound understanding of the financial products and the mathematical models are essential to know where the risks are. This is why some parts of this book can be particularly difficult because risk management is today complex in finance. A companion book is available in order to facilitate learning and knowledge assimilation at the following internet web page:

`http://www.thierry-roncalli.com/RiskManagement.html`

It contains additional information like some detailed calculation and the solution of the tutorial exercises.

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List of Symbols and Notations

Symbol Description

Symbol	Description
$\cdot$	Scalar multiplication
$\circ$	Hadamard product: $(x \circ y)_i = x_i y_i$
$\otimes$	Kronecker product $A \otimes B$
$\lvert\mathcal{E}\rvert$	Cardinality of the set $\mathcal{E}$
$\prec$	Concordance ordering
$\mathbf{1}$	Vector of ones
$\mathbb{1}\{\mathcal{A}\}$	The indicator function is equal to 1 if $\mathcal{A}$ is true, 0 otherwise
$\mathbb{1}_{\mathcal{A}}\{x\}$	The characteristic function is equal to 1 if $x \in \mathcal{A}$, 0 otherwise
$\mathbf{0}$	Vector of zeros
$(A_{i,j})$	Matrix A with entry $A_{i,j}$ in row i and column j
A^{-1}	Inverse of the matrix A
$A^{1/2}$	Square root of the matrix A
$A^{\top}$	Transpose of the matrix A
A^{+}	Moore-Penrose pseudo-inverse of the matrix A
b	Vector of weights $(b_1, \ldots, b_n)$ for the benchmark b
$B_t(T)$	Price of the zero-coupon bond at time t for the maturity T
$\mathcal{B}(p)$	Bernoulli distribution with parameter p
$\mathcal{B}(n,p)$	Binomial distribution with parameter n and p
β_i	Beta of asset i with respect to portfolio w
$\beta_i(w)$	Another notation for the symbol β_i
$\beta(w \mid b)$	Beta of portfolio w when the benchmark is b
$\boldsymbol{c}$	Coupon rate of the CDS premium leg
C (or ρ)	Correlation matrix
$\mathbf{C}(u_1, u_2)$	Copula function
$\mathbf{C}^{-}$	Fréchet lower bound copula
$\mathbf{C}^{\perp}$	Product copula
$\mathbf{C}^{+}$	Fréchet upper bound copula
$\boldsymbol{\mathcal{C}}_t$	Price of the call option at time t
$C(t_m)$	Coupon paid at time t_m
$C_n(\rho)$	Constant correlation matrix $(n \times n)$ with $\rho_{i,j} = \rho$
$\mathrm{CE}(t_0)$	Current exposure at time t_0
$\mathrm{cov}(X)$	Covariance of the random vector X
D	Covariance matrix of idiosyncratic risks
$\det(A)$	Determinant of the matrix A
$\mathrm{diag}\, v$	Build a diagonal matrix with elements $(v_1, \ldots, v_n)$
$\boldsymbol{\Delta}_t$	Delta of the option at time t
Δ_h	Difference operator with lag h, e.g. $\Delta_h V_t = V_t - V_{t-h}$
$\Delta\,\mathrm{CoVaR}_i$	Delta CoVaR of Institution i
Δt_m	Time interval $t_m = t_{m-1}$

$\mathbf{e}_i$	The value of the vector is 1 for the row i and 0 elsewhere
$\mathbb{E}[X]$	Mathematical expectation of the random variable X
$\mathcal{E}(\lambda)$	Exponential probability distribution with parameter λ
$e(t)$	Potential future exposure at time t
$\mathrm{EE}(t)$	Expected exposure at time t
$\mathrm{EEE}(t)$	Effective expected exposure at time t
$\mathrm{EEPE}(0;t)$	Effective expected positive exposure for the time period $[0,t]$
$\mathrm{EPE}(0;t)$	Expected positive exposure for the time period $[0,t]$
$\mathrm{ES}_\alpha(w)$	Expected shortfall of portfolio w at the confidence level α
$\exp A$	Exponential of the matrix A
$f(x)$	Probability density function (pdf)
$\mathbf{F}(x)$	Cumulative distribution function (cdf)
$\mathbf{F}^{-1}(\alpha)$	Quantile function
$\mathbf{F}^{n\star}$	n-fold convolution of the probability distribution $\mathbf{F}$ with itself
$\mathcal{F}$	Vector of risk factors $(\mathcal{F}_1,\ldots,\mathcal{F}_m)$
$\mathcal{F}_j$	Risk factor j
$F_t(T)$	Instantaneous forward rate at time t for the maturity T
$F_t(T,m)$	Forward interest rate at time t for the period $[T,T+m]$
$\mathcal{G}(p)$	Geometric distribution with parameter p
$\mathcal{G}(\alpha,\beta)$	Gamma distribution with parameters α and β
γ_1	Skewness
γ_2	Excess kurtosis
$\boldsymbol{\Gamma}_t$	Gamma of the option at time t
$\Gamma(\alpha)$	Gamma function defined as $\int_0^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t$
$\gamma(\alpha,x)$	Lower incomplete gamma function defined as $\int_0^x t^{\alpha-1}e^{-t}\,\mathrm{d}t$
$\Gamma(\alpha,x)$	Upper incomplete gamma function defined as $\int_x^\infty t^{\alpha-1}e^{-t}\,\mathrm{d}t$
$\mathcal{GEV}(\mu,\sigma,\xi)$	GEV distribution with parameters μ, σ and ξ
$\mathcal{GPD}(\sigma,\xi)$	Generalized Pareto distribution with parameters σ and ξ
h	Holding period
$\boldsymbol{h}$	Kernel or smoothing parameter
i	Asset (or component) i
I_n	Identity matrix of dimension n
$\boldsymbol{\mathcal{K}}$	Regulatory capital
$\boldsymbol{\ell}(\theta)$	Log-likelihood function with θ the vector of parameters to estimate
$\boldsymbol{\ell}_t$	Log-likelihood function for the observation t
L or $L(w)$	Loss of portfolio w
$\ln A$	Logarithm of the matrix A
$\mathcal{LG}(\alpha,\beta)$	Log-gamma distribution with parameters α and β
$\mathcal{LL}(\alpha,\beta)$	Log-logistic distribution with parameters α and β
$\mathcal{LN}(\mu,\sigma^2)$	Log-normal distribution with parameters μ and σ
λ	Parameter of exponential survival times
$\lambda(t)$	Hazard function
$\Lambda(t)$	Markov generator
$\mathrm{MDA}(\mathbf{G})$	Maximum domain of attraction of the extreme value distribution $\mathbf{G}$

MES_i	Marginal expected shortfall of Institution i
$\text{MPE}_\alpha(0;t)$	Maximum peak exposure for the time period $[0,t]$ with a confidence level α
$\mathcal{MR}_i$	Marginal risk of asset i
MtM	Mark-to-market of the portfolio
μ	Vector of expected returns $(\mu_1,\ldots,\mu_n)$
μ_i	Expected return of asset i
μ_{mkt}	Expected return of the market portfolio
$\hat{\mu}$	Empirical mean
$\mu(w)$	Expected return of portfolio w
$\mu(X)$	Mean of the random vector X
$\mu_m(X)$	m-th centered moment of the random vector X
$\mu'_m(X)$	m-th moment of the random vector X
$\mathcal{N}\left(\mu,\sigma^2\right)$	Normal distribution with mean μ and standard deviation σ
$\mathcal{N}(\mu,\Sigma)$	Multivariate normal distribution with mean μ and covariance matrix Σ
n_S	Number of scenarios or simulations
$N(t)$	Poisson counting process for the time interval $[0,t]$
$N(t_1;t_2)$	Poisson counting process for the time interval $[t_1,t_2]$
$\mathcal{NB}(r,p)$	Negative binomial distribution with parameters r and p
Ω	Covariance matrix of risk factors
P	Markov transition matrix
$\mathbb{P}$	Historical probability measure
$\mathcal{P}(\lambda)$	Poisson distribution with parameter λ
$p(n)$	Probability mass function of an integer-valued random variable
$\boldsymbol{\mathcal{P}}_t$	Price of the put option at time t
$\mathcal{P}a(\alpha,x_-)$	Pareto distribution with parameters α and x_-
$\text{PE}_\alpha(t)$	Peak exposure at time t with a confidence level α
$PV_t(\mathcal{L})$	Present value of the leg $\mathcal{L}$
Π or $\Pi(w)$	P&L of the portfolio w
$\phi(x)$	Probability density function of the standardized normal distribution
$\phi_2(x_1,x_2;\rho)$	Probability density function of the bivariate normal distribution with correlation ρ
$\phi_n(x;\Sigma)$	Probability density function of the multivariate normal distribution with covariance matrix Σ
$\Phi(x)$	Cumulative density function of the standardized normal distribution
$\Phi^{-1}(\alpha)$	Inverse of the cdf of the standardized normal distribution
$\Phi_2(x_1,x_2;\rho)$	Cumulative density function of the bivariate normal distribution with correlation ρ
$\Phi_n(x;\Sigma)$	Cumulative density function of the multivariate normal distribution with covariance matrix Σ
$\varphi_X(t)$	Characteristic function of the random variable X
$q_\alpha(n_S)$	Integer part of αn_S
$q_{\bar{\alpha}}(n_S)$	Integer part of $(1-\alpha)n_S$
$\mathbb{Q}$	Risk-neutral probability measure
$\mathbb{Q}_T$	Forward probability measure
$\mathfrak{R}(t)$	Rating of the entity at time t

r	Return of the risk-free asset
R	Vector of asset returns $(R_1, \ldots, R_n)$
R_i	Return of asset i
$R_{i,t}$	Return of asset i at time t
$R(w)$	Return of portfolio w
$\mathcal{R}(w)$	Risk measure of portfolio w
$\mathcal{R}(L)$	Risk measure of loss L
$\mathcal{R}(\Pi)$	Risk measure of P&L Π
$R_t(T)$	Zero-coupon rate at time t for the maturity T
$\mathcal{RC}_i$	Risk contribution of asset i
$\mathcal{RC}_i^\star$	Relative risk contribution of asset i
$\boldsymbol{\mathcal{R}}$	Recovery rate
RPV_{01}	Risky PV01
ρ (or C)	Correlation matrix of asset returns
$\rho_{i,j}$	Correlation between asset returns i and j
$\rho(x,y)$	Correlation between portfolios x and y
$\mathcal{s}$	Credit spread
$\mathbb{S}$	Stress scenario
S_t	Price of the underlying asset at time t
$\mathbf{S}_t(T)$	Survival function of T at time t
SES_i	Systemic expected shortfall of Institution i
$\mathcal{SN}(\xi,\Omega,\eta)$	Skew normal distribution
SRISK_i	Systemic risk contribution of Institution i
$\mathcal{ST}(\xi,\Omega,\eta,\nu)$	Skew t distribution
$SV_t(\mathcal{L})$	Stochastic discounted value of the leg $\mathcal{L}$
Σ	Covariance matrix
$\hat{\Sigma}$	Empirical covariance matrix
σ_i	Volatility of asset i
σ_{mkt}	Volatility of the market portfolio
$\tilde{\sigma}_i$	Idiosyncratic volatility of asset i
$\hat{\sigma}$	Empirical volatility
$\sigma(w)$	Volatility of portfolio w
$\sigma(X)$	Standard deviation of the random variable X
$\mathbf{t}_v(x)$	Probability density function of the univariate t distribution with number of degrees of freedom ν
$\mathbf{t}_n(x;\Sigma,\nu)$	Probability density function of the multivariate t distribution with parameters Σ and ν
$\mathbf{t}_2(x_1,x_2;\rho,\nu)$	Probability density function of the bivariate t distribution with parameters ρ and ν
T	Maturity date
$\mathbf{T}_v(x)$	Cumulative density function of the univariate t distribution with number of degrees of freedom ν
$\mathbf{T}_v^{-1}(\alpha)$	Inverse of the cdf of the Student's t distribution with ν the number of degrees of freedom
$\mathbf{T}_n(x;\Sigma,\nu)$	Cumulative density function of the multivariate t distribution with parameters Σ and ν
$\mathbf{T}_2(x_1,x_2;\rho,\nu)$	Cumulative density function of the bivariate t distribution with parameters ρ and ν
$\mathcal{T}$	return period
$\mathrm{tr}(A)$	Trace of the matrix A
θ	Vector of parameters
$\hat{\theta}$	Estimator of θ
$\mathbf{\Theta}_t$	Theta of the option at time t
$\boldsymbol{\tau}$	Default time
τ	Time to maturity $T-t$
$\mathrm{var}(X)$	Variance of the random variable X
$\mathrm{VaR}_\alpha(w)$	Value-at-risk of portfolio w at the confidence level α

$\boldsymbol{\upsilon}_t$	Vega of the option t
w	Vector of weights $(w_1, \ldots, w_n)$ for portfolio w
w_i	Weight of asset i in portfolio w
X	Random variable
x^+	Maximum value between x and 0
$X_{i:n}$	i^{th} order statistic of a sample of size n
$\mathcal{y}$	Yield to maturity

Abbreviations

ABCP	Asset-backed commercial paper
ABS	Asset-backed security
ADV	Average daily volume
AFME	Association for Financial Markets in Europe
AIFM	Alternative investment fund managers (directive)
AIFMD	Alternative investment fund managers directive
AIRB	Advanced internal rating-based approach (credit risk)
AMA	Advanced Measurement Approaches (operational risk)
AT1	Additional tier 1
BCBS	Basel Committee on Banking Supervision
BCVA	Bilateral CVA
BIA	Basic indicator approach (operational risk)
BIS	Bank for International Settlement
CAD	Capital adequacy directive
CaR	Capital-at-risk
CB	Conservation buffer (CET1)
CBO	Collateralized bond obligation
CCB	Countercyclical capital buffer (CET1)
CCF	Credit conversion factor
CCP	Central counterparty clearing
CCR	Counterparty credit risk
CDF	Cumulative distribution function
CDS	Credit default swap
CDT	Credit default tranche
CDX	Credit default index
CDO	Collateralized debt obligation
CE	Current exposure
CEM	Current exposure method (CEM)
CET1	Common equity tier 1
CLO	Collateralized loan obligation
CMBS	Commercial mortgage-backed security
CoVaR	Conditional value-at-risk
CP	Consultation paper
CRA	Credit rating agency
CRD	Capital requirements directive
CRR	Capital requirements regulation
CRM	Comprehensive risk measure
CVA	Credit valuation adjustment
DVA	Debit valuation adjustment
EAD	Exposure at default
EBA	European Banking Authority
ECB	European Central Bank
EE	Expected exposure
EEE	Effective expected exposure
EEPE	Effective expected positive exposure
EL	Expected loss
EMIR	European market infrastructure regulation
ENE	Expected negative exposure
EPE	Expected positive exposure

ESMA	European Securities and Markets Authority
ETF	Exchange traded fund
EVT	Extreme value theory
FASB	Financial Accounting Standards Board
FIRB	Foundation internal rating-based approach (credit risk)
FRTB	Fundamental review of the trading book
FSB	Financial Stability Board
FtD	First-to-default swap
FVA	Founding valuation adjustment
GAAP	Generally accepted accounting principles (US)
GEV	Generalized extreme value distribution
GFC	Global financial crisis
GMM	Generalized method of moments
GPD	Generalized Pareto distribution
HF	Hedge fund
HLA	Higher loss absorbency
HQLA	High-quality liquid assets
HY	High yield entities
IAIS	International Association of Insurance Supervisors
IAS	International accounting standard
ICAAP	Internal capital adequacy assessment process
ICP	Insurance Core Principles
ICPF	Insurance companies and pension funds
IFRS	International financial reporting standard
IG	Investment grade entities
IMA	Internal model-based approach (market risk)
IMF	International Monetary Fund
IMM	Internal model method (counterparty credit risk)
IOSCO	International Organization of Securities Commissions
IRB	Internal rating-based approach (credit risk)
IRC	Incremental risk charge
IRS	Interest rate swap
IRRBB	Interest rate risk of the banking book
ISDA	International Swaps and Derivatives Association
KRI	Key risk indicator
LCR	Liquidity coverage ratio
LDA	Loss distribution approach
LDCE	Loss data collection exercise
LEE	Loan equivalent exposure
LGD	Loss given default
M	Effective maturity
MBS	Mortgage-backed security
MCR	Minimum capital requirement
MDA	Maximum domain of attraction
MES	Marginal expected shortfall
MF	Mutual fund
MiFID	Markets in financial instruments directive
MiFIR	Markets in financial instruments regulation
ML	Maximum likelihood
MM	Method of moments
MMF	Money market fund
MPE	Maximum peak exposure
MtM	Mark-to-market
NQD	Negative quadrant dependence
NSFR	Net stable funding ratio
ORSA	Own risk and solvency assessment
OTC	Over-the-counter
PD	Probability of Default
PDF	Probability density function
PE	Peak exposure
PFE	Potential future exposure
PMF	Probability mass function
POT	Peak over threshold

PQD	Positive quadrant dependence
QIS	Quantitative impact study
RBC	Risk-based capital (US insurance)
RMBS	Residential mortgage-backed security
RW	Risk weight
RWA	Risk-weighted asset
SA	Standardized approach (credit risk)
SA-CCR	Standardized approach (counterparty credit risk)
SBE	Shadow banking entity
SCR	Solvency capital requirement
SES	Systemic expected shortfall
SIFMA	Securities Industry and Financial Markets Association
SIFI	Systemically important financial institution
SIV	Structured investment vehicle
SLA	Single loss approximation
SME	Small and medium-sized enterprises
SM-CCR	Standardized method (counterparty credit risk)
SMM	Standardized measurement method (market risk)
SRC	Specific risk charge
SREP	Supervisory review and evaluation process
SRISK	Systemic risk contribution
SRP	Supervisory review process
SSM	Single supervisory mechanism
SVaR	Stressed value-at-risk
SPV	Special purpose vehicle
T1	Tier 1
T2	Tier 2
TLAC	Total loss absorbing capacity
TSA	The standardized approach (operational risk)
UCITS	Undertakings for collective investment in transferable securities (directive)
UCVA	Unilateral CVA
UDVA	Unilateral DVA
UL	Unexpected loss
VaR	Value-at-risk
XO	Crossover (or sub-investment grade) entities

Other scientific conventions

YYYY-MM-DD	We use the international standard date notation where YYYY is the year in the usual Gregorian calendar, MM is the month of the year between 01 (January) and 12 (December), and DD is the day of the month between 01 and 31.
\$1 mn	One million dollars.
\$1 bn	One billion dollars.
\$1 tn	One trillion dollars.

Chapter 1

Introduction

The idea that risk management creates value is largely accepted today. However, this has not always been the case in the past, especially in the financial sector (Stulz, 1996). Rather, it has been a long march marked by a number of decisive steps. In this introduction, we present an outline of the most important achievements from a historical point of view. We also give an overview of the current financial regulation, which is a cornerstone in financial risk management.

1.1 The need for risk management

The need for risk management is the title of the first section of the leadership book by Jorion (2007), who shows that risk management can be justified at two levels. At the firm level, risk management is essential for identifying and managing business risk. At the industry level, risk management is a central factor for understanding and preventing systemic risk. In particular, this second need is the '*raison d'être*' of the financial regulation itself.

1.1.1 Risk management and the financial system

The concept of risk management has evolved considerably since its creation, which is believed to be in the early fifties[1]. In November 1955, Wayne Snider gave a lecture entitled "*The Risk Manager*" where he proposed creating an integrated department responsible for risk prevention in the insurance industry (Snider, 1956). Some months later, Gallagher (1956) published an article to outline the most important principles of risk management and to propose the hiring of a full-time risk manager in large companies. For a long time, risk management was systematically associated with insurance management, both from a practical point of view and a theoretical point of view. For instance, the book of Mehr and Hedges (1963) is largely dedicated to the field of insurance with very few applications to other industries. This is explained

[1] See Crockford (1982) or Snider (1991) for a retrospective view on the risk management development.

by the fact that the collective risk model has helped to apply the mathematical and statistical tools for measuring risk in insurance companies since 1930. A new discipline known as actuarial science has been developed at the same time outside the other sciences and has supported the generalization of risk management in the insurance industry.

Simultaneously, risk became an important field of research in economics and finance. Indeed, Arrow (1964) made an important step by extending the Arrow-Debreu model of general equilibrium in an uncertain environment[2]. In particular, he showed the importance of hedging and introduced the concept of payoff. By developing the theory of optimal allocation for a universe of financial securities, Markowitz (1952) pointed out that the risk of a financial portfolio can be diversified. These two concepts, hedging and diversification, together with insurance, are the main pillars of modern risk management. These concepts will be intensively used by academics in the 1960s and 1970s. In particular, Black and Scholes (1973) will show the interconnection between hedging and pricing problems. Their work will have a strong impact on the development of equity, interest rates, currency and commodity derivatives, which are today essential for managing the risk of financial institutions. With the Markowitz model, a new era had begun in portfolio management and asset pricing. First, Sharpe (1964) showed how risk premia are related to non-diversifiable risks and developed the first asset pricing model. Then, Ross (1976) extended the CAPM model of Sharpe and highlighted the role of risk factors in arbitrage pricing theory. These academic achievements will support the further development of asset management, financial markets and investment banking.

In commercial and retail banking, risk management was not integrated until recently. Even though credit scoring models have existed since the fifties, they were rather designed for consumer lending, especially credit cards. When banks used them for loans and credit issuances, they were greatly simplified and considered as a decision-making tool, playing a minor role in the final decision. The underlying idea was that the banker knew his client better than a statistical model could. However, Banker Trust introduced the concept of risk-adjusted return on capital or RAROC under the initiative of Charles Sanford in the late 1970's for measuring risk-adjusted profitability. Gene Guill mentions a memorandum dated February 1979 by Charles Sanford to the head of bank supervision at the Federal Reserve Board of New York that helps to understand the RAROC approach:

> "*We agree that one bank's book equity to assets ratio has little relevance for another bank with a different mix of businesses. Certain activities are inherently riskier than others and more risk capital is required to sustain them. The truly scarce resource is equity, not assets, which is why we prefer to compare and measure*

[2]This paper was originally presented in 1952 and was also published in Cahiers du CNRS (1953).

> *businesses on the basis of return on equity rather than return on assets*" (Guill, 2009, page 10).

RAROC compares the expected return to the economic capital and has become a standard model for combining performance management and risk management. Even if RAROC is a global approach for allocating capital between business lines, it has been mainly used as a credit scoring model. Another milestone was the development of credit portfolio management when Vasicek (1987) adapted the structural default risk model of Merton (1974) to model the loss distribution of a loan portfolio. He then jointly founded KMV Corporation with Stephen Kealhofer and John McQuown, which specializes in quantitative credit analysis tools and is now part of Moody's Analytics.

In addition to credit risk, commercial and retail banks have to manage interest rate and liquidity risks, because their primary activity is to do asset, liquidity and maturity transformations. Typically, a commercial bank has long-term and illiquid liabilities (loans) and short-term and liquid assets (deposits). In such a situation, a bank faces a loss risk that can be partially hedged. This is the role of asset-liability management(ALM). But depositors also face a loss risk that is virtually impossible to monitor and manage. Consequently, there is an information asymmetry between banks and depositors.

In the banking sector, the main issue centered therefore around the deposit insurance. How can we protect depositors against the failure of the bank? The 100% reserve proposal by Fisher in 1935 required banks to keep 100% of demand deposit accounts in cash or government-issued money like bills. Diamond and Dybvig (1983) argued that the mixing policy of liquid and illiquid assets can rationally produce systemic risks, such as bank runs. A better way to protect the depositors is to create a deposit insurance guaranteed by the government. According to the Modigliani-Miller theorem on capital structure[3], this type of government guarantee implied a higher cost of equity capital. Since the eighties, this topic has been highly written about (Admati and Hellwig, 2014). Moreover, banks also differ from other companies, because they create money. Therefore, they are at the heart of the monetary policy. These two characteristics (implicit guarantee and money creation) imply that banks have to be regulated and need regulatory capital. This is all the more valid with the huge development of financial innovations, which has profoundly changed the nature of the banking system and the risk.

1.1.2 The development of financial markets

The development of financial markets has a long history. For instance, the Chicago Board of Trade (CBOT) listed the first commodity futures contract

[3] Under some (unrealistic) assumptions, Modigliani and Miller (1958) showed that the market value of a firm is not affected by how that firm is financed (by issuing stock or debt). They also established that the cost of equity is a linear function of the firm's leverage measured by its debt/equity ratio.

in 1864 (Carlton, 1984). Some authors even consider that the first organized futures exchange was the Dojima Rice Market in Osaka in the 18$^{\text{th}}$ century (Schaede, 1989). But the most important breakthrough came in the seventies with two major financial innovations. In 1972, the Chicago Mercantile Exchange (CME) launched currency futures contracts after the US had decided to abandon the fixed exchange rate system of Bretton Woods (1946). The oil crisis of 1973 and the need to hedge currency risk have considerably helped in the development of this market. After commodity and currency contracts, interest rate and equity index futures have consistently grown. For instance, US Treasury bond, S&P 500, German Bund, and EURO STOXX 50 futures were first traded in 1977, 1982, 1988 and 1998 respectively. Today, the Bund futures contract is the most traded product in the world.

The second main innovation in the seventies concerned option contracts. The CBOT created the Chicago Board of Options (CBOE) in 1973, which was the first exchange specialized in listed stock call options. That same year, Black and Scholes (1973) published their famous formula for pricing a European option. It has been the starting point of the intensive development of academic research concerning the pricing of financial derivatives and contingent claims. The works of Fisher Black, Myron Scholes and Robert Merton[4] are all the more significant in that they consider the pricing problem in terms of risk hedging. Many authors had previously found a similar pricing formula, but Black and Scholes introduced the revolutionary concept of the hedging portfolio. In their model, they derived the corresponding dynamic trading strategy to hedge the option contract, and the option price is therefore equivalent to the cost of the hedging strategy. Their pricing method had a great influence on the development of the derivatives market and more exotic options, in particular path-dependent options[5].

Whereas the primary goal of options is to hedge a directional risk, they will be largely used as underlying assets of investment products. In 1976, Hayne Leland and Mark Rubinstein developed the portfolio insurance concept, which allows for investing in risky assets while protecting the capital of the investment. In 1980, they founded LOR Associates, Inc. with John O'Brien and proposed structured investment products to institutional investors (Tufano and Kyrillos, 1995). They achieved very rapid growth until the 1987 stock market crash[6], and were followed by Wells Fargo, J.P. Morgan and Chase Manhattan as well as other investment banks. This period marks the start of financial engineering applied to structured products and the development of popular

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[4]As shown by Bernstein (1972), the works of Black and Scholes cannot be dissociated from the research of Merton (1973). This explains that they both received the 1997 Nobel Prize in Economics for their option pricing model.

[5]See Box 1 for more information about the rise of exotic options.

[6]In fact, portfolio insurance was blamed by the Brady Commission report (1988) for the stock market crash of October 1987. See for instance Leland and Rubinstein (1988), Shiller (1988), Gennotte and Leland (1990) and Jacklin *et al.* (1992) for a discussions about the impact of portfolio insurance on the October 1987 crash.

trading strategies, such as constant proportion portfolio insurance (CPPI) and option based portfolio insurance (OBPI). Later, they will be extensively used for designing retail investment products, especially capital guaranteed products.

Box 1

Evolution of financial innovations

1864	Commodity futures
1970	Mortgage-backed securities
1971	Equity index funds
1972	Foreign currency futures
1973	Stock options
1977	Put options
1979	Over-the-counter currency options
1980	Currency swaps
1981	Interest rate swaps
1982	Equity index futures
1983	Equity index options
	Interest rate caps/floors
	Collateralized mortgage obligations
1985	Swaptions
	Asset-backed securities
1987	Path-dependent options (Asian, Lookback, etc.)
	Collateralized debt obligations
1992	Catastrophe insurance futures and options
1993	Captions/Floortions
	Exchange-traded funds
1994	Credit default swaps
1997	Weather derivatives
2004	Volatility index futures
2006	Leveraged and inverse ETFs

Source: Jorion (2007) & author's research.

After options, the next great innovation in risk management was the swap. In a swap contract, two counterparties exchange a series of cash flows of one financial instrument for those of another financial instrument. For instance, an interest rate swap (IRS) is an exchange of interest rate cash flows from a fixed rate to a floating rate or between two floating rates. Swaps have become an important tool for managing balance sheets, in particular interest rate and currency risks in the banking book. The original mechanism of cash flow exchanges has been extended to other instruments and underlying assets: inflation-indexed bonds, stocks, equity indexes, commodities, etc. But one of the most significant advances in financial innovations was the creation of credit default swaps (CDS) in the mid-nineties, and more generally credit derivatives. In the simplest case, the cash flows depend on the default of a loan, a bond or a company. We refer then to single-name instruments. Otherwise, they depend on credit events or credit losses of a portfolio (multi-name instruments). However, the development of credit derivatives was made possible thanks to securitization. This is a process through which assets are pooled in a portfolio and securities representing interests in the portfolio are issued. Securities backed by mortgages are called mortgage-backed securities (MBS), while those backed by other types of assets are asset-backed securities (ABS).

Derivatives are traded either in organized markets or in over-the-counter markets (OTC). In organized exchanges, the contracts are standardized and the transactions are arranged by the clearing house, which is in charge of clearing and settlement. By contrast, in OTC markets, the contracts are customized and the trades are done directly by the two counterparties. This implies that OTC trades are exposed to the default risk of the participants. The location of derivatives trades depends on the contract:

Contract	Futures	Forward	Option	Swap
On-exchange	✓		✓	
Off-exchange		✓	✓	✓

For instance, the only difference between futures and forwards is that futures are traded in organized markets whereas forwards are traded over-the-counter. Contrary to options which are negotiated in both markets, swaps are mainly traded OTC. In Table 1.1, we report the amounts outstanding of exchange-traded derivatives concerning futures and options published by the Bank for International Settlement (2015). In December 2014, their notional is equal to \$64.9 tn, composed of \$27.1 tn in futures (41.9%) and \$37.7 tn in options (58.1%). For each instrument, we indicate the split between interest rates, currencies and equity indices. We notice that exchange-traded derivatives on interest rates are the main contributor. The evolution of the global notional is reported in Figure 1.1. The size of exchange-traded derivative markets has grown rapidly since 2000, peaking in June 2007 with an aggregated amount of \$94.9 tn. This trend ended with the financial crisis.

Statistics concerning OTC derivative markets are given in Table 1.2. On average, these markets are ten times bigger than exchange-traded markets in terms of amounts outstanding. In December 2014, the aggregated amount of forwards, swaps and options is equal to \$630.1 tn. Contrary to exchange-traded derivative markets, notional amounts outstanding in OTC derivative markets are higher than before the crisis period (Figure 1.2). In terms of instrument, swaps dominate (66.9% of the total). The main asset class remains fixed-income assets, but its weight is less than in exchange-traded markets. We also notice the impact of the 2008 financial crisis on credit default swaps, which

TABLE 1.1: Amounts outstanding of exchange-traded derivatives

	Dec. 2004	Dec. 2007	Dec. 2010	Dec. 2014
Futures	40.8%	35.6%	32.8%	41.9%
Interest rate	96.1%	95.4%	94.2%	93.3%
Currency	0.6%	0.6%	0.8%	0.9%
Equity index	3.3%	4.0%	5.1%	5.8%
Options	59.2%	64.4%	67.2%	58.1%
Interest rate	89.8%	87.2%	89.6%	84.6%
Currency	0.2%	0.3%	0.3%	0.4%
Equity index	10.0%	12.6%	10.0%	15.0%
Total (in $ tn)	46.3	78.9	68.0	64.9

Source: Bank for International Settlement (2015) & author's calculations.

represented 9.9% of the OTC derivative markets in December 2007. Seven years later, they will represent only 2.6% of these markets.

TABLE 1.2: Amounts outstanding of OTC derivatives

	Dec. 2004	Dec. 2007	Dec. 2010	Dec. 2014
Forwards	11.4%	10.9%	14.0%	19.3%
Swaps	63.9%	65.2%	68.8%	66.9%
Options	14.4%	13.4%	10.6%	10.2%
Unallocated	10.3%	10.5%	6.6%	3.6%
Currency	11.3%	9.6%	9.6%	12.0%
Interest rate	73.7%	67.1%	77.4%	80.2%
Equity	1.7%	1.4%	0.9%	1.3%
Commodity	0.6%	1.4%	0.5%	0.3%
Credit	2.5%	9.9%	5.0%	2.6%
Unallocated	10.3%	10.5%	6.6%	3.6%
Total (in $ tn)	258.6	585.9	601.0	630.1

Source: Bank for International Settlement (2015) & author's calculations.

Whereas notional amounts outstanding is a statistic to understand the size of the derivatives markets, the risk and the activity of these markets may be measured by the gross market value and the turnover:

- The *gross market value* of outstanding derivatives contracts represents "*the cost of replacing all outstanding contracts at market prices prevailing on the reporting date. It corresponds to the maximum loss that market participants would incur if all counterparties failed to meet their*

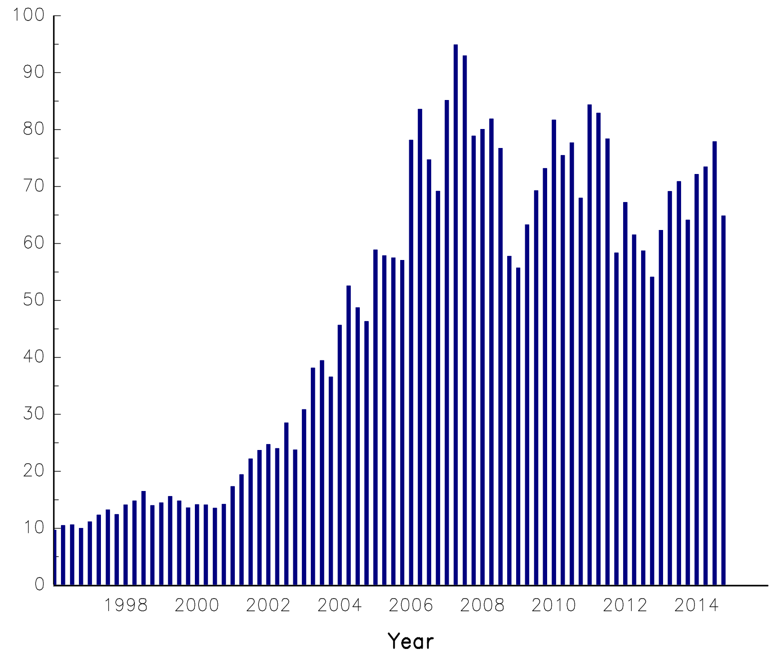

FIGURE 1.1: Notional amount of exchange-traded derivatives ((in $ tn)

Source: Bank for International Settlement (2015).

contractual payments and the contracts were replaced at current market prices" (BIS, 2014).

- The *turnover* is defined as "*the gross value of all new deals entered into during a given period, and is measured in terms of the nominal or notional amount of the contracts. It provides a measure of market activity, and can also be seen as a rough proxy for market liquidity.*" (BIS, 2014).

In December 2014, the gross market value is equal to $20.9 tn for OTC derivatives. It is largely lower than the figure of $35.3 tn in December 2008. This decrease is explained by less complexity in derivatives, but also by a lower volatility regime. If we take into account legally enforceable bilateral netting agreements, the gross counterparty credit is about $3.4 tn. For OTC derivatives, it is difficult to measure a turnover, because the contracts are not standardized. This statistic is more pertinent for exchange-traded markets. For the year 2014, it is equal to $1 450.5 tn for futures and $486.0 tn for options. This means that each day, more than $5 tn of new derivative exposures are negotiated in exchange-traded markets. The consequence of this huge activity is a growing number of financial losses for banks and financial institutions.

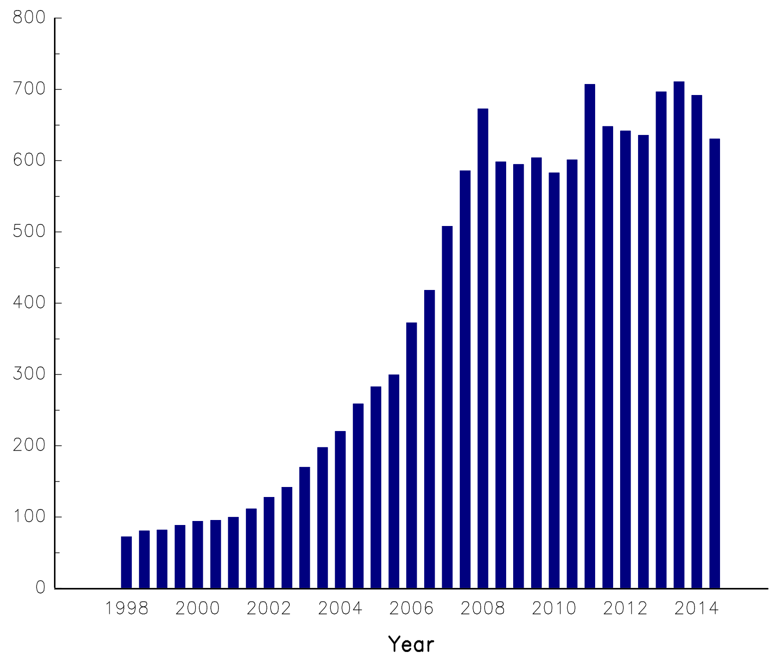

FIGURE 1.2: Notional amount of OTC derivatives (in $ tn)

Source: Bank for International Settlement (2015).

1.1.3 Financial crises and systemic risk

A financial institution generally faces five main risks:

- Market risk
- Credit risk
- Counterparty credit risk
- Operational risk
- Liquidity risk

Market risk is the risk of losses due to changes in financial market prices. We generally distinguish four major types of market risk: equity risk, interest rate risk, currency risk and commodity risk. These risks are present in trading activities, but they also affect all activities that use financial assets. Credit risk is the risk of losses due to the default of a counterparty to fulfill its contractual obligations, that is to make its required payments. It principally concerns debt transactions such as loans and bonds. Counterparty credit risk is another form of credit risk, but concerns the counterparty of OTC transactions. Examples include swaps and options, security lending or repo transactions. Operational risk is the risk of losses resulting from inadequate or failed internal processes, people and systems, or from external events. Examples of operational risk are frauds, natural disasters, business disruption, rogue trading, etc. Finally, liquidity risk is the risk of losses resulting from the failure of the financial institution to meet its obligations on time.

Box 2

An history of financial losses

1974	Herstatt Bank: $620 mn (foreign exchange trading)
1994	Metallgesellschaft: $1.3 bn (oil futures)
1994	Orange County: $1.8 bn (reverse repo)
1994	Procter & Gamble: $160 mn (ratchet swap)
1995	Barings Bank: $1.3 bn (stock index futures)
1997	Natwest: $127 mn (swaptions)
1998	LTCM: $4.6 bn (liquidity crisis)
2001	Dexia Bank: $270 mn (corporate bonds)
2006	Amaranth Advisors: $6.5 bn (gaz forward contracts)
2007	Morgan Stanley: $9.0 bn (credit derivatives)
2008	Société Générale: $7.2 bn (rogue trading)
2008	Madoff: $65 bn (fraud)
2011	UBS: $2.0 bn (rogue trading)
2012	JPMorgan Chase: $5.8 bn (credit derivatives)

Source: Jorion (2007) & author's research.

In Box 2, we have reported some famous financial losses. Most of them are related to the market risk or the operational risk. In this case, these losses are said to be idiosyncratic because they are specific to a financial institution. Idiosyncratic risk is generally opposed to systemic risk: systemic risk refers to the system whereas idiosyncratic risk refers to an entity of the system. For instance, the banking system may collapse, because many banks may be affected by a severe common risk factor and may default at the same time. In financial theory, we generally make the assumption that idiosyncratic and common risk factors are independent. However, there exits some situations where idiosyncratic risk may affect the system itself. It is the case of large financial institutions, for example the default of big banks. In this situation, system risk refers to the propagation of a single bank distressed risk to the other banks.

The case of Herstatt Bank is an example of an idiosyncratic risk that could result in a systemic risk. Herstatt Bank was a privately German bank.

On 26 June 1974, the German Banking Supervisory Office withdrew Herstatt's banking licence after finding that the bank's foreign exchange exposures amounted to three times its capital (BCBS, 2014d). This episode of settlement risk caused heavy losses to other banks, adding a systemic dimension to the individual failure of Herstatt Bank. In response to this turmoil, the central bank governors of the G10 countries established the Basel Committee on Banking Supervision at the end of 1974 with the aim to enhance the financial stability at the global level.

Even if the default of a non-financial institution is a dramatic event for employees, depositors, creditors and clients, the big issue is its impact on the economy. Generally, the failure of a company does not induce a macroeconomic stress and is well located to a particular sector or region. For instance, the decade of the 2000s had faced a lot of bankruptcies, e.g., Pacific Gas and Electric Company (2001), Enron (2001), WorldCom (2002), Arthur Andersen (2002), Parmalat (2003), US Airways (2004), Delta Air Lines (2005), Chrysler (2009), General Motors (2009) and LyondellBasell (2009). However, the impact of these failures was contained within the immediate environment of the company and was not spread to the rest of the economy.

In the financial sector, the issue is different because of the interconnectedness between the financial institutions and the direct impact on the economy. And the issue is especially relevant that the list of bankruptcies in finance is long including, for example: Barings Bank (1995); HIH Insurance (2001); Conseco (2002); Bear Stearns (2008), Lehman Brothers (2008); Washington Mutual (2008); DSB Bank (2008). The number of banking and insurance distresses is even more impressive, for example: Northern Rock (2007); Countrywide Financial (2008); Indy Mac Bank (2008); Fannie Mae/Freddie Mac (2008); Merrill Lynch (2008); AIG (2008); Wachovia (2008); Depfa Bank (2008); Fortis (2009); Icelandic banks (2008-2010); Dexia (2011). In Figure 1.3, we report the number of bank failures computed by the Federal Deposit Insurance Corporation (FDIC), the organization in charge of insuring depositors in the US. We can clearly identify three periods of massive defaults: 1930-1940, 1980-1994 and 2008-2014. Each period corresponds to a banking crisis[7] and lasts long because of delayed effects. Whereas the 1995–2007 period is characterized by a low default rate, with no default in 2005–2006, there is a significant number of bank defaults these last years (517 defaults between 2008 and 2014).

The Lehman Brothers collapse is a case study for understanding the systemic risk. Lehman Brothers filed for Chapter 11 bankruptcy protection on September 15, 2008 after incurring heavy credit and market risk losses implied by the US subprime mortgage crisis. The amount of losses is generally estimated to be about \$600 bn, because Lehman Brothers had at this time \$640 bn in assets and \$620 bn in debt. However, the cost for the system is far greater

[7] They are the Great Depression, the savings and loan crisis of the 1980s and the subprime crisis.

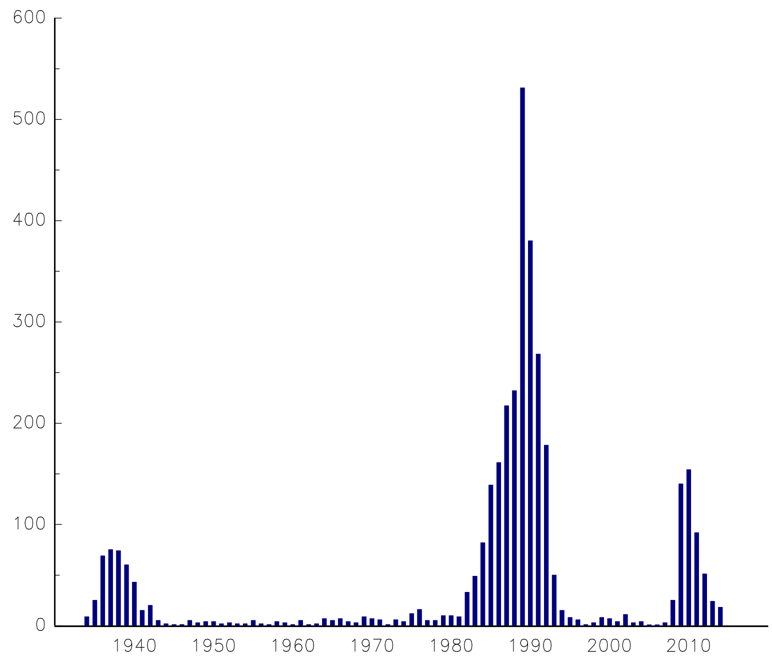

FIGURE 1.3: Number of bank defaults in the US

Source: Federal Deposit Insurance Corporation (2014), Historical Statistics on Banking – Failures & Assistance Transactions, `https://www.fdic.gov/bank/individual/failed/`.

than this figure. On equity markets, about \$10 tn went missing in October 2008. The post-Lehman Brothers default period (from September to December 2008) is certainly one of the most extreme liquidity crisis experienced since many decades. This forced central banks to use unconventional monetary policy measures by implementing quantitative easing (QE) programmes. For instance, the Fed now holds more than five times the amount of securities it had prior before September 2008. The collapse of Lehman Brothers had a huge impact on the banking industry, but also on the asset management industry. For instance, four days after the Lehman Brothers bankruptcy, the US government extended temporary guarantee on money market funds. At the same time, the hedge funds industry suffered a lot because of the stress on the financial markets, but also because Lehman Brothers served at prime broker for many hedge funds.

The bankruptcy of a financial institution can then not be compared to the bankruptcy of a corporate company. Nevertheless, because of the nature of the systemic risk, it is extremely difficult to manage it directly. This explains that the financial supervision is principally a micro-prudential regulation at the firm level. This is only recently that it has been completed by macro-prudential policies in order to mitigate the risk of the financial system as a whole. While the development of risk management was principally due to the advancement of internal models before the 2008 financial crisis, it is now driven by the financial regulation, which complectly reshapes the finance industry.

1.2 Financial regulation

The purpose of supervision and regulatory capital has been to control the riskiness of individual banks and to increase the stability of the financial system. As explained in the previous section, it is a hard task whose bounds are not well defined. Among all the institutions that are participating to this work, four international authorities have primary responsibility of the financial regulation:

1. The Basel Committee on Banking Supervision (BCBS)

2. The International Association of Insurance Supervisors (IAIS)

3. The International Organization of Securities Commissions (IOSCO)

4. The Financial Stability Board (FSB)

The Basel Committee on Banking Supervision provides a forum for regular cooperation on banking supervisory matters. Its main objective is to improve the quality of banking supervision worldwide. The International Association of Insurance Supervisors is the equivalent of the Basel Committee for the insurance industry. Its goal is to coordinate local regulations and to promote a consistent and global supervision for insurance companies. The International Organization of Securities Commissions is the international body that develops and implements standards and rules for securities and market regulation. While these three authorities are dedicated to a specific financial industry (banks, insurers and markets), the FSB is an international body that makes recommendations about the systemic risk of the global financial system. In particular, it is in charge of defining systemically important financial institutions or SIFIs.

These four international bodies define standards at the global level and promote convergence between local supervision. The implementation of the rules is the responsibility of national supervisors. In the case of the European Union, they are the European Banking Authority (EBA), the European Insurance and Occupational Pensions Authority (EIOPA), the European Securities and Markets Authority (ESMA) and the European System of Financial Supervision (ESFS). A fifth authority, the European Systemic Risk Board (ESRB), completes the European supervision system.

The equivalent authorities in the US are the Board of Governors of the Federal Reserve System, also known as the Federal Reserve Board (FRB), the Federal Insurance Office (FIO) and the Securities and Exchange Commission (SEC). In fact, the financial supervision is more complicated in US as shown by Jickling and Murphy (2010). The supervisor of banks is traditionally the Federal Deposit Insurance Corporation (FDIC) for federal banks and the Office of the Comptroller of the Currency (OCC) for national banks. However, the Dodd-Frank Act created the Financial Stability Oversight Council (FSOC) to monitor systemic risk. For banks and other financial institutions designated by the FSOC as SIFI, the supervision is directly done by the FRB. The supervision of markets is shared between the SEC and the Commodity Futures Trading Commission (CFTC), which supervises derivatives trading including futures and options[8].

TABLE 1.3: The supervision institutions in finance

	Banks	Insurers	Markets	All sectors
Global	BCBS	IAIS	IOSCO	FSB
EU	EBA/ECB	EIOPA	ESMA	FRB
US	FDIC/FRB	FIO	SEC	ESFS

1.2.1 Banking regulation

The evolution of the banking supervision has highly evolved since the end of the eighties. Here are the principal dates:

- 1988 Publication of "*International Convergence of Capital Measurement and Capital Standards*", which is better known as "*The Basel Capital Accord*". This text sets the rules of the Cooke ratio.
- 1993 Development of the Capital Adequacy Directive (CAD) by the European Commission.
- 1996 Publication of "*Amendment to the Capital Accord to incorporate Market Risks*". This text includes the market risk to compute the Cooke ratio.
- 2001 Publication of the second consultative document "*The New Basel Capital Accord*".
- 2004 Publication of "*International Convergence of Capital Measurement and Capital Standards – A Revisited Framework*". This text establishes the Basel II framework.

[8] A complete list of supervisory authorities by countries are provided in page 32.

2006 Implementation of Basel II.
2010 Publication of the Basel III framework. Its implementation is scheduled from 2013 until 2019.

In 1998, BCBS introduces the Cooke ratio[9], which is the minimum amount of capital a bank should maintain in case of unexpected losses. Its goal is to:

- provide an adequation between the capital hold by the bank and the risk taken by the bank;
- enhance the soundness and stability of the banking system;
- reduce the competitive inequalities between banks[10].

It is measured as follows:

$$\text{Cooke Ratio} = \frac{C}{\text{RWA}}$$

where C and RWA are the capital and the risk-weighted assets of the bank. A risk-weighted asset is simply defined as a bank's asset weighted by its risk score or risk weight (RW). Because bank's assets are mainly credits, the notional is generally measure by the exposure at default (EAD). To compute risk-weighted assets, we then use the following formula:

$$\text{RWA} = \text{EAD} \times \text{RW}$$

The original Basel Accord only considers credit risk and classifies bank's exposures into four categories depending on the value of the risk weights (0%, 20%, 50% and 100%):

- cash, gold, claims on OECD governments and central banks, claims on governments and central banks outside OECD and denominated in the national currency are risk-weighted at 0%;
- claims on all banks with a residual maturity lower than one year, longer-term claims on OECD incorporated banks, claims on public-sector entities within the OECD are weighted at 20%;
- loans secured on residential property are risk-weighted at 50%;
- longer-term claims on banks incorporated outside the OECD, claims on commercial companies owned by the public sector, claims on private-sector commercial enterprises are weighted at 100%.

[9]This ratio took the name of Peter Cooke, who was the Chairman of the BCBS between 1977 and 1988.

[10]This was particular true between Japanese banks, which were weakly capitalized, and banks in the US and Europe.

For the other assets[11], discretion is given to each national supervisory authority to determine the appropriate weighting factors. Concerning off-balance-sheet exposures, engagements are converted to credit risk equivalents by multiplying the nominal amounts by a credit conversion factor (CCF) and the resulting amounts are risk-weighted according to the nature of the counterparty. Concerning the numerator of the ratio, BCBS distinguishes tier 1 capital and tier 2 capital. Tier 1 capital (or core capital) is composed of[12]:

- common stock (or paid-up share capital);
- disclosed reserves (or retained earnings).

whereas tier 2 capital represent supplementary capital such as[13]:

- undisclosed reserves;
- asset revaluation reserves;
- general loan-loss reserves (or general provisions);
- hybrid debt capital instruments
- subordinated debt.

The Cooke ratio required a minimum capital ratio of 8% when considering both tier 1 and tier 2 capital, whereas tier 1 capital ratio should be at least half of the total capital or 4%.

Example 1 *The assets of a bank are composed of \$100 mn of US treasury bonds, \$100 mn of Brazilian government bonds, \$50 mn of residential mortgage, \$300 mn of corporate loans and \$20 mn of revolving credit loans. The bank liability structure includes \$25 mn of common stock and \$13 mn of subordinated debt.*

For each asset, we compute RWA by choosing the right risk weight factor. We obtain the following results:

Asset	EAD	RW	RWA
US treasury bonds	100	0%	0
Brazilian Gov. bonds	100	100%	100
Residential mortgage	50	50%	25
Corporate loans	300	100%	300
Revolving credit	20	100%	20
Total			445

[11] They are for example claims on governments and central banks outside OECD and denominated in a foreign currency, claims on public-sector entities outside the OECD, etc.

[12] At least 50% of the tier 1 capital should come from the common equity.

[13] The comprehensive definitions and restrictions to define all the elements of capital are defined in Appendix 1 in BCBS (1998).

The risk-weighted assets of the bank are then equal to \$445 mn. We deduce that the capital adequacy ratio is:

$$\text{Cooke Ratio} = \frac{38}{445} = 8.54\%$$

This bank meets the regulatory requirements, because the Cooke ratio is higher than 8% and the tier 1 capital ratio[14] is also higher than 4%. Suppose now that the capital of the bank consists of \$13 mn of common stock and \$25 mn of subordinated debt. In this case, the bank does not satisfy the regulatory requirements, because the tier 2 capital can not exceed the tier 1 capital, meaning that the Cooke ratio is equal to 5.84% and the capital tier 1 ratio is equal to 2.92%.

The Basel Accord, which has been adopted by more than 100 countries, has been implemented in the US by the end of 1992 and in Europe in 1993[15]. In 1996, the Basel Committee published a revision of the original Accord by incorporating market risk. This means that banks have to calculate capital charges for market risk in addition to the credit risk. The major difference with the previous approach to measure credit risk is that banks have the choice between two methods for applying capital charges for the market risk:

- The standardized measurement method (SMM)
- The internal model-based approach[16] (IMA)

With the SMM, the bank apply a fixed capital charge for each asset. The market risk requirement is therefore the sum of the capital charges for all the assets that compose the bank's portfolio. With IMA, the bank estimates the market risk capital charge by computing the 99% value-at-risk of the portfolio's loss for a holding period of 10 trading days. From a statistical point of view, the value-at-risk with confidence level α is defined as the quantile α associated to the probability distribution of the portfolio loss. Its computation is illustrated in Figure 1.4.

Another difference with credit risk is that the bank compute directly the market risk capital requirement $\boldsymbol{\mathcal{K}}_{\mathrm{MR}}$ with these two approaches[17]. Therefore, the Cooke ratio becomes[18]:

$$\frac{C_{\mathrm{Bank}}}{\mathrm{RWA} + 12.5 \times \boldsymbol{\mathcal{K}}_{\mathrm{MR}}} \geq 8\%$$

[14]The tier 1 capital ratio is equal to $25/445 = 5.26\%$.

[15]Under the European directive 93/6/EEC, which is better known as the capital adequacy directive

[16]The use of the internal model method is subject to the approval of the national supervisor.

[17]We use the symbols C and $\boldsymbol{\mathcal{K}}$ in order to make the distinction between the capital of the bank and the regulatory capital.

[18]When considering market risk, the total capital may include tier 3 capital, consisting of short-term subordinated debt with an original maturity of at least 2 years.

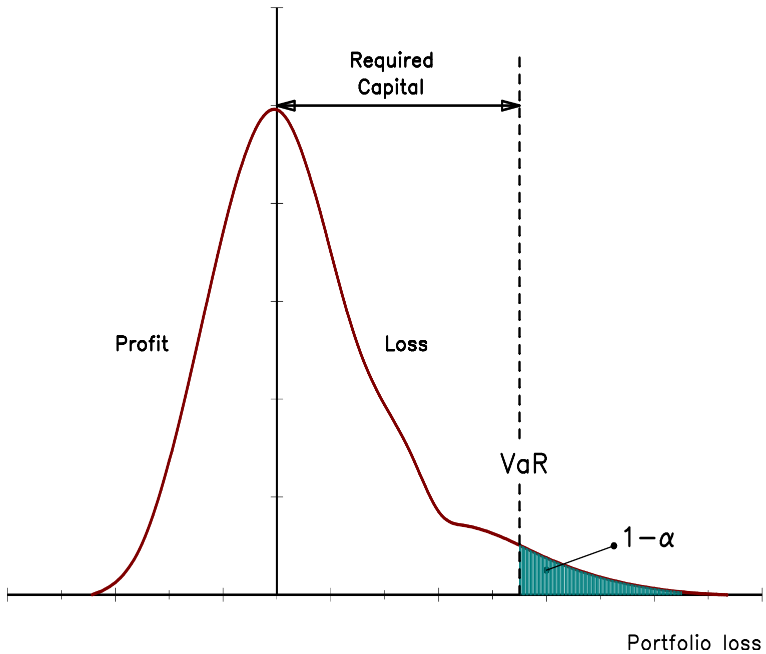

FIGURE 1.4: Probability distribution of the portfolio loss

We deduce that:

$$C_{\text{Bank}} \geq \underbrace{8\% \times \text{RWA}}_{\mathcal{K}_{\text{CR}}} + \mathcal{K}_{\text{MR}}$$

meaning that $8\% \times \text{RWA}$ can be interpreted as the credit risk capital requirement $\mathcal{K}_{\text{CR}}$, which can be compared to the market risk capital charge $\mathcal{K}_{\text{MR}}$.

Example 2 *We consider Example 1 and assume that the bank has a market risk on an equity portfolio of* \$25 *mn. The corresponding risk capital charge for a long exposure on a diversified portfolio of stocks is equal to* 12%. *Using its internal model, the bank estimates that the* 99% *quantile of the portfolio loss is equal to* \$1.71 *mn for a holding period of* 10 *days.*

In the case of the standardized measurement method, the market risk capital requirement is equal to \$3 mn[19] The capital ratio becomes:

$$\text{Cooke Ratio} = \frac{25}{445 + 12.5 \times 3} = 7.88\%$$

[19]We have:

$$\mathcal{K}_{\text{MR}} = 12\% \times 25 = 3$$

In this case, the bank does not meet the minimum capital requirement of 8%. If the bank uses its internal model, the Cooke ratio is satisfied:

$$\text{Cooke Ratio} = \frac{25}{445 + 12.5 \times 1.71} = 8.15\%$$

The Basel Accord has been highly criticized, because the capital charge for credit risk is too simplistic and too little risk sensitive: limited differentiation of credit risk, no maturity, granularity of risk weights, etc. These resulted in regulatory arbitrage through the use of securitization between assets with same regulatory risk but different economic risk. In June 1999, the Basel Committee produced an initial consultative document with the objective to replace the 1998 Accord by a new capital adequacy framework. This paper introduces some features about Basel II, but this is really the publication of the second consultative paper in January 2001 that marks a milestone for the banking regulation. Indeed, the 2001 publication is highly detailed and comprehensive, and the implementation of this new framework seemed very complex at that time. The reaction of the banking industry was negative and somehow hostile at the beginning, in particular because the Basel Committee introduced a third capital charge for operational risk beside credit and market risk and the implementation costs were very high. It has taken a long time until the Basel Committee and the banking industry converge to an accord. Lastly, the finalized Basel II framework is published in June 2004.

TABLE 1.4: The three pillars of the Basel II framework

Pillar 1 Minimum Capital Requirements	Pillar 2 Supervisory Review Process	Pillar 3 Market Discipline
Credit risk Market risk Operational risk	Review & reporting Capital above Pillar 1 Supervisory intervention	Capital structure Capital adequacy Models & parameters Risk management

The new Accord consists of three pillars:

1. the first pillar corresponds to *minimum capital requirements*, that is how to compute the capital charge for credit risk, market risk and operational risk;

2. the second pillar describes the *supervisory review process*; it explains

the role of the supervisor and gives the guidelines to compute additional capital charges for specific risks, which are not covered by the first pillar;

3. the *market discipline* establishes the third pillar and details the disclosure of required information regarding the capital structure and the risk exposures of the bank.

Regarding the first pillar, the Cooke ratio becomes:

$$\frac{C_{\text{Bank}}}{\text{RWA} + 12.5 \times \boldsymbol{\mathcal{K}}_{\text{MR}} + 12.5 \times \boldsymbol{\mathcal{K}}_{\text{OR}}} \geq 8\%$$

where $\boldsymbol{\mathcal{K}}_{\text{OR}}$ is the capital charge for operational risk. This implies that the required capital is directly computed for market risk and operational risk whereas credit risk is indirectly measured by risk-weighted assets[20].

Example 3 *We assume that the risk-weighted assets for the credit risk are equal to \$500 mn, the capital charge for the market risk is equal to \$10 mn and the capital charge for the operational risk is equal to \$3 mn.*

We deduce that the required capital for the bank is:

$$\begin{aligned}
\boldsymbol{\mathcal{K}} &= 8\% \times (\text{RWA} + 12.5 \times \boldsymbol{\mathcal{K}}_{\text{MR}} + 12.5 \times \boldsymbol{\mathcal{K}}_{\text{OR}}) \\
&= 8\% \times \text{RWA} + \boldsymbol{\mathcal{K}}_{\text{MR}} + \boldsymbol{\mathcal{K}}_{\text{OR}} \\
&= 8\% \times 500 + 10 + 3 \\
&= 53
\end{aligned}$$

This implies that credit risk represents 75.5% of the total risk.

With respect to the original Accord, the Basel Committee did not change the market risk approach whereas it profoundly changed the methods to compute the capital charge for the credit risk. Two approaches are proposed:

- The standardized approach (SA)
  This approach, which is more sensitive than Basel I, is based on external ratings provided by credit rating agencies. The capital charge is computed by considering a mapping function between risk weights and ratings.

- The internal rating-based approach (IRB)
  This approach can be viewed as an external risk model with internal and external risk parameters. The key parameter is the default probability of the asset, which is deduced from the internal credit rating model of the bank. The Basel Committee makes the distinction between two methods. In the foundation IRB (FIRB), the bank only estimates the probability

[20] In fact, we can define risk-weighted assets for each category of risk. We have the following relationships $\text{RWA}_{\mathcal{R}} = 12.5 \times \boldsymbol{\mathcal{K}}_{\mathcal{R}}$ and $\boldsymbol{\mathcal{K}}_{\mathcal{R}} = 8\% \times \text{RWA}_{\mathcal{R}}$ where $\boldsymbol{\mathcal{K}}_{\mathcal{R}}$ is the required capital for the risk $\mathcal{R}$. The choice of defining either $\text{RWA}_{\mathcal{R}}$ or $\boldsymbol{\mathcal{K}}_{\mathcal{R}}$ is a mere convention.

of default and uses standard values for the other risk parameters of the model. In the advanced IRB (AIRB), the bank may estimate all the risk parameters.

Regarding operational risk, the Basel Committee propose three approaches to compute the required capital:

- The Basic Indicator Approach (BIA)
  In this case, the capital charge is a fixed percentage α of the gross income.

- The Standardized Approach (TSA)
  This method consists of dividing bank's activities into eight business lines. For each business line, the capital charge is a fixed percentage β of its gross income. The parameter β depends on the riskiness of the business line. The total capital is the sum of the eight regulatory capital charges.

- Advanced Measurement Approaches (AMA)
  In this approach, the bank uses a statistical model with internal data for estimating the total capital.

A summary of the different options is reported in Figure 1.5.

The European Union has adopted the Basel II framework in June 2006 with the capital requirements directive[21] (CRD). In the United States, Basel II is partially applied since 2006 and only concerns the largest banking institutions (Getter, 2014). Since the 2004 publication, more than 40 countries have fully implemented Basel II (Hong Kong in January 2007, Japan in March 2007, Canada in November 2007, South Korea in December 2007, Australia in January 2008, South Africa in January 2008, etc.). However, the subprime crisis in 2007 and the collapse of Lehman Brothers in September 2008 illustrated the limits of the new Accord concerning the issues of leverage and liquidity. In response to the financial market crisis, the Basel Committee enhances then the new Accord by issuing a set of documents between 2008 and 2009. In July 2009, the Basel Committee approved a package of measures to strengthen the rules governing trading book capital, particularly the market risk associated to securitization and credit-related products. Known as the Basel 2.5 framework, these new rules can be summarized into four main elements, which are:

1. the incremental risk charge (IRC), which is an additional capital charge to capture default risk and migration risk for unsecuritized credit products;

2. the stressed value-at-risk requirement (SVaR), which is intended to capture stressed market conditions;

[21] It replaces CAD II (or the 98/31/EEC directive), which is the revision of the original CAD and incorporates market risk.

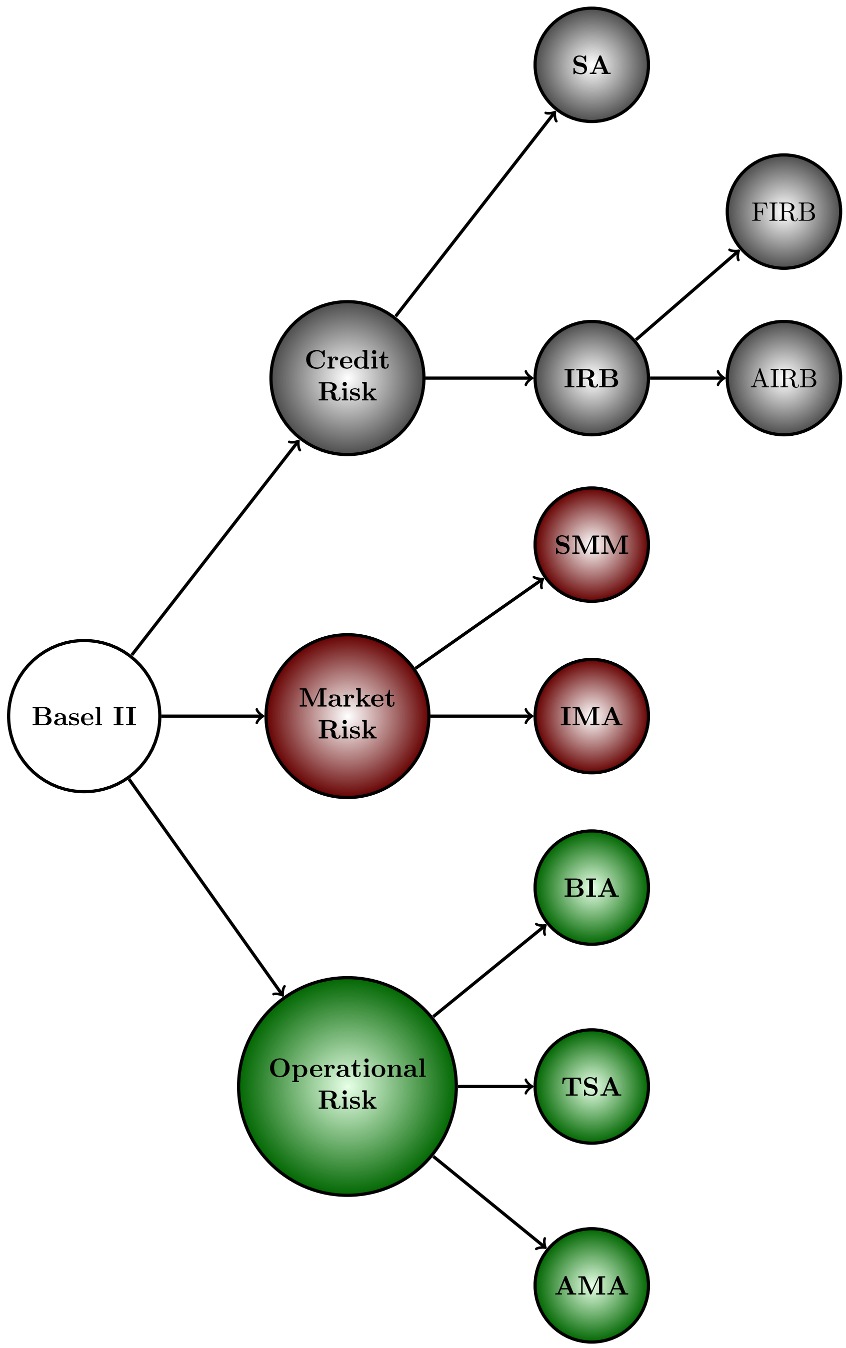

FIGURE 1.5: Minimum capital requirements in the Basel II framework

3. the comprehensive risk measure (CRM), which is an estimate of risk in the credit correlation trading portfolio (CDS baskets, CDO products, etc.);

4. new standardized charges on securitization exposures, which are not covered by CRM.

In addition to these elements affecting the first pillar, the Basel Committee also expand the second pillar (largest exposures and risk concentrations, remuneration policies, governance and risk management) and enhances the third pillar (securitization and re-securitization exposures). The coming into force of Basel 2.5 was December 2011 in the European Union[22] and January 2013 in the United States (BCBS, 2015).

In December 2010, the Basel Committee published a new regulatory framework in order to enhance risk management, increase the stability of the financial markets and improve the banking industry's ability to absorb macroeconomic shocks. The Basel III framework consists of micro-prudential and macro-prudential regulation measures concerning;

- a new definition of the risk-based capital;
- the introduction of a leverage ratio;
- the management of the liquidity.

The capital is redefined as follows. Tier 1 capital is composed of common equity tier 1 (CET1) capital (common equity and retained earnings) and additional tier 1 (AT1) capital. The new capital ratios are 4.5% for CET1, 6% for tier 1 and 8% for total capital (T1 + T2). Therefore, Basel III gives preference to tier 1 capital rather than tier 2 capital whereas the tier 3 risk capital is eliminated. BCBS (2010) introduces also a surplus of CET1, which is "*designed to ensure that banks build up capital buffers outside periods of stress which can be drawn down as losses are incurred*". This capital conservation buffer (CB), which is equal to 2.5% of RWA, applies at all the times outside periods of stress. The aim is to reduce the distribution of earnings and to support the business of bank through periods of stress. A macro-prudential approach completes capital requirements by adding a second capital buffer called the countercyclical capital buffer (CCB). During periods of excessive credit growth, national authorities may require an additional capital charge between 0% and 2.5%, which increases the CET1 ratio until 9.5% (including the conservation buffer). The underlying idea is to smooth the credit cycle, to reduce the procyclicality and to help banks to provide credit during bad periods of economic growth. The implementation of this new framework is progressive from April 2013 until March 2019. A summary of capital requirements and transitional periods is given in Table 1.5.

[22]The Basel 2.5 framework was adopted in two stages: CRD II (or the 2009/111/EC directive) in November 2009 and CRD III (or the 2010/76/EU directive) in December 2010.

TABLE 1.5: Basel III capital requirements

Capital ratio	2013	2014	2015	2016	2017	2018	2019
CET1	3.5%	4.0%	4.5%				**4.5%**
CB				0.625%	1.25%	1.875%	**2.5%**
CET1 + CB	3.5%	4.0%	4.5%	5.125%	5.75%	6.375%	**7.0%**
Tier 1	4.5%	5.5%	6.0%				**6.0%**
Total	8.0%						**8.0%**
Total + CB	8.0%			8.625%	9.25%	9.875%	**10.5%**
CCB				**0% − 2.5%**			

Source: Basel Committee on Banking Supervision (2015),
`www.bis.org/bcbs/basel3.htm`.

Remark 1 *Basel III defines a third capital buffer for systemic banks, which can vary between 1% and 5%. This topic will be presented later on the paragraph dedicated to systemically important financial institutions on page 30.*

Remark 2 *This new definition of the capital is accompanied by a change of the required capital for counterparty credit risk (CCR). In particular, BCBS (2011) adds a credit valuation adjustment charge (CVA) for OTC derivative trades. CVA is defined as the market risk of loss caused by changes in the credit spread of a counterparty due to changes in its credit quality. It also corresponds to the market value of counterparty credit risk.*

Basel III also includes a leverage ratio to prevent the build-up of excessive on- and off-balance sheet leverage in the banking sector. BCBS (2014a) defines this ratio as follows:

$$\text{Leverage ratio} = \frac{\text{Tier 1 capital}}{\text{Total exposures}} \geq 3\%$$

where the total exposures is the sum of on-balance sheet exposures, derivative exposures and some adjustments concerning off-balance sheet items. The leverage ratio can be viewed as the second macro-prudential measure of Basel III. Indeed, during credit boom, we generally observe compression of risk weight assets and a growth of the leverage, because the number of profitable projects increases during economic good times. For instance, Brei and Gambacorta (2014) show that the Basel III leverage ratio is negatively correlated with GDP or credit growth. By introducing a floor value, the Basel Committee expects that the leverage ratio will help to reduce the procyclicality like the countercyclical capital buffer.

The management of the liquidity is another important issue of Basel III. The default of Lehman Brothers was followed by a lack of liquidity, which is one of the main sources of systemic risk. For instance, Brunnermeier and

Pedersen (2009) demonstrated that a liquidity dry-up event arising from a fight-to-quality environment can result in runs, fire sales, and asset liquidations in general transforming the market into a contagion mechanism. In order to prevent such events, the Basel Committee proposed several liquidity rules and introduced in particular two liquidity ratios:

- The liquidity coverage ratio (LCR)
  The objective of the LCR is to promote short-term resilience of the bank's liquidity risk profile. It is expressed as:

  $$\text{LCR} = \frac{\text{HQLA}}{\text{Total net cash outflows}} \geq 100\%$$

  where HQLA is the stock of high quality liquid assets and the denominator is the total net cash outflows over the next 30 calendar days. Therefore, the LCR is designed to ensure that the bank has the necessary assets to face a one-month stressed period of outflows.

- The net stable funding ratio (NSFR)
  NSFR is designed in order to promote long-term resilience of the bank's liquidity profile. It is defined as the amount of available stable funding (ASF) relative to the amount of required stable funding (RFS):

  $$\text{NSFR} = \frac{\text{Available amount of stable funding}}{\text{Required amount of stable funding}} \geq 100\%$$

  The amount of available stable funding is equal to the regulatory capital[23] plus the other liabilities to which we apply a scaling factor between 0% and 100%. The amount of required stable funding is the sum of two components: weighted assets and off-balance sheet exposures.

The implementation of Basel III was due to January 2013, but some countries have delayed the adoption of the full package. According to BCBS (2015c), the rules for risk-based capital are more adopted than those concerning the liquidity ratio or the leverage ratio. In the US, the rules for risk-based capital and the leverage ratio are effective since January 2014, while the LCR rule came into effect in January 2015. In the European Union, the Basel III agreement is transposed on July 2013 into two texts: the CRD IV (or the 2013/36/EU directive) and the capital requirements regulation (CRR) (or the 575/2013 EU regulation). Therefore, Basel III is effective since January 2014 for the rules of risk-based capital and leverage ratio and October 2015 for the LCR rule.

Even before Basel III is fully implemented, the Basel Committee has published a set of consultative documents, which may be viewed as the basis of a future Basel IV. The guiding principle of these works is to simplify the different approaches to compute the regulatory capital and to reduce the risk of

[23]Excluding tier 2 instruments with residual maturity of less than one year.

arbitrage between standardized and advanced methods. These new proposals concern review of the market risk measurement (2013b, 2014h), revision to the standardized approach for credit (2015) and operational risks (2014f), minimum capital requirements for interest rate risk in the banking book (2015d) and a modified framework for CVA risk (2015e). The changes are very significant. For instance, they propose to replace the VaR measure by the expected shortfall measure, give up external ratings or calculate a regulatory capital for interest rate risk in the banking book.

1.2.2 Insurance regulation

Contrary to the banking industry, the regulation in insurance is national. The International Association of Insurance Supervisors (IAIS) is an association to promote globally consistent supervision. For that, the IAIS is responsible for developing principles and standards, which form the Insurance Core Principles (ICP). For instance, the last release of ICP was in October 2013 and contained 26 ICPs[24]. However, its scope of intervention is more limited than this of the BCBS. In particular, the IAIS does not produce any methodologies of risk management or formula to compute risk-based capital. In Europe, the regulatory framework is the Solvency II directive (or the 2009/138/EC directive), which harmonizes the insurance regulation and capital requirements in the European Union. In the US, the supervisor is the National Association of Insurance Commissioners (NAIC). In 2008, it has created a Solvency Modernization Initiative (SMI) in order to reform the current framework in the spirit of Solvency II. However, the convergence across the different jurisdictions is far to being reached.

Solvency I (or the 2002/13/EC directive) is a set of rules to define the insurance solvency regime and was put in place on January 2004 in the European Union. It defined how an insurance company should calculate its liabilities and the required capital. In this framework, the capital is the difference between the book value of assets and the technical provisions (or insurance liabilities). This capital is decomposed in the solvency capital requirement (or SCR) and the surplus (see Figure 1.6). One of the main drawback of Solvency I is that assets and liabilities are evaluated using an accounting approach (historical or amortized cost).

In an address to the European Insurance Forum 2013, Matthew Elderfield, Deputy Governor of the Central Bank of Ireland, justifies the reform of the insurance regulation in Europe as follows:

> "*[...] it is unacceptable that the common regulatory framework for insurance in Europe in the 21st-century is not risk-based and only takes account, very crudely, of one side of the balance sheet. The*

[24] ICP 1 concerns the objectives, powers and responsibilities of the supervisor, ICP 17 is dedicated to capital adequacy, ICP 24 presents the macro-prudential surveillance and insurance supervision, etc.

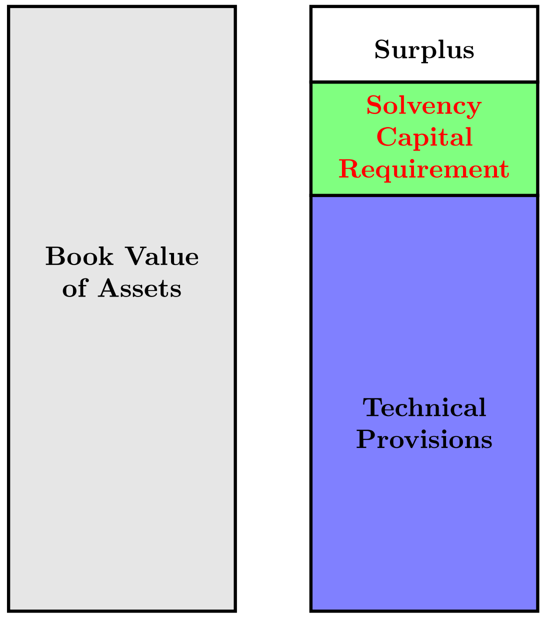

FIGURE 1.6: Solvency I capital requirement

> *European Union urgently needs a new regulatory standard which differentiates solvency charges based on the inherent risk of different lines of business and which provides incentives for enhanced risk management. It urgently needs a framework that takes account of asset risks in an insurance company. It urgently needs a framework that encourages better governance and management of risk. And it urgently needs a framework that provides better disclosure to market participants*" (Elderfield, 2013, page 1).

With Solvency II, capital requirements are then based on an economic valuation of the insurer balance sheet, meaning that:

- assets are valued at their market value;
- liabilities are valued on a best estimate basis.

In this framework, the economic value of liabilities corresponds to the expected present value of the future cash flows. Technical provisions are then the sum of the liabilities best estimate and a risk margin (or prudence margin) in order to take into account non-hedgeable risk components. Solvency II defines two levels of capital requirements. The minimum capital requirement (MCR) is the required capital under which risks are considered as being unacceptable. The solvency capital requirement (SCR) is the targeted required capital (SCR $\geq$ MCR). The underlying idea is to cover the different source of risk at a 99.5%

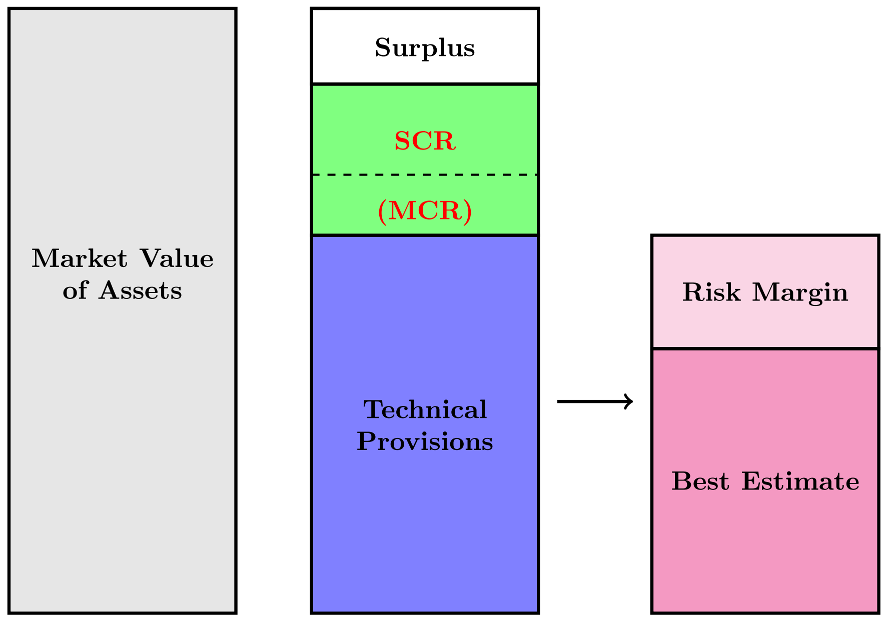

FIGURE 1.7: Solvency II capital requirement

confidence level[25] for a holding period of one year. The insurance company may opt for the standard formula or its own internal model for computing the required capital. In the case of the standard formula method, the SCR of the insurer is equal to:

$$\mathrm{SCR} = \sqrt{\sum_{i,j}^{m} \rho_{i,j} \times \mathrm{SCR}_i \times \mathrm{SCR}_j} + \mathrm{SCR}_{\mathrm{OR}}$$

where SCR_i is the SCR of the risk module i, $\mathrm{SCR}_{\mathrm{OR}}$ is the SCR associated to the operational risk and $\rho_{i,j}$ is the correlation factor between risk modules i and j. Solvency II considers several risk components: underwriting risk (non-life, life, health, etc.), market risk, default and counterpart credit risk[26]. For each risk component, a formula is provided to compute the SCR of the risk factors[27]. Regarding the capital C, own funds are classified into basic own funds and ancillary own funds. The basic own funds consist of the excess of assets over liabilities, and subordinated liabilities. The ancillary own funds corresponds to other items which can be called up to absorb losses. Examples of ancillary own funds are unpaid share capital or letters of credit and guar-

[25] It is set to 85% for the MCR.

[26] Solvency II is an ambitious and complex framework because it mixes both assets and liabilities, risk management and ALM.

[27] See Chapter 7 for the detailed calculations.

antees. Own funds are then divided into tiers depending on their permanent availability and subordination. For instance, tier 1 corresponds to basic own funds which are immediately available and fully subordinated. The solvency ratio is then defined as:

$$\text{Solvency Ratio} = \frac{C}{\text{SCR}}$$

This solvency ratio must be larger than 33% for tier 1 and 100% for the total own funds.

The quantitative approach to compute MCR, SCR and the technical provisions define Pillar 1 (Figure 1.7). As in Basel II framework, it is completed by two other pillars. Pillar 2 corresponds to the governance of the solvency system and concerns qualitative requirements, rules for supervisors and own risk and solvency assessment (ORSA). Pillars 3 includes market disclosures and also supervisory reporting.

Remark 3 *Solvency II is different than other national frameworks. However, we can consider it as the most complete regulation in the insurance industry. This is why we will focus on this approach in Chapter 7 and we will show the differences with the NAIC risk-based capital (RBC) approach, which is implemented in the US.*

1.2.3 Market regulation

Banks and insurers are not the only financial institutions that are regulated and the financial regulatory framework does not reduce to Basel III and Solvency II. In fact, a whole variety of legislation measures helps to regulate the financial market and the participants.

In Europe, the markets in financial instruments directive or MiFID[28] came in force since November 2007. Its goal was to establish a regulatory framework for the provision of investment services in financial instruments (such as brokerage, advice, dealing, portfolio management, underwriting, etc.) and for the operation of regulated markets by market operators. The scope of application concerns various aspects such as passporting, client categorization (retail/professional investor), pre-trade and post-trade transparency or best execution procedures. In August 2012, MiFID is completed by the European market infrastructure regulation (EMIR), which is specifically designed to increase the stability of the OTC derivative markets by promoting central counterparty clearing and trade repositories. In June 2014, MiFID is revised (MiFID 2) and the regulation on markets in financial instruments (MiFIR) replaces EMIR. According to ESMA[29], this supervisory framework concerns 104 European regulated markets at the date of May 2015. On April 2014,

[28] It corresponds to the 2004/39/EC directive

[29] See the website `mifiddatabase.esma.europa.eu`.

the European parliament completes the framework by publishing new rules to protect retail investors (packaged retail and insurance-based investment products or PRIIPS). These rules complete the various UCITS directives, which organize the distribution of mutual funds in Europe.

In the US, the regulation of the market dates back to the 1930s:

- The Securities Act of 1933 concerns the distribution of new securities.
- The Securities Exchange Act of 1934 regulates trading securities, brokers, and exchanges, whereas the Commodity Exchange Act regulates the trading of commodity futures.
- The Trust Indenture Act of 1939 defines the regulating rules for debt securities.
- The Investment Company Act of 1940 is the initial regulation framework of mutual funds.
- The Investment Advisers Act of 1940 is dedicated to investment advisers.

At the same time, the Securities and Exchange Commission (SEC) was created to monitor financial markets (stocks and bonds). Now, the area of SEC supervision is enlarged and concerns stock exchanges, brokers, mutual funds, investment advisors, some hedge funds, etc. In 1974, the Commodities Future Trading Commission Act established the Commodity Futures Trading Commission (CFTC) as the supervisory agency responsible for regulating the trading of futures contracts. The market regulation in the US has not changed significantly until the 2008 financial crisis. In 2010, President Barack Obama signed an ambitious federal law, the Dodd-Frank Wall Street Reform and Consumer Protection Act also named more simply Dodd-Frank, which is viewed as a response to the crisis. This text has an important impact on various areas of regulation (banking, market, investors, asset managers, etc.). It also introduces a new dimension in regulation. It concerns the coordination among regulators with the creation of the Financial Stability Oversight Council (FSOC), whose goal is to monitor the systemic risk.

1.2.4 Systemic risk

The 2008 financial crisis has an unprecedent impact on the financial regulation. It was responsible for Basel III, Dodd-Frank, Volcker rule, etc., but it has also inspired new considerations on the systemic risk. Indeed, the creation of the Financial Stability Board (FSB) in April 2009 was motivated to establish an international body that monitors and makes recommendations about the global financial system, and especially the associated systemic risk. Its area of intervention covers not only banking and insurance, but also all the other financial institutions including asset managers, finance companies, market intermediaries, investors, etc.

The main task of FSB is to develop assessment methodologies for defining systemically important financial institutions (SIFIs) and to make policy recommendations for mitigating the systemic risk of the financial system. According to FSB (2010), SIFIs are institutions whose "*distress or disorderly failure, because of their size, complexity and systemic interconnectedness, would cause significant disruption to the wider financial system and economic activity*". By monitoring SIFIs in a different way than other financial institutions, the objective of the supervisory authorities is obviously to address the '*too big too fail*' problem. A SIFI can be global (G-SIFI) or domestic (D-SIFI). FSB also distinguishes between three types of G-SIFIs:

1. G-SIBs correspond to global systemically important banks.
2. G-SIIs designate global systemically important insurers.
3. The third category is defined with respect to the two previous ones. It incorporates other SIFIs than banks and insurers (non-bank non-insurer global systemically important financial institutions or NBNI G-SIFIs).

The FSB-BCBS framework for identifying G-SIBs is a scoring system based on five categories: size, interconnectedness, substitutability/financial institution infrastructure, complexity and cross-jurisdictional activity (BCBS, 2014g). In November 2015, there is 30 G-SIBs (FSB, 2015b). Depending on the score value, the bank is then assigned to a specific bucket, which is used to calculate the higher loss absorbency (HLA) requirement. This additional capital requirement is part of the Basel III framework and ranges from 1% to 3.5% common equity tier 1. According to FSB (2015b), the two most systemically important banks are HSBC and JPMorgan Chase, which are assigned to an additional capital buffer of 2.5% CET1. This means that the total capital for these two banks can go up to 15.5% with the following decomposition: tier 1 = 6.0%, tier 2 = 2.0%, conservation buffer = 2.5%, countercyclical buffer = 2.5% and systemic risk capital = 2.5%.

For insurers, the assessment methodology is close to the methodology for G-SIBs and is based on five categories: size, global activity, interconnectedness, non-traditional insurance and non-insurance activities and substitutability (IAIS, 2013). However, this quantitative approach is completed by a qualitative analysis and the final list of G-SIIs is the result of the IAIS supervisory judgment. In November 2015, there are 9 G-SIIs (FSB, 2015c). The associated policy measures are documented in IAIS (2014) and consist of three main axes: recovery and resolution planning requirements, enhanced supervision and higher loss absorbency requirements.

Concerning NBNI SIFIs, FSB and IOSCO are still in a consultation process in order to finalize the assessment methodologies (FSB, 2015a). Indeed, the second consultation paper considers three categories of participants in the financial sectors that it identifies as potential NBNI SIFIs:

1. finance companies;

2. market intermediaries, especially securities broker-dealers;
3. investment funds, asset managers and hedge funds.

The final assessment methodology is planned for the end of 2015. However, the fact that FSB already considers that there are other SIFIs than banks and insurers suggests that financial regulation will be strengthened for many financial institutions including the three previous categories but also other financial institutions such as pension funds, sovereign wealth funds, etc.

The identification of SIFIs is not the only task of FSB. The other important objective is to monitor the shadow banking system and to understand how it can pose systemic risk. The shadow banking system can be described as "*credit intermediation involving entities and activities outside the regular banking system*" (FSB, 2011). It is also called non-bank credit intermediation. The shadow banking system may expose the traditional banking system to systemic risk, because they may be spill-over effects between the two systems. Moreover, shadow banking entities (SBE) are not subject to tight regulation like banks. However, it runs bank-like activities such as maturity transformation, liquidity transformation, leverage and credit risk transfer. Examples of shadow banking are for instance money market funds, securitization, securities lending, repos, etc. The task force formed by FSB follows a three-step process:

- the first step is to scan and map the overall shadow banking system and to understand its risks;
- the second step is to identify the aspects of the shadow banking system posing systemic risk or regulatory arbitrage concerns;
- the last step is to assess the potential impact of systemic risk induced by the shadow banking system.

Even if this process is ongoing, shadow banking regulation can be found in Dodd-Frank or 2015 consultation paper of EBA. However, until now regulation is principally focused on money market funds.

1.3 Appendix

1.3.1 List of supervisory authorities

We use the following correspondence: **B** for banking supervision, **I** for insurance supervision, **M** for market supervision and **S** for systemic risk supervision.

International authorities

BCBS	Basel Committee on Banking Supervision; `www.bis.org/bcbs`; **B**
FSB	Financial Stability Board; `www.financialstabilityboard.org`; **S**
IAIS	International Association of Insurance Supervisors; `iaisweb.org`; **I**
IOSCO	International Organization of Securities Commissions; `www.iosco.org`; **M**

European authorities

EBA	European Banking Authority; `www.eba.europa.eu`; **B**
ECB/SSM	European Central Bank/Single Supervisory Mechanism; `www.bankingsupervision.europa.eu`; **B**
EIOPA	European Insurance and Occupational Pensions Authority; `eiopa.europa.eu`; **I**
ESMA	European Securities and Markets Authority; `www.esma.europa.eu`; **M**
ESRB	European Systemic Risk Board; `www.esrb.europa.eu`; **S**

US authorities

CFTC	Commodity Futures Trading Commission; `www.cftc.gov`; **M**
FRB	Federal Reserve Board; `www.federalreserve.gov/bankinforeg`; **B/S**
FDIC	Federal Deposit Insurance Corporation; `www.fdic.gov`; **B**
FIO	Federal Insurance Office; `www.treasury.gov/initiatives/fio`; **I**
FSOC	Financial Stability Oversight Council; `www.treasury.gov/initiatives/fsoc`; **S**
OCC	Office of the Comptroller of the Currency; `www.occ.gov`; **B**
SEC	Securities and Exchange Commission; `www.sec.gov`; **M**

Some national authorities

Canada

CSA	Canadian Securities Administrators; `www.securities-administrators.ca`; **M**
OSFI	Office of the Superintendent of Financial Institutions; `www.osfi-bsif.gc.ca`; **B/I**
IIROC	Investment Industry Regulatory Organization of Canada; `www.iiroc.ca`; **M**

China

CBRC	China Banking Regulatory Commission; `www.cbrc.gov.cn`; **B**

France

AMF	Autorité des Marchés Financiers; `www.amf-france.org`; **M**
ACPR	Autorité de Contrôle Prudentiel et de Résolution; `acpr.banque-france.fr`; **B/I**

Germany

BAFIN	Bundesanstalt für Finanzdienstleistungsaufsicht; `www.bafin.de`; **B/I/M**

Italy

BdI	Banca d'Italia; `www.bancaditalia.it`; **B**
CONSOB	Commissione Nazionale per le Società e la Borsa; `www.consob.it`; **M**
IVASS	Istituto per la Vigilanza sulle Assicurazioni; `www.ivass.it`; **I**

Japan

FSA	Financial Services Agency; `www.fsa.go.jp`; **B/I/M**

Luxembourg

CAA	Commissariat aux Assurances; `www.commassu.lu`; **I**
CSSF	Commission de Surveillance du Secteur Financier; `www.cssf.lu`; **B/M**

Spain

BdE	Banco de España; `www.bde.es`; **B**
CNMV	Comisión Nacional del Mercado de Valores; `www.cnmv.es`; **M**
DGS	Dirección General de Seguros y Pensiones; `www.dgsfp.mineco.es`; **I**

Switzerland

FINMA	Swiss Financial Market Supervisory Authority; `www.finma.ch`; **B/I/M**

United Kingdom

FCA	Financial Conduct Authority; `www.fca.org.uk`; **M**
PRA	Prudential Regulation Authority; `www.bankofengland.co.uk/pra`; **B/I**

1.3.2 Timeline of financial regulation

Before 1980

Before 1980, the financial regulation is mainly developed in the US with several acts, which are voted after the Great Depression in the 1930s. These acts concerns a wide range of financial activities, in particular banking, markets and investment sectors. The Basel Committee on Banking Supervision was established in 1974. In Europe, two directives established a regulatory framework for insurance companies.

Banking Regulation	1913	Federal Reserve Act (establishment of the Federal Reserve System as the central banking system of the US)
	1933	Glass-Steagall Act (separation of commercial and investment banking in the US)
	1933	US Banking Act (creation of FDIC and federal insurance deposit
BCBS	1974	Creation of the Basel Committee on Banking Supervision
Solvency I	1973-07-24	Publication of the non-life insurance directive (73/239/EEC) dedicated to solvency margin requirements
	1979-03-05	Publication of the life insurance directive (79/267/EEC) dedicated to solvency margin requirements
Market Regulation	1933-05-27	Securities Act (registration and prospectus of securities)
	1934-06-06	Securities Exchange Act (regulation of the secondary markets and creation of the SEC)
	1936-06-15	Commodity Exchange Act (regulation of the commodity futures)
	1939-08-03	Trust Indenture Act (regulation of debt securities)
	1940-08-22	Investment Advisers Act (regulation of investment advisers)
	1940-08-22	Investment Company Act (regulation of mutual funds)
	1974-10-23	Commodity Future Trading Commission Act (the CFTC replaces the Commodity Exchange Commission)

The years 1980 - 2000

The years 1980-2000 were marked by the development of the banking regulation and the publication of the Basel Accord. In particular, the end of the 1990s saw the implementation of the regulatory framework concerning market risks. In Europe, the UCITS Directive is also an important step concerning the investment industry. In the US, the insurance regulation is reformed with the risk-based capital framework whereas Solvency I is reinforced in Europe.

Basel I	1987-12-15	Publication of the consultative paper on the Cooke ratio
	1988-07-04	Publication of the Basel Capital Accord
	1996-01-18	Publication of the amendment to incorporate market risks
CAD	1993-03-15	Publication of the Capital Adequacy Directive (93/6/EEC) known as CAD I
	1998-06-22	Revision of the CAD (98/31/EEC) known as CAD II
Solvency I	1988-06-22	Second non-life insurance directive 88/357/EEC
	1990-11-08	Second life insurance directive 90/619/EEC
	1992-06-18	Third non-life insurance directive 92/49/EEC
	1992-11-10	Third life insurance directive 92/96/EEC
RBC	1990	NAIC created the US RBC regime
	1992	Implementation of RBC in US insurance
	1993	Finalization of the RBC formula for life insurance
	1994	Finalization of the RBC formula for property and casuality insurance
	1998	Finalization of the RBC formula for health insurance
Market Regulation	1985-12-20	Publication of the first UCITS Directive (85/611/EEC)
	2000-12-14	Commodity Futures Modernization Act (regulation of OTC derivatives in the US)

The years 2000 - 2008

Over the last decade, banks and regulators have invested significant effort and resources to put in place the Basel II framework. This is during this period that modern risk management was significantly developed in the banking sector. The Solvency II reform emerged in 2004 and intensive work was underway to calibrate this new proposition on insurance regulation.

Basel II	1999-06-02	Publication of the first CP on Basel II
	2001-01-29	Publication of the second CP on Basel II
	2001-11-05	Results of the QIS 2
	2002-06-25	Results of the QIS 2.5
	2003-04-29	Publication of the third CP on Basel II
	2003-05-05	Results of the QIS 3
	2004-06-10	Publication of the Basel II Accord
	2004–2005	Conduct of QIS 4 (national impact study and tests)
	2005-07-30	Publication of "*The Application of Basel II to Trading Activities and the Treatment of Double Default Effects*"
	2006-06-16	Results of the QIS 5
	2006-06-30	Publication of the Basel II Comprehensive Version (including Basel I, Basel II and 2005 revisions)
CRD	2006-05-14	Publication of the directive 2006/48/EC
	2006-05-14	Publication of the directive 2006/49/EC (CRD)
Solvency I	2002-03-05	Non-life insurance directive 2002/13/EC (revision of solvency margin requirements)
	2002-11-05	Life insurance recast directive 2002/83/EC
Solvency II	2004	Initial works on Solvency II
	2006-03-17	Report on the first QIS
	2007	Report on the second QIS
	2007-11-01	Report on the third QIS
Market Regulation	2002-01-22	Publication of the directives 2001/107/EC and 2001/108/EC (UCITS III)
	2004-04-21	Publication of the directive 2004/39/EC (MiFID 1)

The years 2008 - 2015

The financial crisis of 2007-2008 completely changed the landscape of financial regulation. Under political pressures, we assist to a frenetic race of regulatory reforms. For instance, the Basel Committee had published 37 regulatory standards before 2007. From January 2008 to June 2015, this number has dramatically increased with 76 new regulatory standards[30]. With Basel 2.5, new capital requirements are put in place for market risk. The Basel III framework is published at the end of 2010 and introduces new standards for managing the liquidity risk. Revisions of the Basel II Accord is already planned

[30]The list of these publications can be obtained on the BCBS website `http://www.bis.org/bcbs/publications.htm` by selecting the publication type '*Standards*'.

(change of the standardized approach for market, credit and operational risks, review of the internal model-based approach for market risk, etc.). In Europe, market regulation is the new hot topic for regulators. However, the major event of the beginning of this decade concerns systemic risk. New regulations have emerged and new financial activities are under scrutiny (shadow banking system, market infrastructures, investment management).

Basel 2.5	2007-10-12	Publication of the first CP on the incremental risk charge
	2008-07-22	Proposed revisions to the Basel II market risk framework
	2009-07-13	Publication of the final version of Basel 2.5
	2012-05-03	Publication of the CP on the fundamental review of the trading book
	2013-12-13	Capital requirements for banks' equity investments in funds
	2014-04-10	Capital requirements for bank exposures to central counterparties
Basel III	2010-12-16	Publication of the original version of Basel III
	2010-12-16	Results of the comprehensive QIS
	2011-06-01	Revised version of the Basel III capital rules reflecting the CVA modification
	2013-01-07	Publication of the rules concerning the liquidity capital ratio
	2014-10-31	Publication of the rules concerning net stable funding ratio
CRD	2009-09-16	Publication of the directive 2009/111/EC (CRD II)
	2010-09-24	Publication of the directive 2010/76/EU (CRD III)
	2013-06-26	Publication of the directive 2013/36/EU (CRD IV)
	2013-06-26	Publication of the capital requirements regulation 575/2013 (CRR)
Solvency II	2008-11	Report on the fourth QIS
	2009-11-25	Solvency II directive 2009/138/EC
	2011-03-14	Report on the fifth QIS
Market Regulation	2009-07-13	Publication of the directive 2009/65/EC (UCITS IV)
	2010-06-08	Publication of the AIFM directive (2011/61/EU)
	2012-07-04	Publication of the EU regulation 648/2012 (EMIR)
	2014-05-15	Publication of the directive 2014/65/EU (MiFID II)

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Market Regulation	2012-05-15	Publication of the EU regulation 600/2014 (MiFIR)
	2014-07-23	Publication of the directive 2014/91/EU (UCITS V)
	2014-11-26	Publication of the EU regulation 1286/2014 (PRIIPS)
Systemic Risk	2009-04	Creation of the Financial Stability Board (FSB)
	2010-07-21	Dodd-Frank Wall Street Reform and Consumer Protection Act
	2010-07-21	Volcker Rule (§619 of the Dodd-Frank Act)
	2011-11-04	Publication of the G-SIB assessment methodology (BCBS)
	2013-07-03	Update of the G-SIB assessment methodology (BCBS)
	2014-11-06	Update of list of G-SIBs (FSB-BCBS)
	2014-11-06	Update of list of G-SIIs (FSB-IAIS)
	2015-03-04	Second CP on assessment methodologies for identifying NBNI-SIFIs (FSB-IOSCO)

We could also mention the following consultations, which will serve to define the future Basel IV Accord.

Basel IV	2013-10-31	Fundamental review of the trading book (FRTB)
	2014-10-06	Revisions to the simpler approaches for operational risk
	2014-12-22	Capital floors: the design of a framework based on standardized approaches
	2014-12-22	Revisions to the standardized approach for credit risk
	2015-06-08	Interest rate risk in the banking book (IRRBB)
	2015-07-01	Review of the credit valuation adjustment risk framework

Part I

Risk Management in the Banking Sector

Chapter 2

Market Risk

This chapter begins with the presentation of the regulatory framework. It will help us to understand how the supervision on market risk is organized and the capital charge is computed. We will then study the different statistical approaches to measure the value-at-risk. Specifically, a section is dedicated to the risk management of derivatives and exotic products. We will see the main concepts, but we will present the more technical details later in Chapter 12 dedicated to model risk. Advanced topics like Monte Carlo methods and stress testing models will also be addressed in Part III. Finally, the last part of the chapter is dedicated to risk allocation.

2.1 Regulatory framework

We remind that the original Basel Accord only concerned credit risk in 1988. However, the occurrences of shocks were more important and the rapid development of derivatives created some stress events at the end of the eighties and the beginning of the nineties. In October 19, 1987, stock markets crashed and the Dow Jones Industrial Average Index dropped by more than 20% in the day. In 1990, the collapse of the Japanese asset price bubble (both in stock and real estate markets) caused a lot of damage in the Japanese banking system and economy. The unexpected rise of US interest rates in 1994 results in a bond market massacre and difficulties for banks, hedge funds and money managers. In 1994-1995, several financial disasters occurred, in particular the bankruptcy of Barings and the Orange County affair (Jorion, 2007).

In April 1993, the Basel Committee published a first consultative paper to incorporate market risk in the Cooke ratio. Two years later, in April 1995, it accepted the idea to compute the capital charge for market risks with an internal model. This decision is mainly due to the publication of *RiskMetrics* by J.P. Morgan in October 1994. Finally, the Basel Committee published the amendment to the capital accord to incorporate market risks in January 1996. This proposal does not change a lot since its publication and remains the current supervisory framework for market risk. However, a new approach is currently proposed and discussed to fix the future Basel IV Accord (BCBS, 2013b).

According to BCBS (1996a), market risk is defined as "*the risk of losses in on and off-balance sheet positions arising from movements in market prices. The risk subject to this requirements are:*

- *the risks pertaining to interest rate related instruments and equities in the trading book;*
- *foreign exchange risk and commodities risk throughout the bank.*"

The following table summarizes the perimeter of markets risks that require regulatory capital:

Portfolio	Fixed Income	Equity	Currency	Commodity
Trading	✓	✓	✓	✓
Banking			✓	✓

The Basel Committee make the distinction between the trading book and the banking book. The trading book refers to positions in assets held with trading intent or for hedging other elements of the trading book. These assets are valuated on a mark-to-market basis, are actively managed and their holding is intentionally for short-term resale. Examples are proprietary trading, market making, hedging portfolios of derivatives products, etc. The banking book refers to positions in assets that are expected to be held until the maturity. These assets may be valuated at their historic cost.

The first task of the bank is therefore to define trading book assets and banking book assets. For instance, if the bank sells an option on the Libor rate to a client, a capital charge for the market risk is required. If the bank provides a personal loan to a client with a fixed interest rate, there is a market risk if the interest rate risk is not hedged. However, a capital charge is not required in this case, because the exposure concern the banking book. Exposures on stocks may be included in the banking book if the objective is a long-term investment, even if it is a minority shareholding.

To compute the capital charge, banks have the choice between two approaches:

1. The standardized measurement method (SMM)
2. The internal model-based approach (IMA).

Five main risk categories are identified: interest rate risk, equity risk, currency risk, commodity risk and price risk on options and derivatives. For each category, a capital charge is computed to cover the general market risk, but also the specific risk. According to the Basel Committee, specific risk includes the risk "*that an individual debt or equity security moves by more or less than the general market in day-to-day trading and event risk (e.g. takeover risk or default risk)*". The use of internal models is subject to the approval of the supervisor and the bank can mix the two approaches under some conditions. For instance, the bank may use SMM for the specific risk and IMA for the general market risk.

2.1.1 Standardized measurement method

In this approach, the capital charge $\mathcal{K}$ is equal to the risk exposure E times the capital charge weight K:

$$\mathcal{K} = E \times K$$

For the specific risk, the risk exposure corresponds to the notional of the instrument, whether it is a long or a short position. For the general market risk, long and short positions in different instruments can be offset.

2.1.1.1 Interest rate risk

Let us first consider the specific risk. BCBS make the distinction between sovereign and other fixed-income instruments. In the case of government instruments, the capital charge weights are:

Rating	AAA to AA−	A+ to BBB−			BB+ to B−	Below B−	NR
Maturity		0−6M	6M−2Y	2Y+			
K	0%	0.25%	1.00%	1.60%	8%	12%	8%

This capital charge depend on the rating and also the residual maturity for A+ to BBB− issuers[1]. The category NR stands for non-rated issuers. In the case of other instruments issued by public sector entities, banks and corporate companies, the capital charge weights are:

Rating	AAA to BBB−			BB+ to BB−	Below BB−	NR
Maturity	0−6M	6M−2Y	2Y+			
K	0.25%	1.00%	1.60%	8%	12%	8%

We notice that the two scales are close with the following differences. There is no 0% weight, meaning that the weights 0.25%/1.00%/1.60% apply to investment grade (IG) instruments (from AAA to BBB−). The 8% category is more restrictive and only concerns BB+ to BB− instruments.

Example 4 *We consider a trading portfolio with the following exposures: a long position of* \$50 *mn on Euro-Bund futures, a short position of* \$100 *mn on three-month T-Bills and a long position of* \$10 *mn on an investment grade corporate bond with a three-year residual maturity.*

The underlying asset of Euro-Bund futures is a German bond with a long maturity (higher than 6 years). We deduce that the capital charge for specific

[1] Three maturity are defined: 6 months or less, greater than 6 months and up to 24 months, more than 24 months.

risk for the two sovereign exposures is equal to zero, because both Germany and US are rated above A+. Concerning the corporate bond, we obtain:

$$\mathcal{K} = 10 \times 1.60\% = \$160\,000$$

For the general market risk, the bank has the choice between two methods: the maturity approach and the duration approach. In the maturity approach, long and short positions are slotted into a maturity-based ladder comprising fifteen time-bands. The time bands are defined by disjoint intervals $]M^{-}, M^{+}]$. The risk weights depend on the time band t and the value of the coupon[2]:

$K(t)$	0.00%	0.20%	0.40%	0.70%	1.25%	1.75%	2.25%	2.75%
M^{+}_{BC}	1M	3M	6M	1Y	2Y	3Y	4Y	5Y
M^{+}_{SC}	1M	3M	6M	1Y	1.9Y	2.8Y	3.6Y	4.3Y
$K(t)$	3.25%	3.75%	4.50%	5.25%	6.00%	8.00%	12.50%	
M^{+}_{BC}	7Y	10Y	15Y	20Y	$+\infty$			
M^{+}_{SC}	5.7Y	7.3Y	9.3Y	10.6Y	12Y	20Y	$+\infty$	

These risk weights apply to the net exposure on each time band. For reflecting basis and gap risks, the bank must also include a 10% capital charge to the smallest exposure of the matched positions. This adjustment is called the '*vertical disallowance*'. The Basel Committee considers a second adjustment for horizontal offsetting (the '*horizontal disallowance*'). For that, it defines 3 zones (less than 1 year, one year to four years and more than four years). The offsetting can be done within and between the zones. The adjustment coefficients are 30% within the zones 2 and 3, 40% within the zone 1, between the zones 1 and 2, and between the zones 2 and 3, and 100% between the zones 1 and 3.

To compute mathematically the required capital, we note $\mathcal{L}^{\star}(t)$ and $\mathcal{S}^{\star}(t)$ the long and short nominal positions for the time band t. $t = 1$ corresponds to the first time band $[0, 1\mathrm{M}]$, $t = 2$ corresponds to the second time band $]1\mathrm{M}, 3\mathrm{M}[$, etc. The risk weighted positions for the time band t are defined as $\mathcal{L}(t) = K(t) \times \mathcal{L}^{\star}(t)$ and $\mathcal{S}(t) = K(t) \times \mathcal{S}^{\star}(t)$. The required capital for the overall net open position is then equal to:

$$\mathcal{K}^{\mathrm{OP}} = \left| \sum_{t=1}^{15} \mathcal{L}(t) - \sum_{t=1}^{15} \mathcal{S}(t) \right|$$

The matched position $\mathcal{M}(t)$ for the time band t is equal to $\min(\mathcal{L}(t), \mathcal{S}(t))$. We deduce that the additional capital for the vertical disallowance is:

$$\mathcal{K}^{\mathrm{VD}} = 10\% \times \sum_{t=1}^{13} \mathcal{M}(t)$$

[2]Coupons 3% or more are called big coupons (or BC) and coupons less than 3% are called small coupons (SC).

$\mathcal{N}(t) = \mathcal{L}(t) - \mathcal{S}(t)$ is the net exposure for the time band t. We then define the net long and net short exposures for the three zones as follows:

$$\begin{aligned}\mathcal{L}_i &= \sum_{t\in\Delta_i} \max\left(\mathcal{N}(t), 0\right) \\ \mathcal{S}_i &= -\sum_{t\in\Delta_i} \min\left(\mathcal{N}(t), 0\right)\end{aligned}$$

where $\Delta_1 = [0, 1\mathrm{Y}]$, $\Delta_2 =]1\mathrm{Y}, 4\mathrm{Y}]$ and $\Delta_3 =]4\mathrm{Y}, +\infty]$. We define $\mathcal{CF}_{i,j}$ as the exposure of the zone i that can be carried forward to the zone j. We then compute the additional capital for the horizontal disallowance:

$$\begin{aligned}\mathcal{K}^{\mathrm{HD}} = & 40\% \times \min\left(\mathcal{L}_1, \mathcal{S}_1\right) + 30\% \times \min\left(\mathcal{L}_2, \mathcal{S}_2\right) + 30\% \times \min\left(\mathcal{L}_3, \mathcal{S}_3\right) + \\ & 40\% \times \mathcal{CF}_{1,2} + 40\% \times \mathcal{CF}_{2,3} + 100\% \times \mathcal{CF}_{1,3}\end{aligned}$$

The regulatory capital for the general market risk is the sum of the three components:

$$\mathcal{K} = \mathcal{K}^{\mathrm{OP}} + \mathcal{K}^{\mathrm{VD}} + \mathcal{K}^{\mathrm{HD}}$$

Example 5 *We consider a trading portfolio with the following exposures: a long position of* \$100 *mn on four-month instruments, a short position of* \$50 *mn on five-month instruments, a long position of* \$10 *mn on fifteen-year instruments and a short position of* \$50 *mn on twelve-year instruments.*

Let us assume that the instruments correspond to bonds with coupons larger than 3%. For each time band, we report the long, short, matched and net exposures:

Time band	$\mathcal{L}^\star(t)$	$\mathcal{S}^\star(t)$	$K(t)$	$\mathcal{L}(t)$	$\mathcal{S}(t)$	$\mathcal{M}(t)$	$\mathcal{N}(t)$
3M-6M	100	50	0.40%	0.40	0.20	0.20	0.20
7Y-10Y	10	50	3.75%	0.45	2.25	0.45	−1.80

The capital charge for the overall open position is:

$$\begin{aligned}\mathcal{K}^{\mathrm{OP}} &= \left|0.40 + 0.45 - 0.20 - 2.25\right| \\ &= 1.6\end{aligned}$$

whereas the capital for the vertical disallowance is:

$$\begin{aligned}\mathcal{K}^{\mathrm{VD}} &= 10\% \times (0.20 + 0.45) \\ &= 0.065\end{aligned}$$

We now compute the net long and net short exposures for the three zones:

zone	1	2	3
$\mathcal{L}_i$	0.20	0.00	0.00
$\mathcal{S}_i$	0.00	0.00	1.80

It follows that there is no horizontal offsetting within the zones. Moreover, we notice that we can only carry forward the long exposure $\mathcal{L}_1$ to the zone 3 meaning that:

$$\begin{aligned}\mathcal{K}^{\text{HD}} &= 40\% \times 0.00 + 30\% \times 0.00 + 30\% \times 0.00 + \\ &\quad 40\% \times 0.00 + 40\% \times 0.00 + 100\% \times 0.20 \\ &= 0.20\end{aligned}$$

We finally deduce that the required capital is:

$$\begin{aligned}\mathcal{K} &= 1.6 + 0.065 + 0.20 \\ &= \$1.865 \text{ mn}\end{aligned}$$

With the duration approach, the bank computes the price sensitivity of each position with respect to a change in yield Δy, slot the sensitivities into a duration-based ladder and applies adjustments for vertical and horizontal disallowances. The computation is then exactly the same as previously[3], but with the following time bands:

Δy	1.00%	1.00%	1.00%	1.00%	0.90%	0.80%	0.75%	0.75%
M^+	1M	3M	6M	1Y	1.9Y	2.8Y	3.6Y	4.3Y
Zone	1	1	1	1	2	2	2	3
Δy	0.70%	0.65%	0.60%	0.60%	0.60%	0.60%	0.60%	
M^+	5.7Y	7.3Y	9.3Y	10.6Y	12Y	20Y	$+\infty$	
Zone	3	3	3	3	3	3	3	

2.1.1.2 Equity risk

For equity exposures, the capital charge for specific risk is 4% if the portfolio is liquid and well-diversified and 8% otherwise. For the general market risk, the risk weight is equal to 8% and applies to the net exposure.

Remark 4 *In CRD I, stock-index futures which are exchange traded and well-diversified have no capital requirement against specific risk.*

Example 6 *We consider a short exposure on S&P 500 index futures (*\$100 *mn) and a long exposure on the Apple stock (*\$60 *mn).*

The capital charge for specific risk is[4]:

$$\begin{aligned}\mathcal{K}^{\text{Specific}} &= 100 \times 4\% + 60 \times 8\% \\ &= 4 + 4.8 \\ &= 8.8\end{aligned}$$

[3]For the vertical disallowance, the capital charge to capture basis risk is equal to 5% and not 10% as in the case of the maturity approach.

[4]We assume that the S&P 500 index is liquid and well-diversified, whereas the exposure on the Apple stock is not diversified.

The net exposure is $-\$40$ mn. We deduce that the capital charge for the general market risk is:

$$\begin{aligned}\mathcal{K}^{\text{General}} &= \lvert -40 \rvert \times 8\% \\ &= 3.2\end{aligned}$$

It follows that the total capital charge for this equity portfolio is \$12 mn.

Remark 5 *Under Basel 2.5, the capital charge for specific risk is set to 8% whatever the liquidity of the portfolio.*

2.1.1.3 Foreign exchange risk

The Basel Committee includes gold in this category and not in the commodity category because of its specificity in terms of volatility and its status of safe-heaven currency. The bank has first to calculate the net position (long or short) of each currency. The capital charge is then 8% of the global net position defined as the sum of:

- the maximum between the aggregated value $\mathcal{L}_{\text{FX}}$ of long positions and the aggregated value $\mathcal{S}_{\text{FX}}$ of short positions and,
- the absolute value of the net position $\mathcal{N}_{\text{Gold}}$ in gold.

We have:

$$\mathcal{K} = 8\% \times \left(\max\left(\mathcal{L}_{\text{FX}}, \mathcal{S}_{\text{FX}}\right) + \lvert \mathcal{N}_{\text{Gold}} \rvert\right)$$

Example 7 *We consider a bank which has the following long and short positions expressed in \$ mn*[5]*:*

Currency	EUR	JPY	GBP	CHF	CAD	AUD	ZAR	Gold
$\mathcal{L}_i$	170	0	25	37	11	3	8	33
$\mathcal{S}_i$	80	50	12	9	28	0	8	6

We first compute the net exposure $\mathcal{N}_i$ for each currency:

$$\mathcal{N}_i = \mathcal{L}_i - \mathcal{S}_i$$

We obtain the following figures:

Currency	EUR	JPY	GBP	CHF	CAD	AUD	ZAR	Gold
$\mathcal{N}_i$	90	-50	13	28	-17	3	0	27

We then calculate the aggregated long and short positions:

$$\begin{aligned}\mathcal{L}_{\text{FX}} &= 90 + 13 + 28 + 3 + 0 = 134 \\ \mathcal{S}_{\text{FX}} &= 50 + 17 = 67 \\ \mathcal{N}_{\text{Gold}} &= 27\end{aligned}$$

[5] We implicity assume that the reporting currency of the bank is the US dollar.

We finally deduce that the capital charge is equal to \$12.88 mn:

$$\begin{aligned}\mathcal{K} &= 8\% \times (\max(134, 67) + |27|) \\ &= 8\% \times 161 \\ &= 12.88\end{aligned}$$

2.1.1.4 Commodity risk

Commodity risk concerns both physical and derivative positions (forward, futures[6] and options). This includes energy products (oil, gas, ethanol, etc.), agricultural products (grains, oilseeds, fiber, livestock, etc.) and metals (industrial and precious), but excludes gold which is covered under foreign exchange risk. The Basel Committee makes the distinction between the risk of spot or physical trading, which is mainly affected by the directional risk and the risk of derivative trading, which includes the directional risk, the basis risk, the cost of carry and the forward gap (or time spread) risk. The SMM for commodity risk includes two options: the simplified approach and the maturity ladder approach.

Under the simplified approach, the capital charge for directional risk is 15% of the absolute value of the net position in each commodity. For the other three risks, the capital charge is equal to 3% of the global gross position. We have:

$$\mathcal{K} = 15\% \times \sum_{i=1}^{m} |\mathcal{L}_i - \mathcal{S}_i| + 3\% \sum_{i=1}^{m} (\mathcal{L}_i + \mathcal{S}_i)$$

where m is the number of commodities, $\mathcal{L}_i$ is the long position in commodity i and $\mathcal{S}_i$ is the short position in commodity i.

Example 8 *We consider a portfolio of five commodities. The mark-to-market exposures expressed in \$ mn are the following:*

Commodity	Crude Oil	Coffee	Natural Gas	Cotton	Sugar
$\mathcal{L}_i$	23	5	3	8	11
$\mathcal{S}_i$	0	0	19	2	6

The aggregated net exposure $\sum_{i=1}^{5} |\mathcal{L}_i - \mathcal{S}_i|$ is equal to \$55 mn whereas the gross exposure $\sum_{i=1}^{5} (\mathcal{L}_i + \mathcal{S}_i)$ is equal to \$77 mn. We deduce that the required capital is $15\% \times 55 + 3\% \times 77$ or \$10.56 mn.

Under the maturity ladder approach, the bank should spread long and short exposures of each currency to seven time bands: 0-1M, 1M-3M, 3M-6M, 6M-1Y, 1Y-2Y, 2Y-3Y, 3Y+. For each time band, the capital charge for the basis risk is equal to 1.5% of the matched positions (long and short).

[6] The most traded futures contract are crude oil, brent, heating oil, gas oil, natural oil, rbob gasoline silver, platinum, palladium, zinc, lead, aluminium, cocoa, soybeans, corn, cotton, wheat, sugar, live cattle, coffee and soybean oil.

Nevertheless, the residual net position of previous time bands may be carried forward to offset exposures in next time bands. In this case, a surcharge of 0.6% of the residual net position is added at each time band to cover the time spread risk. Finally, a capital charge of 15% is applied to the global net exposure (or the residual unmatched position) for directional risk.

To compute mathematically the required capital, we note $\mathcal{L}_i(t)$ and $\mathcal{S}_i(t)$ the long and short positions of the commodity i for the time band t. $t = 1$ corresponds to the first time band $[0, 1\text{M}]$ and $t = 7$ corresponds to the last time band $]3\text{Y}, +\infty[$. The cumulative long and short exposures are $\mathcal{L}_i^+(t) = \mathcal{L}_i^+(t-1) + \mathcal{L}_i(t)$ with $\mathcal{L}_i^+(0) = 0$ and $\mathcal{S}_i^+(t) = \mathcal{S}_i^+(t-1) + \mathcal{S}_i(t)$ with $\mathcal{S}_i^+(0) = 0$. The cumulative matched position is $\mathcal{M}_i^+(t) = \min\left(\mathcal{L}_i^+(t), \mathcal{S}_i^+(t)\right)$. We deduce that the matched exposition for the time band t is equal to $\mathcal{M}_i(t) = \mathcal{M}_i^+(t) - \mathcal{M}_i^+(t-1)$ with $\mathcal{M}_i^+(0) = 0$. The value of the carried forward $\mathcal{CF}_i(t)$ can be obtained recursively by reporting the unmatched positions at time t which can be offset in the times bands τ with $\tau > t$. The residual unmatched position is $\mathcal{N}_i = \max\left(\mathcal{L}_i^+(7), \mathcal{S}_i^+(7)\right) - \mathcal{M}_i^+(t)$. We finally deduce that the required capital is the sum of the individual capital charges:

$$\mathcal{K}_i = 1.5\% \times \left(\sum_{t=1}^{7} 2 \times \mathcal{M}_i(t)\right) + 0.6\% \times \left(\sum_{t=1}^{6} \mathcal{CF}_i(t)\right) + 15\% \times \mathcal{N}_i$$

We notice that the matched position $\mathcal{M}_i(t)$ is multiplied by 2, because we apply the capital charge 1.5% to the long and short matched positions.

Example 9 *We consider the following positions (in \$):*

Time band	0−1M	1M−3M	3M−6M	6M−1Y	1Y−2Y	2Y−3Y	3Y+
$\mathcal{L}_i(t)$	500	0	0	1 800	300	0	0
$\mathcal{S}_i(t)$	300	900	0	100	600	100	200

We compute the cumulative positions $\mathcal{L}_i^+(t)$ and $\mathcal{S}_i^+(t)$ and deduce the matched expositions $\mathcal{M}_i(t)$:

Time band	t	$\mathcal{L}_i(t)$	$\mathcal{S}_i(t)$	$\mathcal{L}_i^+(t)$	$\mathcal{S}_i^+(t)$	$\mathcal{M}_i(t)$	$\mathcal{CF}_i(t)$
0−1M	1	500	300	500	300	300	200
1M−3M	2	0	900	500	1 200	200	700
3M−6M	3	0	0	500	1 200	0	700
6M−1Y	4	1 800	100	2 300	1 300	800	600
1Y−2Y	5	300	600	2 600	1 900	600	300
2Y−3Y	6	0	100	2 600	2 000	100	200
3Y+	7	0	200	2 600	2 200	200	0

The sum of matched positions is equal to 2 200. This means that the residual unmatched position is 400 (2 600 − 2 200). At time band $t = 1$, we can carry forward 200 of long position in the next time band. At time band $t = 2$, we can carry forward 700 of short position in the times band $t = 4$. This implies that $\mathcal{CF}_i(3) = 700$ and $\mathcal{CF}_i(4) = 700$. At time band $t = 4$, the residual unmatched position is equal to 1 000 (1 800 − 100 − 700). However, we can only carry 600 of this long position in the next time bands (300 for $t = 5$, 100 for $t = 6$ and 200 for $t = 1$). At the end, we verify that the residual position is 400, that is the part of the long position at time band $t = 4$ which can not be carried forward (1 000 − 600). We also deduce that the sum of carried forward positions is 2 700. It follows that the required capital is[7]:

$$\begin{aligned} \mathcal{K}_i &= 1.5\% \times 4\,400 + 0.6\% \times 2\,700 + 15\% \times 400 \\ &= \$142.20 \end{aligned}$$

2.1.1.5 Option's market risk

There are three approaches for the treatment of options and derivatives. The first method, called the simplified approach, consists of calculating separately the capital charge of the position for the option and the associated underlying. In the case of an hedged exposure (long cash and long put, short cash and long call), the required capital is the standard capital charge of the cash exposure less the amount of the in-the-money option. In the case of a non-hedged exposure, the required capital is the minimum value between the mark-to-market of the option and the standard capital charge for the underlying.

Example 10 *We consider a variant of Example 6. We have a short exposure on S&P 500 index futures (*\$100 *mn) and a long exposure on the Apple stock (*\$60 *mn). We assume that the current stock price of Apple is* \$120*. Six months ago, we have bought* 400 000 *put options on Apple with a strike of* \$130 *and a one-year maturity. We also decide to buy* 10 000 *ATM call options on Google. The current stock price of Google is* \$540 *and the market value of the option is* \$45.5*.*

We deduce that we have 500 000 shares of the Apple stock. This implies that \$48 mn of the long exposure on Apple is hedged by the put options. Concerning the derivative exposure on Google, the market value is equal to \$0.455 mn. We can therefore decompose this portfolio into three main exposures:

- a directional exposure composed by the \$100 mn short exposure on the S&P 500 index and the \$12 mn remaining long exposure on the Apple stock;
- a \$48 mn hedged exposure on the Apple stock;
- a \$0.455 mn derivative exposure on the Google stock.

[7] The total matched position is equal to $2 \times 2\,200 = 4\,400$ (long + short).

For the directional exposure, we compute the capital charge for specific and general market risks[8]:

$$\begin{aligned}\mathcal{K} &= (100 \times 4\% + 12 \times 8\%) + 88 \times 8\% \\ &= 4.96 + 7.04 \\ &= 12\end{aligned}$$

For the hedged exposure, we proceed as previously but we deduce the in-the-money value[9]:

$$\begin{aligned}\mathcal{K} &= 48 \times (8\% + 8\%) - 4 \\ &= 3.68\end{aligned}$$

The market value of the Google options is \$0.455 mn. We compare this value to the standard capital charge[10] to determine the capital charge:

$$\begin{aligned}\mathcal{K} &= \min(5.4 \times 16\%, 0.455) \\ &= 0.455\end{aligned}$$

We finally deduce that the required capital is \$16.135 mn.

The second approach is the delta-plus method. In this case, the directional exposure of the option is calculated by its delta. Banks will also required to compute an additional capital charge for gamma and vega risks. We consider different options and we note $j \in \mathcal{A}_i$ when the option j is written on the asset i. We first compute the (signed) capital charge for the 4 risks at the asset level:

$$\begin{aligned}\mathcal{K}_i^{\text{Specific}} &= \left(\sum_{j\in\mathcal{A}_i} N_j \times \boldsymbol{\Delta}_j\right) \times S_i \times K_i^{\text{Specific}} \\ \mathcal{K}_i^{\text{General}} &= \left(\sum_{j\in\mathcal{A}_i} N_j \times \boldsymbol{\Delta}_j\right) \times S_i \times K_i^{\text{General}} \\ \mathcal{K}_i^{\text{Gamma}} &= \frac{1}{2} \times \left(\sum_{j\in\mathcal{A}_i} N_j \times \boldsymbol{\Gamma}_j\right) \times \left(S_i \times K_i^{\text{Gamma}}\right)^2 \\ \mathcal{K}_i^{\text{Vega}} &= \sum_{j\in\mathcal{A}_i} N_j \times \boldsymbol{\upsilon}_j \times (25\% \times \Sigma_j)\end{aligned}$$

where S_i is the current market value of the asset i, K_i^{Specific} and K_i^{General} are the corresponding standard capital charge for specific and general market risk

[8] The net short exposure is equal to \$88 mn.

[9] It is equal to $400\,000 \times \max(0, 130 - 120)$.

[10] It is equal to $10\,000 \times 540 \times (8\% + 8\%)$.

and K_i^{Gamma} is the capital charge for gamma impact[11]. Here, N_j, $\boldsymbol{\Delta}_j$, $\boldsymbol{\Gamma}_j$ and $\boldsymbol{\upsilon}_j$ are the exposure, delta, gamma and vega of the option j. For the vega risk, the shift corresponds to $\pm 25\%$ of the implied volatility Σ_j. For a portfolio of assets, the traditional netting rules apply to specific and general market risks. The total capital charge for gamma risk corresponds to the opposite of the sum of the negative individual capital charges for gamma risk whereas the total capital charge for vega risk corresponds to the sum of the absolute value of individual capital charges for vega risk.

Example 11 *We consider a portfolio of 4 options written on stocks with the following characteristics:*

Option	Stock	Exposure	Type	Price	Strike	Maturity	Volatility
1	A	−5	call	100	110	1.00	20%
2	A	−10	call	100	100	2.00	20%
3	B	10	call	200	210	1.00	30%
4	B	8	put	200	190	1.25	35%

This means that we have 2 assets. For stock A, we have a short exposure of 5 call options with a one-year maturity and a short exposure of 10 call options with a two-year maturity. For stock B, we have a long exposure of 10 call options with a one-year maturity and a long exposure of 8 put options with a maturity of one year and three months.

Using the Black-Scholes model, we first compute the Greek coefficients for each option j. Because the options are written on single stocks, the capital charges K_i^{Specific}, K_i^{General} and K_i^{Gamma} are all equal to 8%. Using the previous formulas, we then deduce the individual capital charges for each option[12]:

j	1	2	3	4
$\boldsymbol{\Delta}_j$	0.45	0.69	0.56	−0.31
$\boldsymbol{\Gamma}_j$	0.02	0.01	0.01	0.00
$\boldsymbol{\upsilon}_j$	39.58	49.91	78.85	79.25
$\boldsymbol{\mathcal{K}}_j^{\text{Specific}}$	−17.99	−55.18	89.79	−40.11
$\boldsymbol{\mathcal{K}}_j^{\text{General}}$	−17.99	−55.18	89.79	−40.11
$\boldsymbol{\mathcal{K}}_j^{\text{Gamma}}$	−3.17	−3.99	8.41	4.64
$\boldsymbol{\mathcal{K}}_j^{\text{Vega}}$	−9.89	−24.96	59.14	55.48

We can now aggregate the previous individual capital charges for each stock. We obtain:

[11] It is equal to 8% for equities, 8% for currencies and 15% for commodities. In the case of interest rate risk, it corresponds to the standard value $K(t)$ for the time band t (see the table in page 46).

[12] For instance, the individual capital charge of the second option for the gamma risk is

$$\boldsymbol{\mathcal{K}}_j^{\text{Gamma}} = \frac{1}{2} \times (-10) \times 0.0125 \times (100 \times 8\%)^2 = -3.99$$

Stock	$\mathcal{K}_i^{\text{Specific}}$	$\mathcal{K}_i^{\text{General}}$	$\mathcal{K}_i^{\text{Gamma}}$	$\mathcal{K}_i^{\text{Vega}}$
A	−73.16	−73.16	−7.16	−34.85
B	49.69	49.69	13.05	114.61
Total	122.85	23.47	7.16	149.46

To compute the total capital charge, we apply the netting rule for the general market risk, but not for the specific risk. This means that $\mathcal{K}^{\text{Specific}} = |-73.16| + |49.69| = 122.85$ and $\mathcal{K}^{\text{General}} = |-73.16 + 49.69| = 23.47$. For gamma risk, we only consider negative impacts and we have $\mathcal{K}^{\text{General}} = |-7.16| = 7.16$. For vega risk, there is no netting rule: $\mathcal{K}^{\text{Vega}} = |-34.85| + |114.61| = 149.46$. We finally deduce that the overall capital is 302.94.

The third method is the scenario approach. In this case, we evaluate the profit and loss (P&L) for simultaneous changes in the underlying price and in the option implied volatility. For defining these scenarios, the ranges are the standard shifts used previously. For instance, we use the following ranges for equities:

		S_i	
		−8%	+8%
Σ_j	−25%		
	+25%		

The scenario matrix corresponds to intermediate points on the 2×2 grid. For each cell of the scenario matrix, we calculate the P&L of the option exposure[13]. The capital charge is then the largest loss.

2.1.1.6 Securitization instruments

The treatment of specific risk of securitization positions is revised in Basel 2.5 and is based on external ratings. For instance, the capital charge for securitization exposures is 1.6% if the instrument is rated from AAA to AA−. For resecuritization exposures, it is equal to 3.2%. If the rating of the instrument is from BB+ to BB−, the risk capital charges becomes respectively 28% and 52%[14].

Remark 6 *In the case of securitization exposures below BB−, the bank does not calculate a risk charge, but applies a deduction from capital.*

2.1.2 Internal model-based approach

The use of an internal model is conditional upon the approval of the supervisory authority. In particular, the bank must meet certain criteria concerning different topics. These criteria concerns the risk management system, the specification of market risk factors, the properties of the internal model, the stress

[13] It may include the cash exposure if the option is used for hedging purposes.

[14] See pages 4-7 of BCBS (2009b) for the other risk capital charges.

testing framework, the treatment of the specific risk and the backtesting procedure. In particular, the Basel Committee considers that the bank must have "*sufficient numbers of staff skilled in the use of sophisticated models not only in the trading area but also in the risk control, audit, and if necessary, back office areas*". We notice that BCBS first insists on the quality of the trading department, meaning that the trader is the first level of risk management. The validation of an internal model does not therefore only concern the risk management department, but the bank as a whole.

2.1.2.1 Qualitative criteria

BCBS (1996a) defines the following qualitative criteria:

- "*The bank should have an independent risk control unit that is responsible for the design and implementation of the bank's risk management system. [...] This unit must be independent from business trading units and should report directly to senior management of the bank*".
- The risk management department produces and analyze daily reports, is responsible for the backtesting procedure and conducts stress testing analysis.
- The internal model must be used to manage the risk of the bank in the daily basis. It must be completed by trading limits expressed in risk exposure.
- The bank must document internal policies, controls and procedures concerning the risk measurement system (including the internal model).

It is today evident that the risk department department should not report to the trading and sales department. Twenty years ago, it was not the case. Most of risk management units were incorporated to business units. It has completely changed because of the regulation and risk management is now independent from the front office. The risk management function has really emerged with the amendment to incorporate market risks and even more with the Basel II reform, whereas the finance function has long been developed in banks. For instance, it's very recent that the head of risk management[15] is also a member of the executive committee of the bank whereas the head of the finance department[16] has always been part of the top management.

From the supervisory point of view, an internal model does not reduce to measure the risk. It must be integrated in the management of the risk. This is why the Basel Committee stresses the importance between the outputs of the model (or the risk measure), the organization of the risk management and the impact on the business.

[15]He is called the chief risk officer or CRO.
[16]He is called the chief financial officer or CFO.

2.1.2.2 Quantitative criteria

The choice of the internal model is left to the bank, but it must respect the following quantitative criteria:

- The value-at-risk (VaR) is computed on a daily basis with a 99% confidence level. The minimum holding period of the VaR is 10 trading days. If the bank computes a VaR with a shorter holding period, it can use the square root rule.
- The risk measure can take into account diversification, that is the correlations between the risk categories.
- The model must capture the relevant risk factors and the bank must pay attention to the specification of the appropriate set of market risk factors.
- The sample period for calculating the value-at-risk is at least one year and the bank must update the data set frequently (every month at least).
- In the case of options, the model must capture the non-linear effects with respect to the risk factors and the vega risk.
- "*Each bank must meet, on a daily basis, a capital requirement expressed as the higher of (i) its previous day's value-at-risk number [...] and (ii) an average of the daily value-at-risk measures on each of the preceding sixty business days, multiplied by a multiplication factor*".
- The value of the multiplication factor depends on the quality of the internal model with a range between 3 and 4. The quality of the internal model is related to its ex-post performance measured by the backtesting procedure.

The holding period to define the capital is 10 trading days. However, it is difficult to compute the value-at-risk for such holding period. In practice, the bank computes the one-day value-at-risk and converts this number into a ten-day value-at-risk using the square-root-of-time rule:

$$\mathrm{VaR}_\alpha\left(w;10\mathrm{D}\right)=\sqrt{10}\times\mathrm{VaR}_\alpha\left(w;1\mathrm{D}\right)$$

This rule comes from the scaling property of the volatility associated to a geometric Brownian motion. It has the advantage to be simple and objective, but it generally underestimates the risk when the loss distribution exhibits fat tails[17].

The required capital at time t is equal to:

$$\mathcal{K}_t=\max\left(\mathrm{VaR}_{t-1},(3+\xi)\times\frac{1}{60}\sum_{i=1}^{60}\mathrm{VaR}_{t-i}\right)\tag{2.1}$$

[17]See for instance Diebold *et al.* (1998), Daníelsson and Zigrand (2006) or Wang *et al.* (2011).

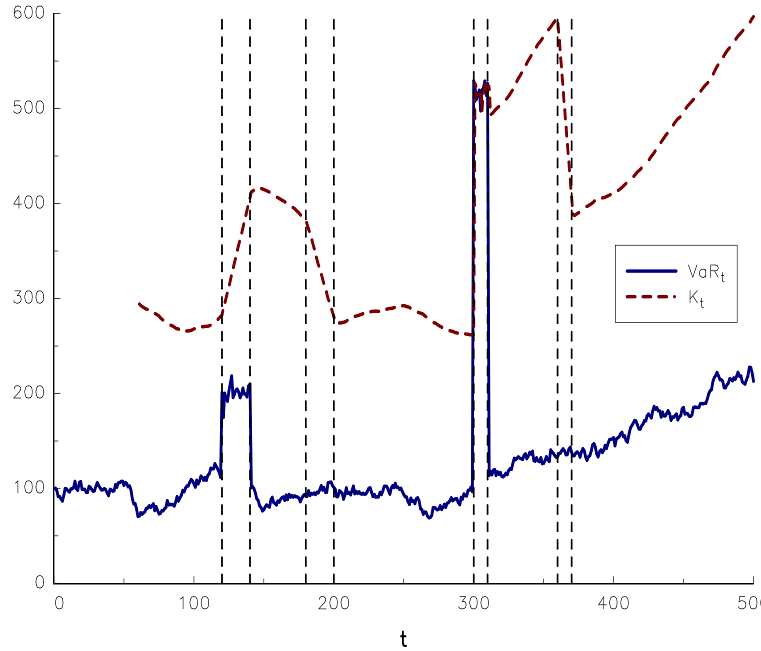

FIGURE 2.1: Calculation of the required capital with the VaR

where VaR_t is the value-at-risk calculated at time t and ξ is the penalty coefficient ($0 \leq \xi \leq 1$). In normal periods where $\mathrm{VaR}_{t-1} \simeq \mathrm{VaR}_{t-i}$, the required capital is the average of the last 60 value-at-risk values times the multiplication factor[18] $m_c = 3 + \xi$. In this case, we have:

$$\mathcal{K}_t = \mathcal{K}_{t-1} + \frac{m_c}{60}\left(\mathrm{VaR}_{t-1} - \mathrm{VaR}_{t-61}\right)$$

The impact of VaR_{t-1} is limited because the factor $(3+\xi)/60$ is smaller than 6.7%. The required capital can only be equal to the previous day's value-at-risk if the bank faces a stress $\mathrm{VaR}_{t-1} \gg \mathrm{VaR}_{t-i}$. We also notice that a shock on the VaR vanishes after 60 trading days. To understand the calculation of the capital, we report an illustration in Figure 2.1. The solid blue line corresponds to the value-at-risk VaR_t whereas the dashed red line corresponds to the capital $\mathcal{K}_t$. We assume that $\xi = 0$ meaning that the complementary factor is equal to 3. When $t < 120$, the value-at-risk varies around a constant. The capital is then relatively smooth and is three times the average VaR. At time $t = 120$, we observe a shock on the value-at-risk, which lasts 20 days. Immediately, the capital increases until $t \leq 140$. Indeed, at this time, the capital takes into account the full period of the shocked VaR (between $t = 120$ and $t = 139$). The full effect of this stressed period continues until

[18]The complementary factor will be explained later in page 92.

$t \leq 180$, but this effect becomes partial when $t > 180$. The impact of the shock vanishes when $t = 200$. We then observe a period of 100 days where the capital is smooth because the daily value-at-risk does not change a lot. A second shock on the value-at-risk occurs at time $t = 300$, but the magnitude of the shock is larger than previously. During 10 days, the required capital is exactly equal to the previous day's value-at-risk. After 10 days, the bank succeeds to reduce the risk of its portfolio. However, the daily value-at-risk increases from $t = 310$ to $t = 500$. As previously, the impact of the second shock vanishes 60 days after the end of shock. However, the capital increases strongly at the end of the period. This is due to the effect of the multiplication factor m_c on the value-at-risk.

2.1.2.3 Stress testing

Stress testing is a simulation method to identify events that could have a great impact on the soundness of the bank. The framework consists of applying stress scenarios and low-probability events on the trading portfolio of the bank and to evaluate the maximum loss. Contrary to the value-at-risk[19], stress testing is not used to compute the required capital. The underlying idea is more to identify the adverse scenarios for the bank, to evaluate the corresponding losses, to eventually reduce the too risky exposures and to anticipate the management of such stress periods.

Stress tests should incorporate both market and liquidity risks. The Basel Committee considers two types of stress tests:

1. supervisory stress scenarios;
2. stress scenarios developed by the bank itself.

The supervisory stress scenarios are standardized and apply to the different banks. This allows the supervisor to compare the vulnerability between the different banks. The bank must complement them by its own scenarios in order to evaluate the vulnerability of its portfolio according to the characteristics of the portfolio. In particular, the bank may be exposed to some political risks, regional risks or market risks that are not taken into account by standardized scenarios. The banks must report their test results to the supervisors in a quarterly basis.

Stress scenarios may be historical or hypothetical. In the case of historical scenarios, the bank computes the worst-case loss associated to different typical crisis: the Black Monday (1987), the European monetary system crisis (1992), the bond market sell-off (1994), the internee bubble (2000), the subprime mortgage crisis (2007), the liquidity crisis due to Lehman Brothers collapse (2008), etc. Hypothetical scenarios are more difficult to calibrate, because they

[19] The 99% VaR is considered as a risk measure in normal markets and therefore ignore stress events.

must correspond to extreme but also plausible events. Moreover, the multi-dimension aspect of stress scenarios is an issue. Indeed, the stress scenario is defined by the extreme event, but the corresponding loss is evaluated with respect to the shocks on market risk factors. For instance, if we consider a severe middle east crisis, this event will have a direct impact on the oil price, but also indirect impacts on other market risk factors (equity prices, US dollar, interest rates). Whereas historical scenarios are objective, hypothetical scenarios are by construction subjective and their calibration will differ from one financial institution to another. In the case of the middle east crisis, one bank may consider that the oil price could fall by 30% whereas another bank may use a price reduction of 50%.

In 2009, the Basel Committee revised the market risk framework. In particular, it introduces the stressed value-at-risk measure. The stressed VaR has the same characteristics than the traditional VaR (99% confidence level and 10-day holing period), but the model inputs are "*calibrated to historical data from a continuous 12-month period of significant financial stress relevant to the bank's portfolio*". For instance, a typical period is the 2008 year which both combines the subprime mortgage crisis and the Lehman Brothers bankruptcy. This implies that the historical period to compute the SVaR is completely different than the historical period to compute the VaR (see Figure 2.2). In Basel 2.5, the capital requirement for stressed VaR is:

$$\mathcal{K}_t^{\mathrm{SVaR}} = \max\left(\mathrm{SVaR}_{t-1}, m_s \times \frac{1}{60}\sum_{i=1}^{60}\mathrm{SVaR}_{t-i}\right)$$

where SVaR_t is the stressed VaR measure computed at time t. Like the coefficient m_c, the complementary factor m_s for the stressed VaR is also calibrated with respect to the backtesting outcomes, meaning that we have $m_s = m_c$ in many cases.

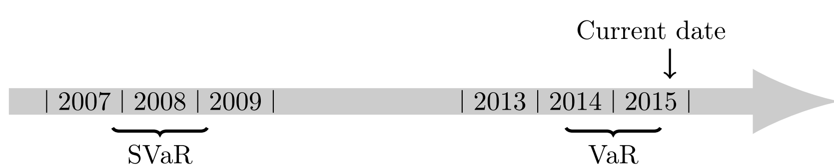

FIGURE 2.2: Two different periods to compute the VaR and the SVaR

2.1.2.4 Specific risk and other risk charges

In the case where the internal model does not take into account the specific risk, the bank must compute a specific risk charge (SRC) using the standardized measurement method. To be validated as a value-at-risk measure with specific risks, the model must satisfy at least the following criteria: it captures concentrations (magnitude and changes in composition), it captures name-related basis and event risks and it considers the assessment of the liquidity risk. For instance, an internal model built with a general market risk factor[20] does not capture specific risk. Indeed, the risk exposure of the portfolio is entirely determined by the beta of the portfolio with respect to the market risk factor. This implies that two portfolios with the same beta but with a different composition, concentration or liquidity have the same value-at-risk.

Basel 2.5 established a new capital requirement "*in response to the increasing amount of exposure in banks' trading books to credit-risk related and often illiquid products whose risk is not reflected in value-at-risk*" (BCBS, 2009b). The incremental risk charge (IRC) measures the impact of rating migrations and defaults, corresponds to a 99.9% value-at-risk for a one-year horizon time and concerns portfolios of credit vanilla trading (bonds and CDS). The IRC may be incorporated into the internal model or it may be treated as a surcharge from a separate calculation. Also under Basel 2.5, BCBS introduced the comprehensive risk measure (CRM), which corresponds to a supplementary capital charge for credit exotic trading portfolios[21]. The CRM is also a 99.9% value-at-risk for a one-year time horizon. For IRC and CRM, the capital charge is the maximum between the most recent risk measure and the average of the risk measure over 12 weeks[22]. We finally obtain the following formula to compute the capital charge for the market risk under Basel 2.5:

$$\mathcal{K}_t = \mathcal{K}_t^{\mathrm{VaR}} + \mathcal{K}_t^{\mathrm{SVaR}} + \mathcal{K}_t^{\mathrm{SRC}} + \mathcal{K}_t^{\mathrm{IRC}} + \mathcal{K}_t^{\mathrm{CRM}}$$

where $\mathcal{K}_t^{\mathrm{VaR}}$ is given by Equation (2.1) and $\mathcal{K}_t^{\mathrm{SRC}}$ is the specific risk charge. In this formula, $\mathcal{K}_t^{\mathrm{SRC}}$ and/or $\mathcal{K}_t^{\mathrm{IRC}}$ may be equal to zero if the modeling of these two risks is included in the value-at-risk internal model.

2.1.2.5 Backtesting and the ex-post evaluation of the internal model

The backtesting procedure is described in the document *Supervisory Framework for the Use of Backtesting in Conjunction with the Internal Models Approach to Market Risk Capital Requirements* published by the Basel Committee in January 1996. It consists of verifying that the internal model is consistent with a 99% confidence level. The idea is then to compare the outcomes of the risk model with realized loss values. For instance, we expect that the realized loss exceeds the VaR number once every 100 observations on average.

The backtesting is based on the one-day holding period and compares the previous day's value-at-risk with the daily realized profit and loss. An exception occurs if the loss exceeds the value-at-risk. For a given period, we

[20]This is the case of the capital asset pricing model (CAPM) developed by Sharpe (1964).

[21]This concerns correlation trading activities on credit derivatives.

[22]Contrary to the VaR and SVaR measures, the risk measure is not scaled by a complementary factor for IRC and CRM.

compute the number of exceptions. Depending of the frequency of exceptions, the supervisor determines the value of the penalty function between 0 and 1. In the case of a sample based on 250 trading days, the Basel Committee defines three zones and proposes the values given in Table 2.1. The green zone corresponds to a number of exceptions less or equal to 4. In this case, BCBS considers that there is no problem and the penalty coefficient ξ is set to 0. If the number of exceptions belongs to the yellow zone (between 5 and 9 exceptions), it may indicate that the confidence level of the internal model could be lower than 99% and implies that ξ is greater than zero. For instance, if the number of exceptions for the last 250 trading days is 6, BCBS proposes that the penalty coefficient is set to 0.50, meaning that the multiplication coefficient m_c is equal to 3.50. The red zone is a concern. In this case, the supervisor must investigate the reasons of such large number of exceptions. If the problem comes from the relevancy of the model, the supervisor can invalidate the internal model-based approach.

TABLE 2.1: Value of the penalty coefficient ξ for a sample of 250 observations

Zone	Number of exceptions	ξ
Green	0	0.00
	1	0.00
	2	0.00
	3	0.00
	4	0.00
Yellow	5	0.40
	6	0.50
	7	0.65
	8	0.75
	9	0.85
Red	10+	1.00

The definition of the color zones comes from the statistical analysis of the exception frequency. Let L_t and VaR_t be respectively the daily loss and the value-at-risk at time t. By definition, L_t is the opposite of the P&L Π_t:

$$\begin{aligned} L_t &= -\Pi_t \\ &= \mathrm{MtM}_{t-1} - \mathrm{MtM}_t \end{aligned}$$

where MtM_t is the mark-to-market of the trading portfolio at time t. By definition, we have:

$$\Pr\left\{L_t \geq \mathrm{VaR}_{t-1}\right\} = 1 - \alpha$$

where α is the confidence level of the value-at-risk. Let e_t be the random variable which is equal to 1 if there is an exception and 0 otherwise. e_t is a

Bernoulli random variable with parameter p:

$$\begin{aligned} p &= \Pr\{e_t = 1\} \\ &= \Pr\{L_t \geq \mathrm{VaR}_{t-1}\} \\ &= 1 - \alpha \end{aligned}$$

In the case of the Basel framework, α is set to 99% meaning that we have a probability of 1% to observe an exception every trading day. For a given period $[t_1, t_2]$ of n trading days, the probability to observe exactly m exceptions is given by the binomial formula:

$$\Pr\{N_e(t_1; t_2) = m\} = \binom{n}{m} (1-\alpha)^m \alpha^{n-m}$$

where $N_e(t_1; t_2) = \sum_{t=t_1}^{t_2} e_t$ is the number of exceptions for the period $[t_1, t_2]$. We obtain this result under the assumption that the exceptions are independent across time. $N_e(t_1; t_2)$ is then the binomial random variable $\mathcal{B}(n; 1-\alpha)$. We deduce that the probability to have up to m exceptions is:

$$\Pr\{N_e(t_1; t_2) \leq m\} = \sum_{i=0}^{m} \binom{n}{i} (1-\alpha)^i \alpha^{n-i}$$

The three previous zones are then defined with respect to the statistical confidence level of the assumption $\mathcal{H} : \alpha = 99\%$. The green zone corresponds to the 95% confidence level: $\Pr\{N_e(t_1; t_2) \leq m\} < 95\%$. In this case, the hypothesis $\mathcal{H} : \alpha = 99\%$ is not rejected at the 95% confidence level. The yellow and red zones are respectively defined by $95\% \leq \Pr\{N_e(t_1; t_2) \leq m\} < 99.99\%$ and $\Pr\{N_e(t_1; t_2) \leq m\} \geq 99.99\%$. This implies that the hypothesis $\mathcal{H} : \alpha = 99\%$ is rejected at the 99.99% confidence level if the number of exceptions belongs to the red zone.

If we apply the previous statistical analysis when n is equal to 250 trading days, we obtain the results given in Table 2.2. For instance, the probability to have zero exception is 8.106%, the probability to have one exception is 2.469%, etc. We retrieve the three color zones determined by the Basel Committee. The green zone corresponds to the interval $[0, 4]$, the yellow zone is defined by the interval $[5, 9]$ and the red zone involves the interval $[10, 250]$. We notice that the color zones can vary significantly if the confidence level of the value-at-risk is not equal to 99%. For instance, if it is equal to 98%, the green zone corresponds to less than 9 exceptions. In Figure 2.3, we have reported the color zones with respect to the size n of the sample.

Exercise 12 *Calculate the color zones when n is equal to 1000 trading days and $\alpha = 99\%$.*

We have $\Pr\{N_e \leq 14\} = 91.759\%$ and $\Pr\{N_e \leq 15\} = 95.213\%$. This implies that the green zones ends at 14 exceptions whereas the yellow zone begins at 15 exceptions. Because $\Pr\{N_e \leq 23\} = 99.989\%$ and $\Pr\{N_e \leq 24\} = 99.996\%$, we also deduce that the red zone begins at 24 exceptions.

TABLE 2.2: Probability distribution (in %) of the number of exceptions ($n = 250$ trading days)

	$\alpha = 99\%$		$\alpha = 98\%$	
m	$\Pr\{N_e = m\}$	$\Pr\{N_e \le m\}$	$\Pr\{N_e = m\}$	$\Pr\{N_e \le m\}$
0	8.106	8.106	0.640	0.640
1	20.469	28.575	3.268	3.908
2	25.742	54.317	8.303	12.211
3	21.495	75.812	14.008	26.219
4	13.407	89.219	17.653	43.872
5	6.663	95.882	17.725	61.597
6	2.748	98.630	14.771	76.367
7	0.968	99.597	10.507	86.875
8	0.297	99.894	6.514	93.388
9	0.081	99.975	3.574	96.963
10	0.020	99.995	1.758	98.720

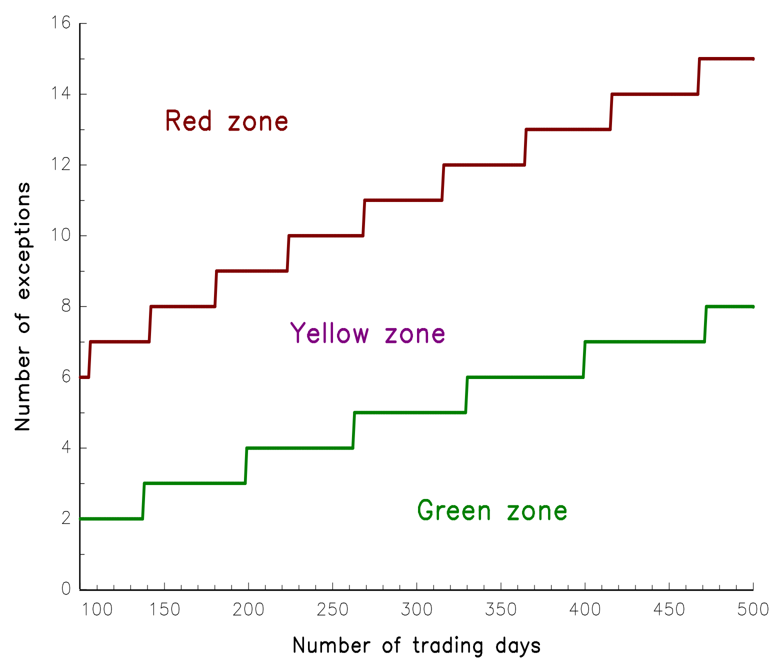

FIGURE 2.3: Color zones of the backtesting procedure ($\alpha = 99\%$)

Remark 7 *The statistical approach of backtesting ignores the effects of intra-day trading. Indeed, we make the assumption that the portfolio remains unchanged from $t-1$ to t, which is not the case in practice. This is why the Basel Committee proposes to compute the loss in two different ways. The first approach uses the official realized P&L, whereas the second approach consists in separating the P&L of the previous's day portfolio and the P&L due to the intra-day trading activities.*

2.2 Value-at-risk method

The value-at-risk $\mathrm{VaR}_\alpha(w;h)$ is defined as the potential loss which the portfolio w can suffer for a given confidence level α and a fixed holding period h. Three parameters are necessary to compute this risk measure:

- the holding period h, which indicates the time period to calculate the loss;
- the confidence level α, which gives the probability that the loss is lower than the value-at-risk;
- the portfolio w, which gives the allocation in terms of risky assets and is related to the risk factors.

Without the first two parameters, it is not possible to interpret the amount of the value-at-risk, which is expressed in monetary units. For instance, a portfolio with a VaR of $100 mn may be regarded as highly risky if the VaR corresponds to a 90% confidence level and a one-day holding period, but it may be a low risk investment if the confidence level is 99.9% and the holding period is one year.

We note $P_t(w)$ the mark-to-market value of the portfolio w at time t. The profit and loss between t and $t+h$ is equal to:

$$\Pi(w) = P_{t+h}(w) - P_t(w)$$

We define the loss of the portfolio as the opposite of the P&L: $L(w) = -\Pi(w)$. At time t, the loss is not known and is therefore random. From a statistical point of view, the value-at-risk $\mathrm{VaR}_\alpha(w;h)$ is the quantile[23] of the loss for the probability α:

$$\Pr\{L(w) \leq \mathrm{VaR}_\alpha(w;h)\} = \alpha$$

[23] If the distribution of the loss is not continuous, the statistical definition of the quantile function is:

$$\mathrm{VaR}_\alpha(w;h) = \inf\{x : \Pr\{L(w) \leq x\} \geq \alpha\}$$

This means that the probability that the random loss is lower than the VaR is exactly equal to the confidence level. We finally obtain:

$$\mathrm{VaR}_\alpha(w;h) = \mathbf{F}_L^{-1}(\alpha)$$

where $\mathbf{F}_L$ is the distribution function of the loss[24].

We notice that the previous analysis assumes that the portfolio remains unchanged between t and $t+h$. In practice, it is not the case because of trading and rebalancing activities. The holding period h depends then on the nature of the portfolio. The Basel Committee has set h to one trading day for performing the backtesting procedure in order to minimize rebalancing impacts. However, h is equal to 10 trading days for capital requirements. It is the period which is considered necessary to ensure the rebalancing of the portfolio if it is too risky or if it costs too much regulatory capital. The confidence level α is equal to 99% meaning that there is an exception every 100 trading days. It is obvious that it does not correspond to an extreme risk measure. From the point of view of regulators, the 99% value-at-risk gives then a measure of the market risk in the case of normal conditions.

To calculate the VaR, we first have to identify the risk factors that affect the future value of the portfolio. Their number can be larger or smaller depending on the market, but also on the portfolio. For instance, in the case of an equity portfolio, we can use the one-factor model (CAPM), a multi-factor model (industry risk factors, Fama-French risk factors, etc.) or we can have a risk factors for each individual stock. For interest-rate products, the Basel Committee imposes that the bank use at least six factors to model the yield curve risk. This contrasts with currency and commodity portfolios where one must take into account one risk factor by exchange rate and by currency. Let $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$ be the vector of risk factors. We assume that there is a function g such that:

$$P_t(w) = g(\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t}; w)$$

g is called the pricing function. We deduce that the expression of the random loss is:

$$\begin{aligned} L(w) &= P_t(w) - g(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}; w) \\ &= \ell(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}; w) \end{aligned}$$

where ℓ is the loss function. The big issue is then to model the future values of risk factors. In practice, the distribution $\mathbf{F}_L$ is not known because the multi-dimensional distribution of the risk factors is not known. This is why we have to estimate $\mathbf{F}_L$ meaning that the calculated VaR is also an estimate:

$$\widehat{\mathrm{VaR}}_\alpha(w;h) = \hat{\mathbf{F}}_L^{-1}(\alpha) = -\hat{\mathbf{F}}_\Pi^{-1}(1-\alpha)$$

[24] In a similar way, we have $\Pr\{\Pi(w) \geq -\mathrm{VaR}_\alpha(w;h)\} = \alpha$ and $\mathrm{VaR}_\alpha(w;h) = -\mathbf{F}_\Pi^{-1}(1-\alpha)$ where $\mathbf{F}_\Pi$ is the distribution function of the P&L.

In practice, there are three approaches to calculate $\widehat{\mathrm{VaR}}_\alpha(w;h)$ depending on the method used to estimate $\hat{\mathbf{F}}_L$:

1. the historical value-at-risk, which is also called the empirical or non-parametric VaR;
2. the analytical (or parametric) value-at-risk;
3. the Monte Carlo (or simulated) value-at-risk.

Remark 8 *In this book, we use the statistical expression* $\mathrm{VaR}_\alpha(w;h)$ *in place of* $\widehat{\mathrm{VaR}}_\alpha(w;h)$ *in order to reduce the amount of notation.*

2.2.1 Historical value-at-risk

The historical VaR corresponds to a non-parametric estimate of the value-at-risk. For that, we consider the empirical distribution of the risk factors observed in the past. Let $(\mathcal{F}_{1,s},\ldots,\mathcal{F}_{m,s})$ be the vector of risk factors observed at time $s<t$. If we calculate the future P&L with this historical scenario, we obtain:

$$\Pi_s(w)=g(\mathcal{F}_{1,s},\ldots,\mathcal{F}_{m,s};w)-P_t(x)$$

If we consider n_S historical scenarios ($s=1,\ldots,n_S$), the empirical distribution $\hat{\mathbf{F}}_\Pi$ is described by the following probability distribution:

$\Pi(w)$	$\Pi_1(w)$	$\Pi_2(w)$	$\cdots$	$\Pi_{n_S}(w)$
p_s	$1/n_S$	$1/n_S$		$1/n_S$

because each probability of occurrence is the same for all the historical scenarios. To calculate the empirical quantile $\hat{\mathbf{F}}_L^{-1}(\alpha)$, we can use two approaches: the order statistics approach and the kernel density approach.

2.2.1.1 The order statistics approach

Theorem 1 (Lehmann, 1999) *Let* $X_1,\ldots,X_n$ *be a sample from a continuous distribution* $\mathbf{F}$*. Suppose that for a given scalar* $\alpha\in\left]0,1\right[$*, there exist a sequence* $\{a_n\}$ *such that* $\sqrt{n}(a_n-n\alpha)\to 0$*. Then, we have:*

$$\sqrt{n}\left(X_{(a_n:n)}-\mathbf{F}^{-1}(\alpha)\right)\to\mathcal{N}\left(0,\frac{\alpha(1-\alpha)}{f^2\left(\mathbf{F}^{-1}(\alpha)\right)}\right)\tag{2.2}$$

This result implies that we can estimate the quantile $\mathbf{F}^{-1}(\alpha)$ by the mean of the $n\alpha^{\mathrm{th}}$ order statistic. Let us apply the previous result to our problem. We calculate the order statistics associated to the P&L sample $\{\Pi_1(w),\ldots,\Pi_{n_S}(w)\}$:

$$\min_s\Pi_s(w)=\Pi_{(1:n_S)}\leq\Pi_{(2:n_S)}\leq\cdots\leq\Pi_{(n_S-1:n_S)}\leq\Pi_{(n_S:n_S)}=\max_s\Pi_s(w)$$

The value-at-risk for a confidence level α is then equal to the opposite of the $n_S(1-\alpha)^{\text{th}}$ order statistic of the P&L:

$$\mathrm{VaR}_\alpha(w;h) = -\Pi_{(n_S(1-\alpha):n_S)} \tag{2.3}$$

If $n_S(1-\alpha)$ is not an integer, we consider an interpolation scheme:

$$\mathrm{VaR}_\alpha(x;h) = -\left(\Pi_{(q:n_S)} + (n_S(1-\alpha) - q)\left(\Pi_{(q+1:n_S)} - \Pi_{(q:n_S)}\right)\right)$$

where $q = [n_S(1-\alpha)]$ is the integer part of $n_S(1-\alpha)$. For instance, if $n_S = 100$, the 99% value-at-risk corresponds to the largest loss. In the case where we use 250 historical scenarios, the 99% value-at-risk is the mean between the second and third largest losses:

$$\begin{aligned}
\mathrm{VaR}_\alpha(x;h) &= -\left(\Pi_{(2:250)} + (2.5-2)\left(\Pi_{(3:250)} - \Pi_{(2:250)}\right)\right) \\
&= -\frac{1}{2}\left(\Pi_{(2:250)} + \Pi_{(3:250)}\right) \\
&= \frac{1}{2}\left(L_{(249:250)} + L_{(248:250)}\right)
\end{aligned}$$

Remark 9 *We remind that* $\mathrm{VaR}_\alpha(x;h)$ *defined by Equation (2.3) is an estimator with an asymptotic variance given by Theorem 1. Suppose that the loss of the portfolio is Gaussian and* $L(w) \sim \mathcal{N}(0,1)$. *The exact value-at-risk is* $\Phi^{-1}(\alpha)$ *and takes the values* 1.28 *or* 2.33 *if* α *is equal to 90% or 99%. The standard deviation of the estimator depends on the number* n_S *of historical scenarios:*

$$\sigma(\mathrm{VaR}_\alpha) \approx \frac{\sqrt{\alpha(1-\alpha)}}{\sqrt{n_S}\,\phi\left(\Phi^{-1}(\alpha)\right)}$$

In Figure 2.4, we have reported the density function of the VaR estimator. We notice that the estimation error decreases with n_S. *Moreover, it is lower for* $\alpha = 90\%$ *than for* $\alpha = 99\%$, *because the density of the Gaussian distribution at the point* $x = 1.28$ *is larger than at the point* $x = 2.33$.

Example 13 *We consider a portfolio composed of* 10 *stocks Apple and* 20 *stocks Coca-Cola. The current date is January* 2^{nd}*, 2015.*

The mark-to-market of the portfolio is:

$$P_t(w) = 10 \times P_{1,t} + 20 \times P_{2,t}$$

where $P_{1,t}$ and $P_{2,t}$ are the stock prices of Apple and Coca-Cola. We assume that the market risk factors corresponds to the daily stock returns $R_{1,t}$ and $R_{2,t}$. We deduce that the P&L for the scenario s is equal to:

$$\Pi_s(w) = \underbrace{10 \times P_{1,s} + 20 \times P_{2,s}}_{g(R_{1,s},R_{2,s};w)} - P_t(w)$$

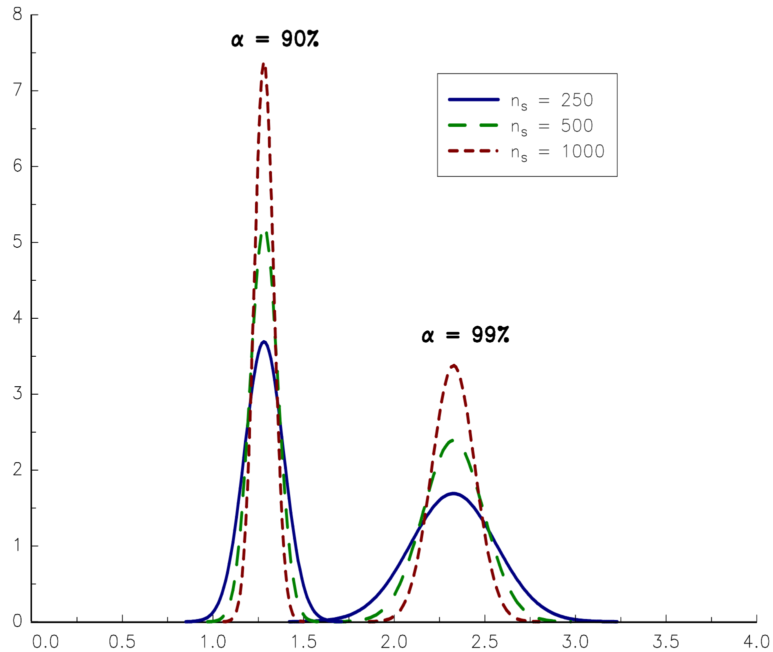

FIGURE 2.4: Density of the VaR estimator

where $P_{i,s} = P_{i,t} \times (1 + R_{i,s})$ is the simulated price of stock i for the scenario s. In Table 2.3, we have reported the values of the 10 first historical scenarios[25]. Using these scenarios, we can calculate the simulated price $P_{i,s}$ using the current price of the stocks (\$109.33 for Apple and \$42.14 for Coca-Cola). For instance, in the case of the ninth scenario, we obtain:

$$\begin{aligned} P_{1,s} &= 109.33 \times (1 - 0.77\%) = \$108.49 \\ P_{2,s} &= 42.14 \times (1 - 1.04\%) = \$41.70 \end{aligned}$$

We then deduce the simulated market-to-market $\mathrm{MtM}_s(w) = g(R_{1,s}, R_{2,s}; w)$, the current value of the portfolio[26] and the P&L $\Pi_s(w)$. These data are given in Table 2.4. In addition to the 10 first historical scenarios, we also report the results for the five worst cases and the last scenario[27]. We notice that

[25] For instance, the market risk factor for the first historical scenario and for Apple is calculated as follows:

$$R_{1,1} = \frac{109.33}{110.38} - 1 = -0.95\%$$

[26] We have:

$$P_t(w) = 10 \times 109.33 + 20 \times 42.14 = \$1\,936.10$$

[27] We assume that the value-at-risk is calculated using 250 historical scenarios (from 2015-01-02 to 2014-01-07)

the largest loss is reached for the 236$^{\text{th}}$ historical scenario at the date of January 28$^{\text{th}}$, 2014. If we rank the scenarios, the worst P&L are −84.34, −51.46, −43.31, −40.75 and −35.91. We deduce that the daily historical VaR is equal to:

$$\mathrm{VaR}_{99\%}\left(w;\text{one day}\right)=\frac{1}{2}\left(51.46+43.31\right)=\$47.39$$

If we assume that $m_c = 3$, the corresponding capital charge represents 23.22% of the portfolio's value:

$$\mathcal{K}_t^{\mathrm{VaR}}=3\times\sqrt{10}\times 47.39=\$449.54$$

TABLE 2.3: Computation of the market risk factors $R_{1,s}$ and $R_{2,s}$

s	Date	Apple		Coca-Cola	
		Price	$R_{1,s}$	Price	$R_{2,s}$
1	2015-01-02	109.33	−0.95%	42.14	−0.19%
2	2014-12-31	110.38	−1.90%	42.22	−1.26%
3	2014-12-30	112.52	−1.22%	42.76	−0.23%
4	2014-12-29	113.91	−0.07%	42.86	−0.23%
5	2014-12-26	113.99	1.77%	42.96	0.05%
6	2014-12-24	112.01	−0.47%	42.94	−0.07%
7	2014-12-23	112.54	−0.35%	42.97	1.46%
8	2014-12-22	112.94	1.04%	42.35	0.95%
9	2014-12-19	111.78	−0.77%	41.95	−1.04%
10	2014-12-18	112.65	2.96%	42.39	2.02%

Under Basel 2.5, we have to compute a second capital charge for the stressed VaR. If we assume that the stressed period is from October 9$^{\text{th}}$, 2007 to March 9$^{\text{th}}$, 2009, we obtain 356 stressed scenarios. By applying the previous method, the six largest simulated losses are[28] 219.20 (29/09/2008), 127.84 (17/09/2008), 126.86 (07/10/2008), 124.23 (14/10/2008), 115.24 (23/01/2008) and 99.55 (29/09/2008). The 99% SVaR corresponds to the 3.56$^{\text{th}}$ order statistic. We deduce that:

$$\begin{aligned}\mathrm{SVaR}_{99\%}\left(w;\text{one day}\right) &= 126.86+(3.56-3)\times(124.23-126.86)\\ &= \$125.38\end{aligned}$$

It follows that:

$$\mathcal{K}_t^{\mathrm{SVaR}}=3\times\sqrt{10}\times 125.38=\$1\,189.49$$

The total capital requirement under Basel 2.5 is then:

$$\mathcal{K}_t=\mathcal{K}_t^{\mathrm{VaR}}+\mathcal{K}_t^{\mathrm{SVaR}}=\$1\,639.03$$

It represents 84.6% of the current mark-to-market!

[28]We indicate in brackets the scenario day of the loss.

TABLE 2.4: Computation of the simulated P&L $\Pi_s(w)$

s	Date	Apple $R_{1,s}$	Apple $P_{1,s}$	Coca-Cola $R_{2,s}$	Coca-Cola $P_{2,s}$	$\mathrm{MtM}_s(w)$	$\Pi_s(w)$
1	2015-01-02	−0.95%	108.29	−0.19%	42.06	1 924.10	−12.00
2	2014-12-31	−1.90%	107.25	−1.26%	41.61	1 904.66	−31.44
3	2014-12-30	−1.22%	108.00	−0.23%	42.04	1 920.79	−15.31
4	2014-12-29	−0.07%	109.25	−0.23%	42.04	1 933.37	−2.73
5	2014-12-26	1.77%	111.26	0.05%	42.16	1 955.82	19.72
6	2014-12-24	−0.47%	108.82	−0.07%	42.11	1 930.36	−5.74
7	2014-12-23	−0.35%	108.94	1.46%	42.76	1 944.57	8.47
8	2014-12-22	1.04%	110.46	0.95%	42.54	1 955.48	19.38
9	2014-12-19	−0.77%	108.49	−1.04%	41.70	1 918.91	−17.19
10	2014-12-18	2.96%	112.57	2.02%	42.99	1 985.51	49.41
23	2014-12-01	−3.25%	105.78	−0.62%	41.88	1 895.35	−40.75
69	2014-09-25	−3.81%	105.16	−1.16%	41.65	1 884.64	−51.46
85	2014-09-03	−4.22%	104.72	0.34%	42.28	1 892.79	−43.31
236	2014-01-28	−7.99%	100.59	0.36%	42.29	1 851.76	−84.34
242	2014-01-17	−2.45%	106.65	−1.08%	41.68	1 900.19	−35.91
250	2014-01-07	−0.72%	108.55	0.30%	42.27	1 930.79	−5.31

Remark 10 *As the previous example has shown, directional exposures are highly penalized under Basel 2.5. More generally, it is not always evident that capital requirements are lower with IMA than with SMM (Crouhy* et al.*, 2013).*

2.2.1.2 The kernel density approach

Description of the kernel density estimation We consider the estimation of the density function f of the random variable X. Let $\{x_1, \ldots, x_n\}$ be a sample of X. The empirical distribution $\hat{\mathbf{F}}(x) = n^{-1}\sum_{i=1}^{n} \mathbb{1}\{x_i \leq x\}$ indicates the frequency of observations whose values are smaller than x. Because we have $\mathrm{d}\hat{F}(x) = \hat{f}(x)\,\mathrm{d}x$, we deduce that:

$$\begin{aligned}\hat{f}(x) &\simeq \frac{\hat{\mathbf{F}}(x+\boldsymbol{h}) - \hat{\mathbf{F}}(x-\boldsymbol{h})}{2\boldsymbol{h}} \\ &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{2\boldsymbol{h}}\mathbb{1}\{x-\boldsymbol{h} \leq x_i \leq x+\boldsymbol{h}\}\end{aligned}$$

The nonparametric estimator $\hat{f}(x)$ is then the histogram of the sample. We notice that we can write $\hat{f}(x)$ as follows:

$$\hat{f}(x) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\boldsymbol{h}}\mathcal{K}\left(\frac{x-x_i}{\boldsymbol{h}}\right)$$

$\mathcal{K}(u) = \frac{1}{2}\mathbb{1}\{-1 \leq u \leq 1\}$ is called the uniform (or rectangular) kernel. Non-parametric estimation consists in replacing the histogram kernel by a smooth function with the following desirable properties (Silverman, 1986):

- $\mathcal{K}(u) \geq 0$ (to ensure the positivity of the density function);
- $\int \mathcal{K}(u)\,\mathrm{d}u = 1$ (to ensure that $\hat{\mathbf{F}}(x) = \int_{-\infty}^{x} \hat{f}(y)\,\mathrm{d}y$ is a probability distribution function).

We generally add the symmetry property $\int u\,\mathcal{K}(u)\,\mathrm{d}u = 0$ in order to satisfy some empirical statistics. We then show that:

$$\mathbb{E}\left[\hat{f}_n(x) - f(x)\right] \approx \frac{\boldsymbol{h}^2}{2} f''(x) \int u^2 \mathcal{K}(u)\,\mathrm{d}u$$

and:

$$\operatorname{var}\left(\hat{f}_n(x)\right) \approx \frac{1}{n\boldsymbol{h}} f(x) \int \mathcal{K}^2(u)\,\mathrm{d}u$$

The bias and the variance of the estimator depend on the density function f to estimate, the bandwidth $\boldsymbol{h}$ and the kernel function $\mathcal{K}$. The bias is sensitive to the curvature f'' while the variance mainly depends on the size n of the sample. The estimator $\hat{\mathbf{F}}$ is then defined as:

$$\begin{aligned}\hat{\mathbf{F}}(x) &= \int_{-\infty}^{x} \frac{1}{n}\sum_{i=1}^{n} \frac{1}{\boldsymbol{h}}\mathcal{K}\left(\frac{y - x_i}{\boldsymbol{h}}\right)\mathrm{d}y \\ &= \frac{1}{n}\sum_{i=1}^{n} \mathcal{I}\left(\frac{x - x_i}{\boldsymbol{h}}\right)\end{aligned}$$

where $\mathcal{I}$ is the integrated kernel function:

$$\mathcal{I}(u) = \int_{-\infty}^{u} \mathcal{K}(t)\,\mathrm{d}t$$

The most used functions are the Gaussian[29] and Epanechnikov[30] kernels. The greatest challenge is the choice of the bandwidth $\boldsymbol{h}$, which controls the trade-off between bias and variance[31] (Jones *et al.*, 1996).

Application to the historical value-at-risk To estimate the value-at-risk for the confidence level α, Gouriéroux *et al.* (2000) solves the equation $\hat{\mathbf{F}}_L(\mathrm{VaR}_\alpha(w; h)) = \alpha$ or:

$$\frac{1}{n_S}\sum_{s=1}^{n_S} \mathcal{I}\left(\frac{-\mathrm{VaR}_\alpha(w; h) - \Pi_s(w)}{\boldsymbol{h}}\right) = 1 - \alpha$$

[29] We have $\mathcal{K}(u) = \phi(u)$ and $\mathcal{I}(u) = \Phi(u)$.

[30] It is also called the optimal kernel because it minimizes the mean squared error. We have $\mathcal{K}(u) = 3/4 \times (1 - u^2) \times \mathbb{1}\{|u| \leq 1\}$ and $\mathcal{I}(u) = \min\left(1/4 \times \left(3u - u^3 + 2\right) \times \mathbb{1}\{u > -1\}, 1\right)$.

[31] An optimal value is $1.364 \times n^{-1/5} \times \hat{s}$ where $\hat{s}$ is the standard deviation of the sample.

If we consider Example 13 with the last 250 historical scenarios, we obtain the results given in Figure 2.5. We have reported the estimated distribution $\hat{\mathbf{F}}_{\Pi}$ of $\Pi(w)$ with order statistics and Gaussian kernel methods[32]. We verify that the kernel approach produces a smoother distribution. If we zoom on the 1% quantile, we notice that the two methods give similar results. The daily VaR with the kernel approach is equal to \$47.44 whereas it was equal to \$47.39 with the order statistics approach.

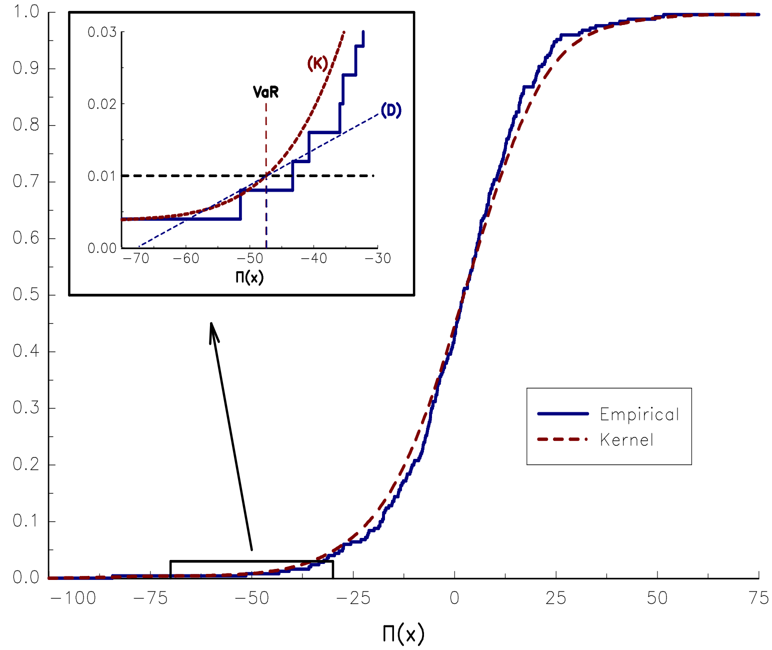

FIGURE 2.5: Kernel estimation of the historical VaR

Remark 11 *In practice, the kernel approach gives similar figures than the order statistics approach, especially when the number of scenarios is large. However, the two estimators may differ in the presence of fat tails. For large confidence levels, the order statistics approach seems to be more conservative.*

2.2.2 Analytical value-at-risk

2.2.2.1 Derivation of the closed-form formula

We speak about analytical value-at-risk when we are able to find a closed-form formula of $\mathbf{F}_L^{-1}(\alpha)$. Suppose that $L(w) \sim \mathcal{N}\left(\mu(L), \sigma^2(L)\right)$. In this case,

[32]The estimated standard deviation $\hat{s}$ is equal to 17.7147. We consider the bandwidth $\boldsymbol{h} = 1.364 \times n^{-1/5} \times \hat{s} = 8.0027$.

we have $\Pr\left\{L(w) \leq \mathbf{F}_L^{-1}(\alpha)\right\} = \alpha$ or:

$$\Pr\left\{\frac{L(w) - \mu(L)}{\sigma(L)} \leq \frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)}\right\} = \alpha \Leftrightarrow \Phi\left(\frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)}\right) = \alpha$$

We deduce that:

$$\frac{\mathbf{F}_L^{-1}(\alpha) - \mu(L)}{\sigma(L)} = \Phi^{-1}(\alpha) \Leftrightarrow \mathbf{F}_L^{-1}(\alpha) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L)$$

The expression of the value-at-risk is then[33]:

$$\mathrm{VaR}_\alpha(w;h) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L) = -\mu(\Pi) + \Phi^{-1}(\alpha)\,\sigma(\Pi) \tag{2.4}$$

This formula is known as the Gaussian value-at-risk. For instance, if $\alpha = 99\%$ (resp. 95%), $\Phi^{-1}(\alpha)$ is equal to 2.33 (resp. 1.65) and we have:

$$\mathrm{VaR}_\alpha(w;h) = \mu(L) + 2.33 \times \sigma(L) = -\mu(\Pi) + 2.33 \times \sigma(\Pi)$$

Remark 12 *We notice that the value-at-risk depends on the parameters μ_L and σ_L. This is why the analytical value-at-risk is also called the parametric value-at-risk. In practice, we don't know these parameters and we have to estimate them. This implies that the analytical value-at-risk is also an estimator. For the Gaussian distribution, we obtain:*

$$\widehat{\mathrm{VaR}}_\alpha(w;h) = \hat{\mu}(L) + \Phi^{-1}(\alpha)\,\hat{\sigma}(L)$$

In practice, it is extremely difficult to estimate the mean and we set $\hat{\mu}(L) = 0$.

Exercise 14 *We consider a short position of \$1 mn in the S&P 500 futures contract. We estimate that the annualized volatility $\hat{\sigma}_{\mathrm{SPX}}$ is equal to 35%. Calculate the daily value-at-risk with a 99% confidence level.*

The portfolio loss is equal to $L(w) = N \times R_{\mathrm{SPX}}$ where N is the exposure amount ($-\$1$ mn) and R_{SPX} is the (Gaussian) return of the S&P 500 index. We deduce that the annualized loss volatility is $\hat{\sigma}(L) = |N| \times \hat{\sigma}_{\mathrm{SPX}}$. The value-at-risk for a one-year holding period is:

$$\mathrm{VaR}_{99\%}(w;\text{one year}) = 2.33 \times 10^6 \times 0.35 = \$815\,500$$

By using the square-root-of-time rule, we deduce that:

$$\mathrm{VaR}_{99\%}(w;\text{one day}) = \frac{815\,500}{\sqrt{260}} = \$50\,575$$

This means that we have a 1% probability to lose more than \$50 575 per day.

[33] We remind that the P&L $\Pi(x)$ is the opposite of the portfolio loss $L(x)$ meaning that $\mu(\Pi) = -\mu(L)$ and $\sigma(\Pi) = \sigma(L)$.

In finance, the standard model is the Black-Scholes model where the price S_t of the asset is a geometric Brownian motion:

$$\mathrm{d}S_t = \mu_S S_t \,\mathrm{d}t + \sigma_S S_t \,\mathrm{d}W_t$$

with W_t a Wiener process. We can show that:

$$\ln S_{t_2} - \ln S_{t_1} = \left(\mu_S - \frac{1}{2}\sigma_s^2\right)(t_2 - t_1) + \sigma_S \left(W_{t_2} - W_{t_1}\right)$$

for $t_2 \geq t_1$. We have $W_{t_2} - W_{t_1} = \sqrt{t_2 - t_1}\varepsilon$ with $\varepsilon \sim \mathcal{N}(0,1)$. We finally deduce that $\operatorname{var}\left(\ln S_{t_2} - \ln S_{t_1}\right) = \sigma_S^2 (t_2 - t_1)$. Let $R_s(\Delta t)$ be a sample of log-returns measured at a regular time interval Δt. It follows that:

$$\hat{\sigma}_S = \frac{1}{\sqrt{\Delta t}}\sigma\left(R_s(\Delta t)\right)$$

If we consider two sample periods Δt and $\Delta t'$, we obtain the following relationship:

$$\sigma\left(R_s(\Delta t')\right) = \sqrt{\frac{\Delta t'}{\Delta t}}\sigma\left(R_s(\Delta t)\right)$$

For the mean, we have $\hat{\mu}_S = \Delta t^{-1}\mathbb{E}\left[R_s(\Delta t)\right]$ and $\mathbb{E}\left(R_s(\Delta t')\right) = (\Delta t'/\Delta t)\,\mathbb{E}\left(R_s(\Delta t)\right)$. We notice that the square-root-of-time rule is only valid for the volatility and therefore for risk measures that are linear with respect to the volatility. In practice, there is no other solutions and this explains why this rule continues to be used even if we know that the approximation is poor if the portfolio loss is not Gaussian.

2.2.2.2 Gaussian VaR and Linear factor models

We consider a portfolio of n assets and a pricing function g which is linear with respect to the asset prices. We have

$$g(\mathcal{F}_t; w) = \sum_{i=1}^n w_i P_{i,t}$$

We deduce that the random P&L is:

$$\begin{aligned}
\Pi(w) &= P_{t+h}(w) - P_t(w) \\
&= \sum_{i=1}^n w_i P_{i,t+h} - \sum_{i=1}^n w_i P_{i,t} \\
&= \sum_{i=1}^n w_i \left(P_{i,t+h} - P_{i,t}\right)
\end{aligned}$$

Here, $P_{i,t}$ is known whereas $P_{i,t+h}$ is random. The first idea is to choose the factors as the future prices. The problem is that prices are far to be stationary meaning that we will face some issues to model the distribution $\mathbf{F}_\Pi$. Another idea is to write the future price as follows:

$$P_{i,t+h} = P_{i,t}\left(1 + R_{i,t+h}\right)$$

where $R_{i,t+h}$ is the asset return between t and $t+h$. In this case, we obtain:

$$\Pi\left(w\right) = \sum_{i=1}^{n} w_i P_{i,t} R_{i,t+h}$$

In this approach, the asset returns are the market risk factors and each asset has its own risk factor.

The covariance model Let R_t be the vector of asset returns. We note $W_{i,t} = w_i P_{i,t}$ the wealth invested (or the nominal exposure) in asset i and $W_t = \left(W_{1,t}, \ldots, W_{n,t}\right)$. It follows that:

$$\Pi\left(w\right) = \sum_{i=1}^{n} W_{i,t} R_{i,t+h} = W_t^\top R_{t+h}$$

If we assume that $R_{t+h} \sim \mathcal{N}\left(\mu, \Sigma\right)$, we deduce that $\mu\left(\Pi\right) = W_t^\top \mu$ and $\sigma^2\left(\Pi\right) = W_t^\top \Sigma W_t$. Using Equation (2.4), the expression of the value-at-risk is:

$$\mathrm{VaR}_\alpha\left(w; h\right) = -W_t^\top \mu + \Phi^{-1}\left(\alpha\right)\sqrt{W_t^\top \Sigma W_t}$$

In this approach, we only need to estimate the covariance matrix of asset returns to compute the value-at-risk. This explains the popularity of this model, especially when the P&L of the portfolio is a linear function of the asset returns[34].

Let us consider our previous example. The nominal exposures[35] are \$1 093.3 (Apple) and \$842.8 (Coca-Cola). If we consider the historical prices from 2014-01-07 to 2015-01-02, the estimated standard deviation of daily returns is equal to 1.3611% for Apple and 0.9468% for Coca-Cola, whereas the cross-correlation is equal to 12.0787%. It follows that:

$$\begin{aligned}
\sigma^2\left(\Pi\right) &= W_t^\top \Sigma W_t \\
&= 1\,093.3^2 \times \left(\frac{1.3611}{100}\right)^2 + 842.8^2 \times \left(\frac{0.9468}{100}\right)^2 + \\
&\quad 2 \times \frac{12.0787}{100} \times 1\,093.3 \times 842.8 \times \frac{1.3611}{100} \times \frac{0.9468}{100} \\
&= 313.80
\end{aligned}$$

[34]For instance, this approach is frequently used by asset managers to measure the risk of equity portfolios.

[35]These figures are equal to 10×109.33 and 20×42.14.

If we omit the term of expected return $-W_t^\top \mu$, we deduce that the 99% daily value-at-risk[36] is equal to \$41.21. We obtain a lower figure than with the historical value-at-risk, which was equal to \$47.39. We explain this result, because the Gaussian distribution underestimates the probability of extreme events and is not adapted to take into account tail risk.

The linear factor model We consider the standard linear factor model where asset returns R_t are related to a set of risk factors $\mathcal{F}_t = (\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t})$ in the following way:

$$R_t = B\mathcal{F}_t + \varepsilon_t$$

where $\mathbb{E}(\mathcal{F}_t) = \mu(\mathcal{F})$, $\operatorname{cov}(\mathcal{F}_t) = \Omega$, $\mathbb{E}(\varepsilon_t) = \mathbf{0}$ and $\operatorname{cov}(\varepsilon_t) = D$. $\mathcal{F}_t$ represents the common risks whereas ε_t is the vector of specific or idiosyncratic risks. This implies that $\mathcal{F}_t$ and ε_t are independent and D is a diagonal matrix[37]. B is a $(n \times m)$ matrix that measures the sensitivity of asset returns with respect to the risk factors. The first two moments of R_t are given by:

$$\mu = \mathbb{E}[R_t] = B\mu(\mathcal{F})$$

and[38]:

$$\Sigma = \operatorname{cov}(R_t) = B\Omega B^\top + D$$

If we assume that asset returns are Gaussian, we deduce that:

$$\mathrm{VaR}_\alpha(w; h) = -W_t^\top B\mu(\mathcal{F}) + \Phi^{-1}(\alpha)\sqrt{W_t^\top \left(B\Omega B^\top + D\right) W_t}$$

The linear factor model plays a major role in financial modeling. The capital asset pricing model (CAPM) developed by Sharpe (1964) is a particular case of this model when there is a single factor, which corresponds to the market portfolio. In the arbitrage pricing theory (APT) of Ross (1976), $\mathcal{F}_t$ corresponds to a set of (unknown) arbitrage factors. They may be macroeconomic, statistical or characteristic-based factors. The three-factor model of Fama and French (1993) is certainly the most famous application of APT. In this case, the factors are the market factor, the size factor corresponding

[36]We have:

$$\mathrm{VaR}_{99\%}(w; \text{one day}) = \Phi^{-1}(0.99)\sqrt{313.80} = \$41.21$$

[37]In the following, we note $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)$ with $\tilde{\sigma}_i$ the idiosyncratic volatility of asset i.

[38]We have:

$$\begin{aligned}
\Sigma &= \mathbb{E}\left[(R_t - \mu)(R_t - \mu)^\top\right] \\
&= \mathbb{E}\left[(B(\mathcal{F}_t - \mu(\mathcal{F}) + \varepsilon_t))(B(\mathcal{F}_t - \mu(\mathcal{F}) + \varepsilon_t))^\top\right] \\
&= B\mathbb{E}\left[(\mathcal{F}_t - \mu(\mathcal{F}))(\mathcal{F}_t - \mu(\mathcal{F}))^\top\right] B_t^\top + \mathbb{E}\left[\varepsilon_t \varepsilon_t^\top\right] \\
&= B\Omega B^\top + D
\end{aligned}$$

to a long/short portfolio between small stocks and large stocks and the value factor, which is the return of stocks with high book-to-market values minus the return of stocks with low book-to-market values. Since its publication, the original Fama-French factor has been extended to many other factors including momentum, quality or liquidity factors[39].

BCBS (1996a) makes direct reference to CAPM. In this case, we obtain a single-factor model:

$$R_t = \alpha + \beta R_t^{\mathrm{mkt}} + \varepsilon_t$$

where R_t^{mkt} is the return of the market and $\beta = (\beta_1, \ldots, \beta_n)$ is the vector of beta coefficients. Let σ_{mkt} be the volatility of the market risk factor. We have $\operatorname{var}(R_{i,t}) = \beta_i^2 \sigma_{\mathrm{mkt}}^2 + \tilde{\sigma}_i^2$ and $\operatorname{cov}(R_{i,t}, R_{j,t}) = \beta_i \beta_j \sigma_{\mathrm{mkt}}^2$. By omitting the mean, we obtain:

$$\mathrm{VaR}_\alpha (w; h) = \Phi^{-1}(\alpha) \sqrt{\sigma_{\mathrm{mkt}}^2 \sum_{i=1}^n \tilde{\beta}_i^2 + 2 \sum_{j>i} \tilde{\beta}_i \tilde{\beta}_j + \sum_{i=1}^n W_{i,t}^2 \tilde{\sigma}_i^2}$$

where $\tilde{\beta}_i = W_{i,t} \beta_i$ is the beta exposure of asset i expressed in \$. With the previous formula, we can deduce the VaR only due to the market risk factor by omitting the specific risk[40].

If we consider our previous example, we can choose the S&P 500 index as the market risk factor. For the period 2014-01-07 to 2015-01-02, the beta coefficient is equal to 0.8307 for Apple and 0.4556 for Coca-Cola, whereas the corresponding idiosyncratic volatilities are 1.2241% (Apple) and 0.8887% (Coca-Cola). As the market volatility is estimated at 0.7165%, the daily value-at-risk is equal to \$41.68 if we include specific risks. Otherwise, it is equal to \$21.54 if we only consider the effect of the market risk factor.

Application to a bond portfolio We consider a portfolio of bonds from the same issuer. In this instance, we can model the bond portfolio by a stream of n_C coupons $C(t_m)$ with fixed dates $t_m \geq t$. Figure 2.6 presents an example of aggregating cash flows with two bonds with a fixed coupon rate and two short exposures. We note $B_t(T)$ the price of a zero-coupon bond at time t for the maturity T. We have $B_t(T) = e^{-(T-t)R_t(T)}$ where $R_t(T)$ is the zero-coupon rate. The sensitivity of the zero-coupon bond is

$$\frac{\partial \, B_t(T)}{\partial \, R_t(T)} = -(T-t) B_t(T)$$

For a small change in yield, we obtain:

$$\Delta_h B_{t+h}(T) \approx -(T-t) B_t(T) \Delta_h R_{t+h}(T)$$

The value of the portfolio is:

[39] See Cazalet and Roncalli (2014) for a survey.

[40] We set $\tilde{\sigma}_i$ to 0.

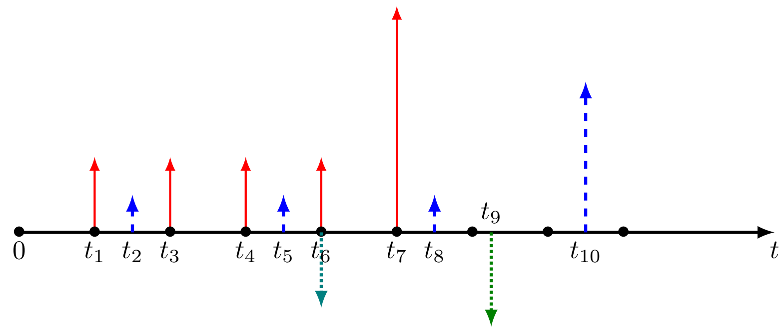

FIGURE 2.6: Cash flows of two bonds and two short exposures

$$P_t(w) = \sum_{m=1}^{n_C} C(t_m) B_t(t_m)$$

We deduce that:

$$\begin{aligned}
\Pi(w) &= P_{t+h}(w) - P_t(w) \\
&= \sum_{m=1}^{n_C} C(t_m)\left(B_{t+h}(t_m) - B_t(t_m)\right) \\
&\approx -\sum_{m=1}^{n_C} C(t_m)(t_m - t) B_t(t_m) \Delta_h R_{t+h}(t_m) \\
&= \sum_{m=1}^{n_C} W_{i,t_m} \Delta_h R_{t+h}(t_m)
\end{aligned}$$

with $W_{i,t_m} = C(t_m)(t_m - t) B_t(t_m)$. This expression of the P&L is similar to this obtained with a portfolio of stocks. If we assume that the yield variations are Gaussian, the value-at-risk is equal to:

$$\mathrm{VaR}_\alpha(w;h) = -W_t^\top \mu + \Phi^{-1}(\alpha)\sqrt{W_t^\top \Sigma W_t}$$

where μ and Σ are the mean and the covariance matrix of the vector of yield changes $(\Delta_h R_{t+h}(t_1), \ldots, \Delta_h R_{t+h}(t_{n_c}))$.

Example 15 *We consider an exposure on a US bond at December* 31^{st}*, 2014. The notional of the bond is* 100 *whereas the annual coupons are equal to* 5*. The remaining maturity is five years and the fixing dates are at the end of December. The number of bonds hold in the portfolio is* 10 000*.*

Using the US zero-coupon rates[41], we obtain the following figures for one

[41]The data comes from the Datastream database. The zero-coupon interest rate of maturity yy years and mm months corresponds to the code USyyYmm.

bond at December 31st, 2014:

$t_m - t$	$C(t_m)$	$R_t(t_m)$	$B_t(t_m)$	W_{t_m}
1	5	0.431%	0.996	−4.978
2	5	0.879%	0.983	−9.826
3	5	1.276%	0.962	−14.437
4	5	1.569%	0.939	−18.783
5	105	1.777%	0.915	−480.356

At the end of December 2014, the one-year zero-coupon rate is 0.431%, the two-year zero-coupon rate is 0.879%, etc. We deduce that the bond price is \$115.47 and the total exposure is \$1 154 706. Using the historical period of year 2014, we estimate the covariance matrix between daily changes of the five zero-coupon rates[42]. We deduce that the Gaussian VaR of the bond portfolio is equal to \$4 971. If the multiplicative factor m_c is set to 3, the required capital $\mathcal{K}_t^{\mathrm{VaR}}$ is equal to \$47 158 or 4.08% of the mark-to-market. We can compare these figures with those obtained with the historical value-at-risk. In this instance, the daily value-at-risk is higher and equal to \$5 302.

Remark 13 *The previous analysis assumes that the risk factors correspond to the yield changes, meaning that the calculated value-at-risk only concerns interest-rate risk. Therefore, it can not capture all the risks if the bond portfolio is subject to credit risk.*

Defining risk factors with the principal component analysis In the previous paragraph, the bond portfolio was very simple with only one bond and one yield curve. In practice, the bond portfolio contains streams of coupons for many maturities and yield curves. It is therefore necessarily to reduce the dimension of the VaR calculation. The underlying idea is that we don't need to use the comprehensive set of zero-coupon rates to represent the set of risk factors that affects the yield curve. For instance, Nelson and Siegel (1987) proposes a three-factor parametric model to define the yield curve. Another representation of the yield curve has been formulated by Litterman and Scheinkman (1991), who propose to characterize the factors using the principal component analysis (PCA).

Let Σ be the covariance matrix associated to the random vector X_t of dimension n. We consider the eigendecomposition $\Sigma = V\Lambda V^\top$ where $\Lambda =$

[42] The standard deviation is respectively equal to 0.746 bps for $\Delta_h R_t(t+1)$, 2.170 bps for $\Delta_h R_t(t+2)$, 3.264 bps for $\Delta_h R_t(t+3)$, 3.901 bps for $\Delta_h R_t(t+4)$ and 4.155 bps for $\Delta_h R_t(t+5)$ where h corresponds to one trading day. For the correlation matrix, we get:

$$\rho = \begin{pmatrix} 100.000 & & & & \\ 87.205 & 100.000 & & & \\ 79.809 & 97.845 & 100.000 & & \\ 75.584 & 95.270 & 98.895 & 100.000 & \\ 71.944 & 92.110 & 96.556 & 99.219 & 100.000 \end{pmatrix}$$

diag $(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and V is an orthornormal matrix. In the principal component analysis, the (endogenous) risk factors are $\mathcal{F}_t = V^\top X_t$. The reduction method by PCA consists in selecting the first m risk factors with $m \leq n$. When applied to the value-at-risk calculation, it can be achieved in two different ways:

1. In the parametric approach, the covariance matrix Σ is replaced by $\Sigma^\star = V \Lambda^\star V^\top$ where $\Lambda^\star = \operatorname{diag}(\lambda_1, \ldots, \lambda_m, 0, \ldots, 0)$.
2. In the historical method, we only consider the first m PCA factors $\mathcal{F}_t^\star = (\mathcal{F}_{1,t}, \ldots, \mathcal{F}_{m,t})$ or equivalently the modified random vector[43] $X_t^\star = V \mathcal{F}_t^\bullet$ with $\mathcal{F}_t^\bullet = (\mathcal{F}_t^\star, \mathbf{0})$.

If we apply this extracting method of risk factors to Example 15, the eigenvalues are equal to 47.299×10^8, 0.875×10^8, 0.166×10^8, 0.046×10^8, 0.012×10^8 whereas the matrix V of eigenvectors is:

$$V = \begin{pmatrix} 0.084 & -0.375 & -0.711 & 0.589 & 0.002 \\ 0.303 & -0.610 & -0.215 & -0.690 & -0.114 \\ 0.470 & -0.389 & 0.515 & 0.305 & 0.519 \\ 0.567 & 0.103 & 0.195 & 0.223 & -0.762 \\ 0.599 & 0.570 & -0.381 & -0.183 & 0.371 \end{pmatrix}$$

We deduce that:

$$\begin{aligned} \mathcal{F}_{1,t} &= 0.084 \times R_t(t+1) + 0.303 \times R_t(t+2) + \cdots + 0.599 \times R_t(t+5) \\ &\vdots \\ \mathcal{F}_{5,t} &= 0.002 \times R_t(t+1) - 0.114 \times R_t(t+2) + \cdots + 0.371 \times R_t(t+5) \end{aligned}$$

We retrieve the three factors of Litterman and Scheinkman, which are a factor of general level ($\mathcal{F}_{1,t}$), a slope factor ($\mathcal{F}_{2,t}$) and a convexity or curvature factor ($\mathcal{F}_{3,t}$). In the following table, we report the incremental VaR of each risk factor, which is defined as difference between the value-at-risk including the risk factor and the value-at-risk excluding the risk factor:

VaR	$\mathcal{F}_{1,t}$	$\mathcal{F}_{2,t}$	$\mathcal{F}_{3,t}$	$\mathcal{F}_{4,t}$	$\mathcal{F}_{5,t}$	Sum
Gaussian	4934.71	32.94	2.86	0.17	0.19	4970.87
Historical	5857.39	−765.44	216.58	−7.98	1.41	5301.95

We notice that the value-at-risk is principally explained by the first risk factor, that is the general level of interest rates, whereas the contribution of the slope and convexity factors is small and the contribution of the remaining risk factors is negligible. This result can be explained by the long-only characteristics of the portfolio. Nevertheless, even if we consider a more complex bond portfolio, we generally observed than a few number of factors is sufficient to model

[43] Because we have $V^{-1} = V^\top$.

all the risk dimensions of the yield curve. An example is provided in Figure 2.7 with a stream of long and short exposures[44]. Using the period January 2014 – December 2014, the convergence of the value-at-risk is achieved with six factors. This result is connected to the requirement of the Basel Committee that "*banks must model the yield curve using a minimum of six risk factors*".

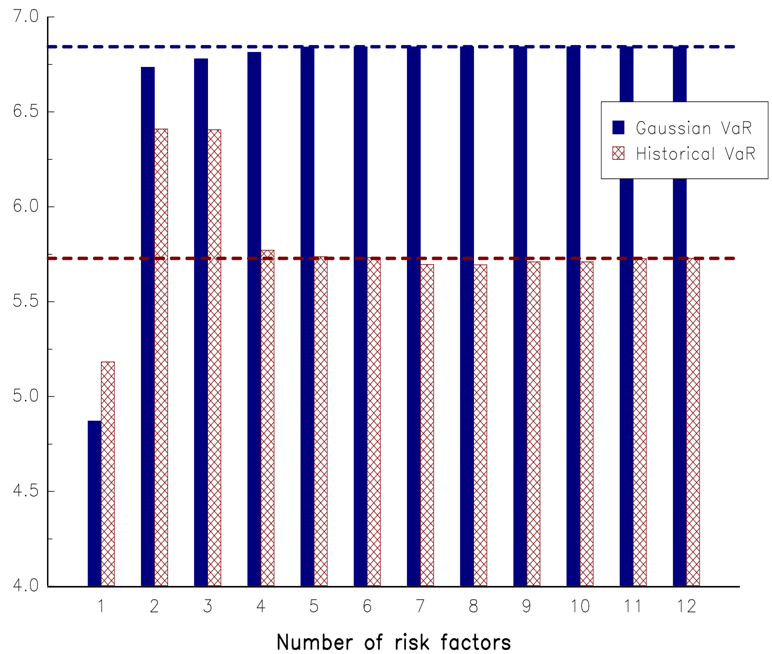

FIGURE 2.7: Convergence of the VaR with PCA risk factors

2.2.2.3 Volatility forecasting

The challenge of the Gaussian value-at-risk is the estimation of the loss volatility or the covariance matrix of asset returns/risk factors. The issue is not to consider the best estimate for describing the past, but to use the best estimate for forecasting the loss distribution. In the previous illustrations, we use the empirical covariance matrix or the empirical standard deviation. However, other estimators have been proposed by academics and professionals.

The original approach implemented in RiskMetrics uses an exponentially weighted moving average (EWMA) for modeling the covariance between asset

[44]We have $C_t(t+1/2) = 400$, $C_t(t+1) = 300$, $C_t(t+3/2) = 200$, $C_t(t+2) = -200$, $C_t(t+3) = -300$, $C_t(t+4) = -500$, $C_t(t+5) = 500$, $C_t(t+6) = 400$, $C_t(t+7) = -300$, $C_t(t+10) = -700$, $C_t(t+10) = 300$ and $C_t(t+30) = 700$.

returns[45]:

$$\hat{\Sigma}_t = \lambda \hat{\Sigma}_{t-1} + (1 - \lambda) R_{t-1} R_{t-1}^\top$$

where the parameter $\lambda \in [0, 1]$ is the decay factor, which represents the degree of weighting decrease. Using a finite sample, the previous estimate is equivalent to a weighted estimator:

$$\hat{\Sigma}_t = \sum_{s=1}^{n_S} \varpi_s R_{t-s} R_{t-s}^\top$$

with:

$$\varpi_s = \frac{(1 - \lambda)}{(1 - \lambda^{n_S})} \lambda^{s-1}$$

In Figure 2.8, we represent the weights ϖ_s for different values of λ when the number n_S of historical scenarios is equal to 250. We verify that this estimator gives more importance to the current values than to the past values. For instance, if λ is equal to 0.94[46], 50% of the weights corresponds to the twelve first observations and the half-life is 16.7 days. We also observe that the case $\lambda = 1$ corresponds to the standard covariance estimator with uniform weights.

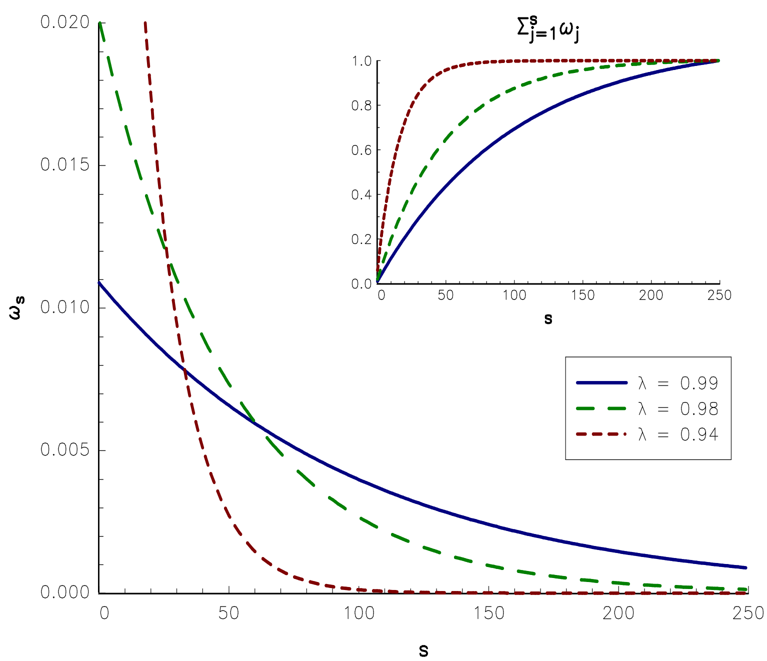

FIGURE 2.8: Weights of the EWMA estimator

[45]We assume that the mean of expected returns is equal to **0**.

[46]It is the original value of the RiskMetrics system (J.P. Morgan, 1996).

Another approach to model volatility in risk management is to consider that the volatility is time-varying. In 1982, Engle introduced a class of stochastic processes in order to take into account the heteroscedasticity of asset returns:

$$R_{i,t} = \mu_i + \varepsilon_t \quad \text{with} \quad \varepsilon_t = \sigma_t e_t \quad \text{and} \quad e_t \sim \mathcal{N}(0,1)$$

where the time-varying variance $h_t = \sigma_t^2$ satisfies the following equation:

$$h_t = \kappa + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2$$

with $\kappa > 0$ and $\alpha_j \geq 0$ for all $j > 0$. We note that the conditional variance of ε_t is not constant and depends on the past values of ε_t. A substantial impact on the asset return $R_{i,t}$ implies an increase of the conditional variance of ε_{t+1} at time $t+1$ and therefore an increase of the probability to observe another substantial impact on $R_{i,t+1}$. Therefore, this means that the volatility is persistent, which is a well-known stylized fact in finance (Chou, 1988). This type of stochastic processes, known as ARCH models (Autoregressive Conditional Heteroscedasticity), has been extended by Bollerslev (1986) in the following way:

$$\begin{aligned} h_t &= \kappa + \delta_1 h_{t-1} + \delta_2 h_{t-2} + \cdots + \delta_p h_{t-p} + \\ &\quad \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2 \end{aligned}$$

In this case, the conditional variance depends also on its past values and we obtain a GARCH(p,q) model. If $\sum_{i=1}^p \delta_i + \sum_{i=1}^q \alpha_i = 1$, we may show that the process ε_t^2 has a unit root and the model is called an integrated GARCH (or IGARCH) process. If we neglect the constant term, the expression of the IGARCH(1,1) process is $h_t = (1-\alpha) h_{t-1} + \alpha R_{i,t-1}^2$ or equivalently:

$$\sigma_t^2 = (1-\alpha)\sigma_{t-1}^2 + \alpha R_{i,t-1}^2$$

This estimator is then an exponentially weighted moving average with a factor λ equal to $1-\alpha$.

In Figure 2.9, we have reported the annualized volatility of the S&P 500 index estimated using the GARCH model (first panel). The ML estimates of the parameters are $\hat{\delta}_1 = 0.8954$ and $\hat{\alpha}_1 = 0.0929$. We verify that this estimated model is close to an IGARCH process. In the other panels, we compare the GARCH volatility with the empirical one-year historical volatility, the EWMA volatility (with $\lambda = 0.94$) and a short volatility based on 20 trading days. We observe large differences between the GARCH volatility and the one-year historical volatility, but the two others estimators (EWMA and short volatility) give similar results to the GARCH estimator. To compare the out-of-sample forecasting accuracy of these different models, we consider respectively a long and a short exposure on the S&P 500 index. At time t, we compute the value-at-risk for the next day and we compare this figure with the realized mark-to-market. Table 2.5 show the number of exceptions per year

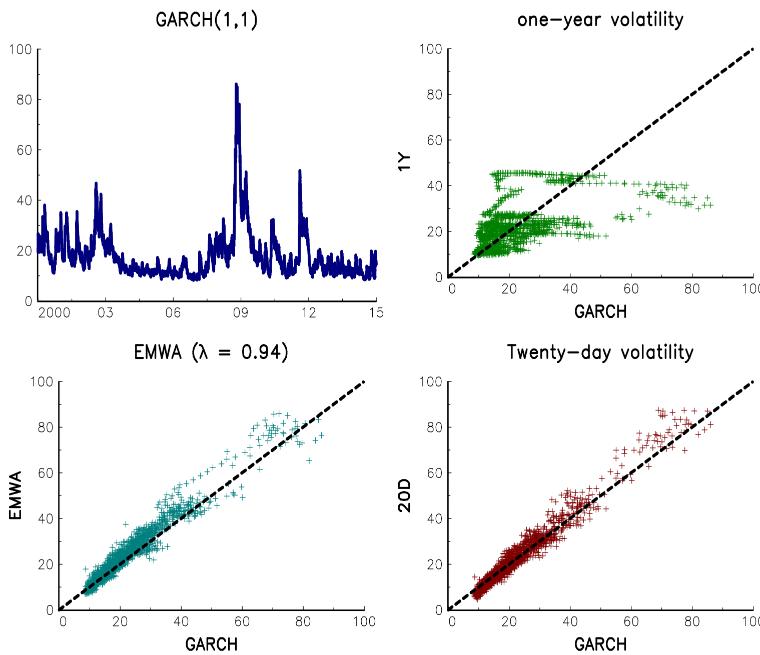

FIGURE 2.9: Comparison of GARCH and EWMA volatilities (S&P 500)

TABLE 2.5: Number of exceptions per year for long and short exposures on the S&P 500 index

Year	Long exposure					Short exposure				
	GARCH	1Y	EWMA	20D	H	GARCH	1Y	EWMA	20D	H
2000	5	5	2	4	4	5	8	4	6	4
2001	4	3	2	3	2	2	4	2	5	2
2002	2	5	2	4	3	5	9	4	6	5
2003	1	0	0	2	0	1	0	1	4	0
2004	2	0	2	6	0	0	0	0	2	1
2005	1	1	2	4	3	1	4	1	6	3
2006	2	4	3	4	4	2	5	3	5	3
2007	6	15	6	10	7	1	9	0	3	7
2008	7	23	5	7	10	4	12	4	3	8
2009	5	0	1	6	0	2	2	2	3	0
2010	7	6	5	8	3	3	5	2	7	3
2011	6	8	6	7	4	2	8	1	6	3
2012	5	1	4	5	0	3	1	2	7	1
2013	4	2	3	9	2	2	2	2	4	1
2014	6	9	7	11	2	2	4	2	2	4

for the different models: the GARCH (1,1) model, the Gaussian value-at-risk with a one-year historical volatility, the EWMA model with $\lambda = 0.94$, the Gaussian value-at-risk with a twenty-day short volatility and the historical value-at-risk based on the last 260 trading days. We observe that the GARCH model produces the smallest number of exceptions, whereas the largest number of exceptions occurs in the case of the Gaussian value-at-risk with the one-year historical volatility. We also notice that the number of exceptions is smaller for the short exposure than for the long exposure. This is due to the asymmetry of returns, because extreme negative returns are larger than extreme positive returns on average.

2.2.2.4 Extension to other probability distributions

The Gaussian value-at-risk has been strongly criticized because it depends only on the first two moments of the loss distribution. Indeed, there is a lot of evidence that asset returns and risk factors are not Gaussian (Cont, 2001). They generally present fat tails and skew effects. It is therefore interesting to consider alternative probability distributions, which are more appropriate to take into account these stylized facts.

Let $\mu_r = \mathbb{E}\left[(X - \mathbb{E}[X])^r\right]$ be the centered r-order moment of the random variable X. The skewness $\gamma_1 = \mu_3/\mu_2^{3/2}$ is the measure of the asymmetry of the loss distribution. If $\gamma_1 < 0$ (resp. $\gamma_1 > 0$), the distribution is left-skewed (resp. right-skewed) because the left (resp. right) tail is longer. For the Gaussian distribution, γ_1 is equal to zero. To characterize whether the distribution is peaked or flat relative to the normal distribution, we consider the excess kurtosis $\gamma_2 = \mu_4/\mu_2^2 - 3$. If $\gamma_2 > 0$, the distribution presents heavy tails. In the case of the Gaussian distribution, γ_2 is exactly equal to zero. We have illustrated the skewness and kurtosis statistics in Figure 2.10. Whereas we generally encounter skewness risk in credit and hedge fund portfolios, kurtosis risk has a stronger impact in equity portfolios. For example, if we consider the daily returns of the S&P 500 index, we obtain an empirical distribution[47] which has a higher kurtosis than the fitted Gaussian distribution (Figure 2.11).

An example of fat-tail distributions is the Student's t probability distribution. If $X \sim t_\nu$, we have $\mathbb{E}[X] = 0$ and $\operatorname{var}(X) = \nu/(\nu - 2)$ for $\nu > 2$. Because X has a fixed mean and variance for a given degrees of freedom, we need to introduce location and scale parameters to model the future loss:

$$L(w) = \xi + \omega X$$

[47] It is estimated using the kernel approach.

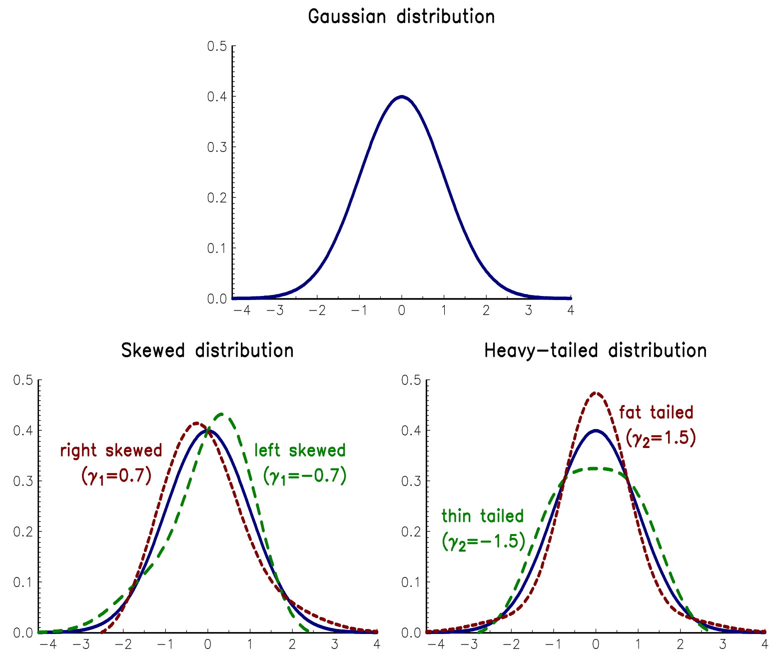

FIGURE 2.10: Examples of skewed and fat tailed distributions

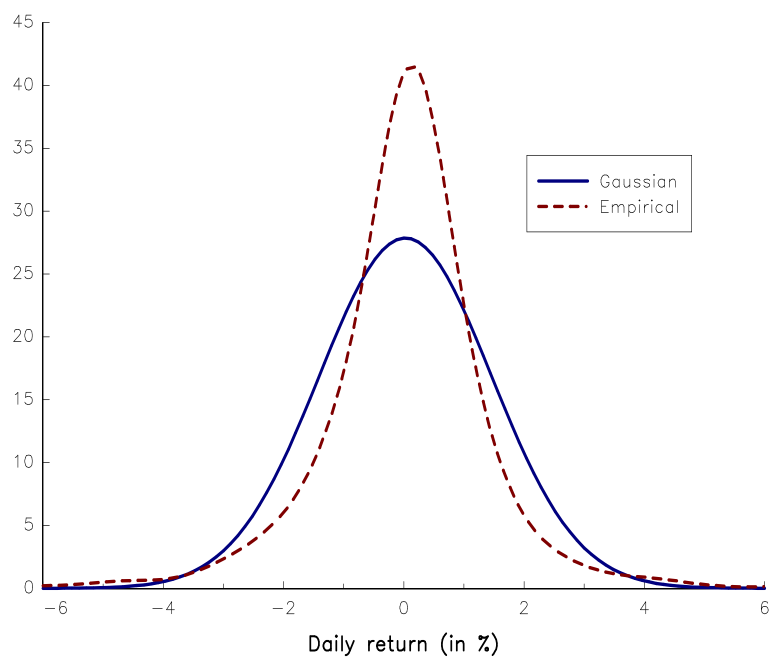

FIGURE 2.11: Estimated distribution of S&P 500 daily returns (2007-2014)

To calculate the value-at-risk, we proceed as in the Gaussian case. We have:

$$\begin{aligned}
\Pr\left\{L(w) \leq \mathbf{F}_L^{-1}(\alpha)\right\} = \alpha \quad &\Leftrightarrow \quad \Pr\left\{X \leq \frac{\mathbf{F}_L^{-1}(\alpha) - \xi}{\omega}\right\} = \alpha \\
&\Leftrightarrow \quad \mathbf{T}_v\left(\frac{\mathbf{F}_L^{-1}(\alpha) - \xi}{\omega}\right) = \alpha \\
&\Leftrightarrow \quad \mathbf{F}_L^{-1}(\alpha) = \xi + \mathbf{T}_v^{-1}(\alpha)\,\omega
\end{aligned}$$

In practice, the parameters ξ and ω are estimated by the method of moments[48]. We finally deduce that:

$$\mathrm{VaR}_\alpha(w;h) = \mu(L) + \mathbf{T}_v^{-1}(\alpha)\,\sigma(L)\sqrt{\frac{\nu-2}{\nu}} = -\mu(\Pi) + \mathbf{T}_v^{-1}(\alpha)\,\sigma(\Pi)\sqrt{\frac{\nu-2}{\nu}}$$

Let us illustrate the impact of the probability distribution with Example 13. By using different values of ν, we obtain the following results.

ν	3.00	3.50	4.00	5.00	6.00	10.00	1000	∞
ω	10.23	11.60	12.53	13.72	14.46	15.84	17.70	17.71
$\mathrm{VaR}_\alpha(w;1\mathrm{D})$	46.44	47.09	46.93	46.17	45.46	43.79	41.24	41.21

If $\nu \to \infty$, we verify that the Student's t value-at-risk converges to the Gaussian value-at-risk (\$41.21). If the degrees of freedom is equal to 4, it is closer to the historical value-at-risk (\$47.39).

We can derive closed-form formula for several probability distributions. However, most of them are not used in practice, because these methods are not appealing from a professional point of view. Nevertheless, one approach is very popular among professionals. Using the Cornish-Fisher expansion of the normal distribution, Zangari (1996) proposes to estimate the value-at-risk in the following way:

$$\mathrm{VaR}_\alpha(w;h) = \mu(L) + \mathfrak{z}(\alpha;\gamma_1(L),\gamma_2(L)) \times \sigma(L) \tag{2.5}$$

where:

$$\mathfrak{z}(\alpha;\gamma_1,\gamma_2) = z_\alpha + \frac{1}{6}\left(z_\alpha^2 - 1\right)\gamma_1 + \frac{1}{24}\left(z_\alpha^3 - 3z_\alpha\right)\gamma_2 - \frac{1}{36}\left(2z_\alpha^3 - 5z_\alpha\right)\gamma_1^2 \tag{2.6}$$

and $z_\alpha = \Phi^{-1}(\alpha)$. This is the same formula as the one used for the Gaussian value-at-risk but with another scaling parameter[49]. In Equation (2.5), the skewness and excess kurtosis coefficients are those of the loss distribution[50]. Table 2.6 shows the value of the Cornish-Fisher quantile $\mathfrak{z}(99\%;\gamma_1,\gamma_2)$ for different values of skewness and excess kurtosis. We can not always calculate

[48] We have $\mathbb{E}[\xi + \omega X] = \xi$ and $\mathrm{var}(\xi + \omega X) = \left(\omega^2\nu\right)/(\nu-2)$.

[49] If $\gamma_1 = \gamma_2 = 0$, we retrieve the Gaussian value-at-risk with $\mathfrak{z}(\alpha;0,0) = \Phi^{-1}(\alpha)$.

[50] If we prefer to use the moments of the P&L, we have to consider the relationships $\gamma_1(L) = -\gamma_1(\Pi)$ and $\gamma_2(\Pi) = \gamma_2(\Pi)$.

TABLE 2.6: Value of the Cornish-Fisher quantile $\mathfrak{z}\,(99\%;\gamma_1,\gamma_2)$

γ_1	γ_2							
	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00
−2.00								0.99
−1.00			1.68	1.92	2.15	2.38	2.62	2.85
−0.50		2.10	2.33	2.57	2.80	3.03	3.27	3.50
0.00	2.33	2.56	2.79	3.03	3.26	3.50	3.73	3.96
0.50		2.83	3.07	3.30	3.54	3.77	4.00	4.24
1.00			3.15	3.39	3.62	3.85	4.09	4.32
2.00								3.93

the quantile because Equation (2.6) does not define necessarily a probability distribution if the parameters γ_1 and γ_2 does not satisfy the following condition:

$$\frac{\partial\,\mathfrak{z}\,(\alpha;\gamma_1,\gamma_2)}{\partial\, z_\alpha} \geq 0 \Leftrightarrow \frac{\gamma_1^2}{9} - 4\left(\frac{\gamma_2}{8} - \frac{\gamma_1^2}{6}\right)\left(1 - \frac{\gamma_2}{8} + \frac{5\gamma_1^2}{36}\right) \leq 0$$

We have reported the domain of definition in Figure 2.12. For instance, Equation (2.6) is not valid if the skewness is equal to 2 and the excess kurtosis is equal to 3. If we analyze results in Table 2.6, we do not observe that there is a monotone relationship between the skewness and the quantile. To understand this curious behavior, we report the partial derivatives of $\mathfrak{z}\,(\alpha;\gamma_1,\gamma_2)$ with respect to γ_1 and γ_2 in Figure 2.12. We notice that their signs depend on the confidence level α, but also on the skewness for $\partial_{\gamma_1}\,\mathfrak{z}\,(\alpha;\gamma_1,\gamma_2)$. Another drawback of the Cornish-Fisher approach concerns the statistical moments, which are not necessarily equal to the input parameters if the skewness and the kurtosis are not close to zero[51]. Contrary to what professionals commonly think, the Cornish-Fisher expansion is therefore difficult to implement.

When we consider other probability distribution than the normal distribution, the difficulty concerns the multivariate case. In the previous examples, we directly model the loss distribution, that is the reduced form of the pricing system. To model the joint distribution of risk factors, two main approaches are available. The first approach considers copula functions and the value-at-risk is calculated using the Monte Carlo simulation method (see Chapters 15 and 17). The second approach consists in selecting a multivariate probability distribution, which have some appealing properties. For instance, it should

[51]Let Z be a Cornish-Fisher random variable satisfying $\mathbf{F}^{-1}(\alpha) = \mathfrak{z}\,(\alpha;\gamma_1,\gamma_2)$. A direct application of the result in Appendix A.2.2.2 gives:

$$\mathbb{E}\left[Z^r\right] = \int_0^1 \mathfrak{z}^r\,(\alpha;\gamma_1,\gamma_2)\,\mathrm{d}\alpha$$

Using numerical integration, we can show that $\gamma_1(Z) \neq \gamma_1$ and $\gamma_2(Z) \neq \gamma_2$ if γ_1 and γ_2 are large enough.

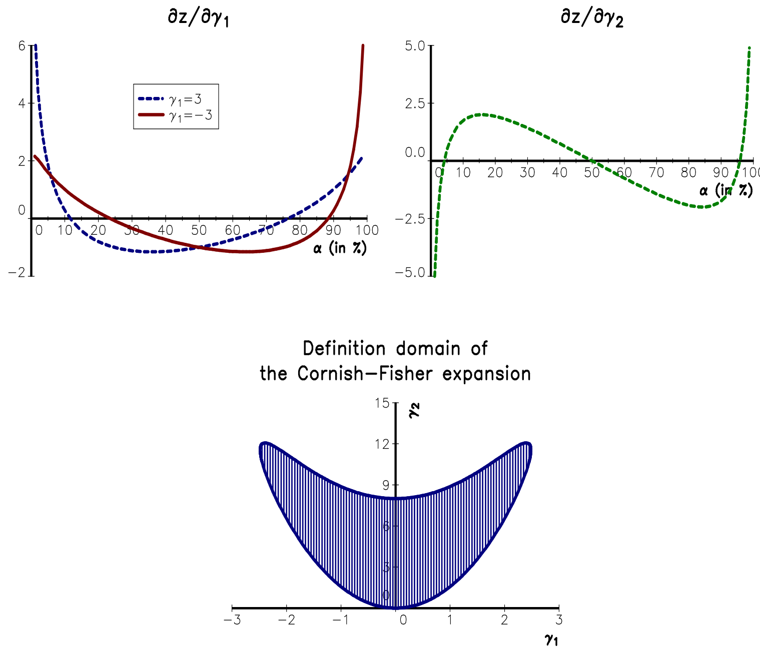

FIGURE 2.12: Derivatives and definition domain of the Cornish-Fisher expansion

be flexible enough to calibrate the first two moments of the risk factors and should also include asymmetry (positive and negative skewness) and fat tails (positive excess kurtosis) in a natural way. In order to obtain an analytical formula for the value-at-risk, it must be tractable and verify the closure property under affine transformation. This implies that if the random vector X follows a certain class of distribution, then the random vector $Y = A + BX$ belongs also to the same class. These properties reduce dramatically the set of eligible multivariate probability distributions, because the potential candidates are mostly elliptical distributions. Such examples are the skew normal and t distributions presented in Appendix A.2.1 in page 451.

Exercise 16 *We consider a portfolio of three assets and assume that their annualized returns follows a multivariate skew normal distribution. The location parameters are equal to* 1%, -2% *and* 15% *whereas the scale parameters are equal to* 5%, 10% *and* 20%. *The correlation parameters* $C_{i,j}$ *to describe the dependence between the skew normal variables are given by the following matrix:*

$$C = \begin{pmatrix} 1.00 & & \\ 0.35 & 1.00 & \\ 0.20 & -0.50 & 1.00 \end{pmatrix}$$

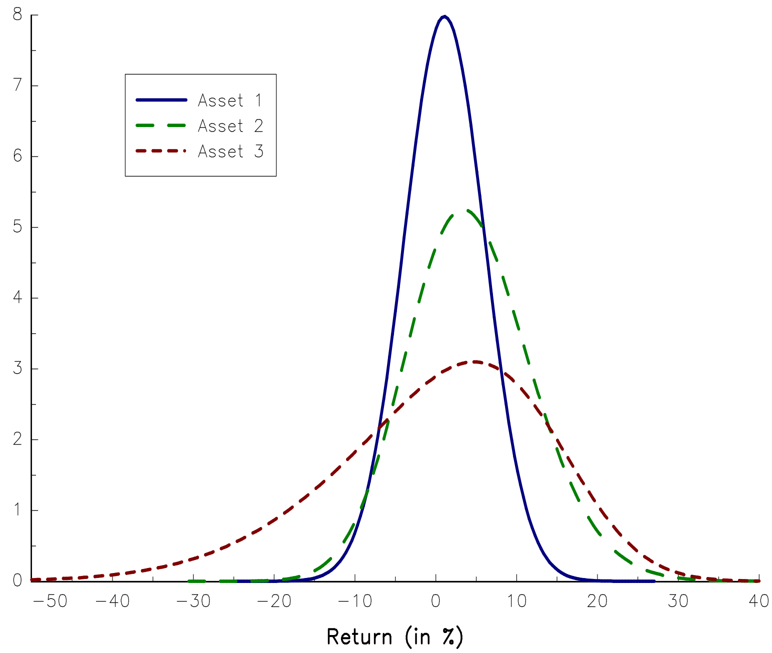

FIGURE 2.13: Skew normal distribution of asset returns

The three assets have different skewness profiles, and the shape parameters are equal to 0, 10 *and* −15.50.

In Figure 2.13, we have reported the density function of the three asset returns[52]. The return of the first asset is close to be Gaussian whereas the two other assets exhibit respectively negative and positive skews. Moments are given in the table below:

Asset i	μ_i (in %)	σ_i (in %)	$\gamma_{1,i}$	$\gamma_{2,i}$
1	1.07	5.00	0.00	0.00
2	4.36	7.72	0.24	0.13
3	0.32	13.58	−0.54	0.39

Let us consider the portfolio $w = (\$500, \$200, \$300)$. The annualized P&L $\Pi(w)$ is equal to $w^\top R$ with $R \sim \mathcal{SN}(\xi, \Omega, \eta)$. We deduce that $\Pi(w) \sim \mathcal{SN}(\xi_w, \omega_w, \eta_w)$ with $\xi_w = 46.00$, $\omega_w = 66.14$ and $\eta_w = -0.73$. We finally deduce that the one-year 99% value-at-risk is equal to \$123.91. If we use the multivariate skew t distribution in place of the multivariate skew normal distributions to model asset returns and if we use the same parameter values, the one-year 99% value-at-risk becomes \$558.35 for $\nu = 2$, \$215.21 for $\nu = 5$

[52]We also show the density functions in the case of the skew t distribution with $\nu = 1$ and $\nu = 4$.

and \$130.47 for $\nu = 50$. We verify that the skew t value-at-risk converges to the skew normal value-at-risk as the number of degrees of freedom ν tends to $+\infty$.

The choice of the probability distribution is an important issue and raises the question of model risk. In this instance, the Basel Committee justifies the introduction of the penalty coefficient in order to reduce the risk of a wrong specification (Stahl, 1997). For example, imagine that we calculate the value-at-risk with a probability distribution $\mathbf{F}$ while the true probability distribution of the loss portfolio is $\mathbf{H}$. The multiplication factor m_c defines then a capital buffer such that we are certain that the confidence level of the value-at-risk will be at least equal to α:

$$\Pr\{L(w) \leq \underbrace{m_c \times \mathrm{VaR}_\alpha^{(\mathbf{F})}(w)}_{\text{Capital}}\} \geq \alpha \tag{2.7}$$

This implies that $\mathbf{H}\left(m_c \times \mathrm{VaR}_\alpha^{(\mathbf{F})}(w)\right) \geq \alpha$ and $m_c \times \mathrm{VaR}_\alpha^{(\mathbf{F})}(w) \geq \mathbf{H}^{-1}(\alpha)$. We finally deduce that:

$$m_c \geq \frac{\mathrm{VaR}_\alpha^{(\mathbf{H})}(w)}{\mathrm{VaR}_\alpha^{(\mathbf{F})}(w)}$$

In the case where $\mathbf{F}$ and $\mathbf{H}$ are the normal and Student's t distributions, we obtain[53]:

$$m_c \geq \sqrt{\frac{\nu-2}{2}}\frac{\mathbf{T}_\nu^{-1}(\alpha)}{\Phi^{-1}(\alpha)}$$

Below is the lower bound of m_c for different values of α and ν.

α/ν	3	4	5	6	10	50	100
90%	0.74	0.85	0.89	0.92	0.96	0.99	1.00
95%	1.13	1.14	1.12	1.10	1.06	1.01	1.01
99%	1.31	1.26	1.21	1.18	1.10	1.02	1.01
99.9%	1.91	1.64	1.48	1.38	1.20	1.03	1.02
99.99%	3.45	2.48	2.02	1.76	1.37	1.06	1.03

For instance, we have $m_c \geq 1.31$ when $\alpha = 99\%$ and $\nu = 3$.

Stahl (1997) considers the general case when $\mathbf{F}$ is the normal distribution and $\mathbf{H}$ is an unknown probability distribution. Let X be a given random variable. The Chebyshev's inequality states that:

$$\Pr\{(|X-\mu(X)| > k \times \sigma(X))\} \leq k^{-2}$$

for any real number $k > 0$. If we apply this theorem to the value-at-risk, we obtain[54]:

$$\Pr\left\{L(w) \leq \sqrt{\frac{1}{1-\alpha}}\sigma(L)\right\} \geq \alpha$$

[53] We remind that the Gaussian value-at-risk is equal to $\Phi^{-1}(\alpha)\sigma(L)$ whereas the Student's t value-at-risk is equal to $\sqrt{(\nu-2)/2} \cdot \mathbf{T}_\nu^{-1}(\alpha)\sigma(L)$.

[54] We set $\alpha = 1 - k^{-2}$.

Using Equation (2.7), we deduce that:

$$m_c = \sqrt{\frac{1}{1-\alpha}} \frac{\sigma(L)}{\mathrm{VaR}_\alpha^{(\mathbf{F})}(w)}$$

In the case of the normal distribution, we finally obtain that the multiplicative factor is:

$$m_c = \frac{1}{\Phi^{-1}(\alpha)} \sqrt{\frac{1}{1-\alpha}}$$

This ratio is the multiplication factor to apply in order to be sure that the confidence level of the value-at-risk is at least equal to α if we use the normal distribution to model the portfolio loss. In the case where the probability distribution is symmetric, this ratio becomes:

$$m_c = \frac{1}{\Phi^{-1}(\alpha)} \sqrt{\frac{1}{2-2\alpha}}$$

In Table 2.7, we report the values of m_c for different confidence level. If α is equal to 99%, the multiplication factor is equal to 3.04 if the distribution is symmetric and 4.30 otherwise.

TABLE 2.7: Value of the multiplication factor m_c deduced from the Chebyshev's inequality

α (in %)	90.00	95.00	99.00	99.25	99.50	99.75	99.99
Symmetric	1.74	1.92	3.04	3.36	3.88	5.04	19.01
Asymmetric	2.47	2.72	4.30	4.75	5.49	7.12	26.89

Remark 14 *Even if the previous analysis justifies the multiplication factor from a statistical point of view, we face two main issues. First, the multiplication factor assumes that the bank uses a Gaussian value-at-risk. It was the case for many banks in the early 1990s, but they use today historical value-at-risk measures. Some have suggested that the multiplication factor has been introduced in order to reduce the difference in terms of regulatory capital between SMM and IMA and it is certainly the case. The second issue concerns the specificity of the loss distribution. For many positions like long-only unlevered portfolios, the loss is bounded. If we use a Gaussian value-at-risk, the regulatory capital satisfies*[55] $\mathcal{K} = \mathcal{K}^{\mathrm{VaR}} + \mathcal{K}^{\mathrm{SVaR}} > 13.98 \cdot \sigma(L)$ *where* $\sigma(L)$ *is the non-stressed loss volatility. This implies that the value-at-risk is larger than the portfolio value if* $\sigma(L) > 7.2\%$*! There is a direct contradiction here.*

[55] Because we have $2 \times m_c \times 2.33 > 13.98$

2.2.3 Monte Carlo value-at-risk

In this approach, we postulate a given probability distribution $\mathbf{H}$ for the risk factors:

$$(\mathcal{F}_{1,t+h}, \ldots, \mathcal{F}_{m,t+h}) \sim \mathbf{H}$$

Then, we simulate n_S scenarios of risk factors and calculate the simulated P&L $\Pi_s(w)$ for each scenario s. Finally, we estimate the value-at-risk by the method of order statistics. The Monte Carlo method to calculate the value-at-risk is therefore close to the historical method. The only difference is that it uses simulated scenarios instead of historical scenarios. This implies that the Monte Carlo approach is not limited by the number of scenarios. By construction, the Monte Carlo value-at-risk is also similar to the analytical value-at-risk, because they both specify the parametric probability distribution of risk factors. In summary, we can say that:

- the Monte Carlo VaR is an historical VaR with simulated scenarios;
- the Monte Carlo VaR is a parametric VaR for which it is difficult to find an analytical formula.

Let us consider Example 16. The expression of the P&L is:

$$\Pi(w) = 500R_1 + 200R_2 + 300R_3$$

Because we know that the combination of the componoents of a skew normal random vector is a skew normal random variable, we were able to compute the analytical quantile of $\Pi(w)$ at the 1% confidence level. Suppose now that we don't know the analytical distribution of $\Pi(w)$. We can repeat the exercise by using the Monte Carlo method. At each simulation s, we generate the random variates $(R_{1,s}, R_{2,s}, R_{3,s})$ such that:

$$(R_{1,s}, R_{2,s}, R_{3,s}) \sim SN(\xi, \Omega, \eta)$$

and the corresponding P&L $\Pi_s(w) = 500R_{1,s} + 200R_{2,s} + 300R_{3,s}$. The Monte Carlo value-at-risk is the $n_s(1-\alpha)^{\text{th}}$ order statistic:

$$\widehat{\mathrm{VaR}}_\alpha(n_S) = -\Pi_{(n_s(1-\alpha):n_s)}(w)$$

Using the law of large numbers, we can show that the MC estimator converges to the exact VaR:

$$\lim_{n_S \to \infty} \widehat{\mathrm{VaR}}_\alpha(n_S) = \mathrm{VaR}_\alpha$$

In Figure 2.14, we report four Monte Carlo runs with 10 000 simulated scenarios. We notice that the convergence of the Monte Carlo VaR to the analytical VaR is slow[56], because asset returns present high skewness. The convergence will be faster if the probability distribution of risk factors is close to be normal and has no fat tails.

[56]We have previously found that the exact VaR is equal to \$123.91.

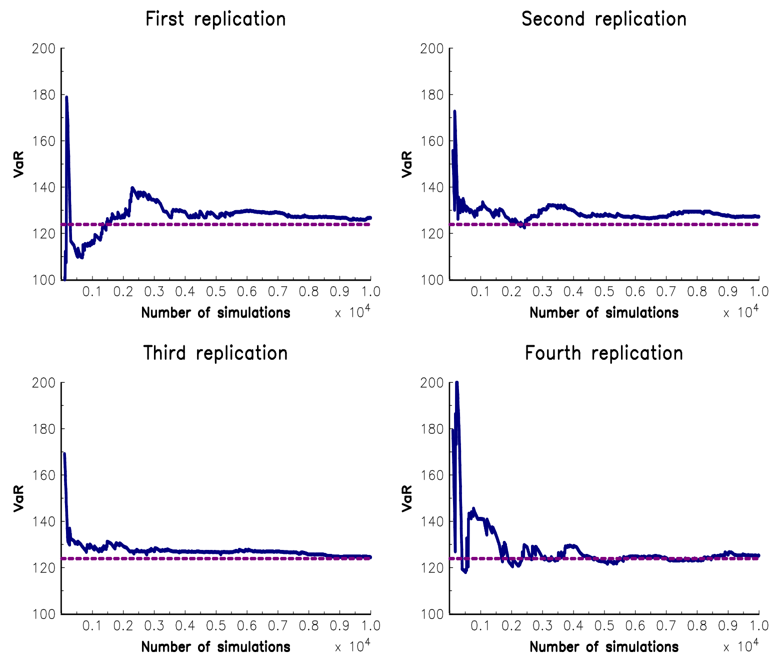

FIGURE 2.14: Convergence of the Monte Carlo VaR when asset returns are skew normal

Remark 15 *The Monte Carlo value-at-risk has been extensively studied with heavy-tailed risk factors (Dupire, 1998; Eberlein* et al. *(1998); Glasserman* et al.*, 2002). In those cases, one needs to use advanced and specific methods to reduce the variance of the estimator*[57].

Example 17 *We use a variant of Example 15 in page 79. We consider that the bond is exposed to credit risk. In particular, we assume that the current default intensity of the bond issuer is equal to* 200 *bps whereas the recovery rate is equal to* 50%.

In the case of a defaultable bond, the coupons and the notional are paid until the issuer does not default whereas a recovery rate is applied if the issuer defaults before the maturity of the bond. If we assume that the recovery is paid at maturity, we can show that the bond price under default risk is:

$$P_t = \sum_{t_m \geq t} C(t_m) B_t(t_m) \mathbf{S}_t(t_m) + N B_t(T) \left(\mathbf{S}_t(T) + \boldsymbol{\mathcal{R}}_t (1 - \mathbf{S}_t(T))\right)$$

where $\mathbf{S}_t(t_m)$ is the survival function at time t_m and $\boldsymbol{\mathcal{R}}_t$ is the current recovery rate. We retrieve the formula of the bond price without default risk if $\mathbf{S}_t(t_m) =$

[57]These techniques will be presented in Chapter 16.

1. Using the numerical values of the parameters, the bond price is equal to \$109.75 and is lower than the non-defaultable bond price[58]. If we assume that the default time is exponential with $\mathbf{S}_t(t_m) = e^{-\lambda_t(t_m - t)}$, we have:

$$\begin{aligned} P_{t+h} &= \sum_{t_m \geq t} C(t_m) e^{(t_m - t - h)R_{t+h}(t_m)} e^{-\lambda_{t+h}(t_m - t - h)} + \\ & \quad N e^{(T-t-h)R_{t+h}(T)} \left(\boldsymbol{\mathcal{R}}_{t+h} + (1 - \boldsymbol{\mathcal{R}}_{t+h}) e^{-\lambda_{t+h}(T-t-h)} \right) \end{aligned}$$

We define the risk factors as the zero-coupon rates, the default intensity and the recovery rate:

$$\begin{aligned} R_{t+h}(t_m) &\simeq R_t(t_m) + \Delta_h R_{t+h}(t_m) \\ \lambda_{t+h} &= \lambda_t + \Delta_h \lambda_{t+h} \\ \boldsymbol{\mathcal{R}}_{t+h} &= \boldsymbol{\mathcal{R}}_t + \Delta_h \boldsymbol{\mathcal{R}}_{t+h} \end{aligned}$$

We assume that the three risk factors are independent and follow the following probability distributions:

$$\begin{aligned} &(\Delta_h R_{t+h}(t_1), \ldots, \Delta_h R_{t+h}(t_n)) \sim \mathcal{N}(0, \Sigma) \\ &\Delta_h \lambda_{t+h} \sim \mathcal{N}\left(0, \sigma_\lambda^2\right) \\ &\Delta_h \boldsymbol{\mathcal{R}}_{t+h} \sim \mathcal{U}(a, b) \end{aligned}$$

We can then simulate the daily P&L $\Pi(w) = w(P_{t+h} - P_t)$ using the above specifications. For the numerical application, we use the covariance matrix given in Footnote 42 whereas the values of σ_λ, a and b are equal to 20 bps, -10% and 10%. In Figure 2.15, we have estimated the density of the daily P&L using 100 000 simulations. IR corresponds to the case when risk factors are only the interest rates[59]. The case IR/S considers that both $R_t(t_m)$ and λ_t are risk factors whereas $\boldsymbol{\mathcal{R}}_t$ is assumed to be constant. Finally, we include the recovery risk in the case IR/S/RR. Using 10 millions of simulations, we find that the daily value-at-risk is equal to \$4 730 (IR), \$13 460 (IR/S) and \$18 360 (IR/S/RR). We see the impact of taking into account default risk in the calculation of the value-at-risk.

2.2.4 The case of options and derivatives

Special attention should be paid to portfolios of derivatives, because their risk management is much more complicated than a long-only portfolio of traditional assets (Duffie and Pan, 1997). They involve non-linear exposures to risk factors that are difficult to measure, they are sensitive to parameters that are not always observable and they are generally traded on OTC markets. In this section, we provide an overview of the challenges that arise when measuring and managing the risk of these assets. Chapter 12 complements it with

[58] We remind that it was equal to \$115.47.

[59] This implies that we set $\Delta_h \lambda_{t+h}$ and $\Delta_h \boldsymbol{\mathcal{R}}_{t+h}$ to zero in the Monte Carlo procedure.

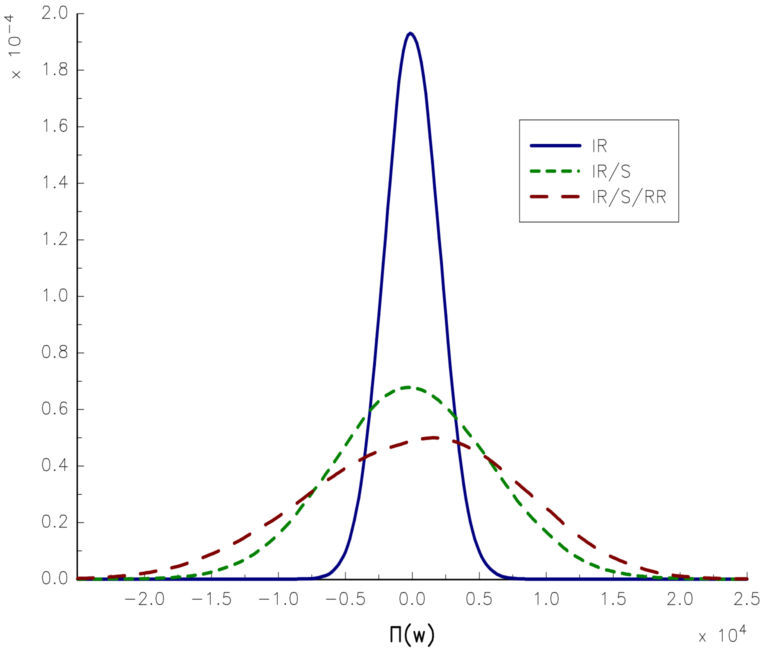

FIGURE 2.15: Probability density function of the daily P&L with credit risk

a more exhaustive treatment of hedging and pricing issues as well as model risk.

2.2.4.1 Identification of risk factors

Let us consider an example of a portfolio containing w_S stocks and w_C call options on this stock. We note S_t and $\boldsymbol{\mathcal{C}}_t$ the stock and option prices at time t. The P&L for the holding period h is equal to:

$$\Pi(w) = w_S (S_{t+h} - S_t) + w_C (\boldsymbol{\mathcal{C}}_{t+h} - \boldsymbol{\mathcal{C}}_t)$$

If we use asset returns as risk factors, we get:

$$\Pi(w) = w_S S_t R_{S,t+h} + w_C \boldsymbol{\mathcal{C}}_t R_{C,t+h}$$

where $R_{S,t+h}$ and $R_{C,t+h}$ are the returns of the stock and the option for the period $[t, t+h]$. In this approach, we identify two risk factors. The problem is that the option price $\boldsymbol{\mathcal{C}}_t$ is a non-linear function of the underlying price S_t:

$$\boldsymbol{\mathcal{C}}_t = f_C(S_t)$$

This implies that:

$$\begin{aligned}
\Pi(w) &= w_S S_t R_{S,t+h} + w_C (f_C(S_{t+h}) - \boldsymbol{\mathcal{C}}_t) \\
&= w_S S_t R_{S,t+h} + w_C (f_C(S_t(1 + R_{S,t+h})) - \boldsymbol{\mathcal{C}}_t)
\end{aligned}$$

The P&L depends then on a single risk factor R_S. We notice that we can write the return of the option price as a non-linear function of the stock return:

$$R_{C,t+h} = \frac{f_C\left(S_t\left(1 + R_{S,t+h}\right)\right) - \boldsymbol{\mathcal{C}}_t}{\boldsymbol{\mathcal{C}}_t}$$

The problem is that the probability distribution of R_C is non-stationary and depends on the value of S_t. Therefore, the risk factors can not be the random vector (R_S, R_C) because they require too complex modeling.

Risk factors are often explicit in primary financial assets (equities, bonds, currencies), which is not the case with derivatives. In the previous example, we have identified the return of the underlying asset as a risk factor for the call option. In the Black-Scholes, the price of the call option is given by:

$$C_{\mathrm{BS}}\left(S_t, K, \Sigma_t, T, b_t, r_t\right) = S_t e^{(b_t - r_t)\tau}\Phi\left(d_1\right) - K e^{-r_t\tau}\Phi\left(d_2\right)$$

where S_t is the current price of the underlying asset, K is the option strike, Σ_t is the volatility parameter, T is the maturity date, b_t is the cost-of-carry[60] and r_t is the interest rate. The parameter $\tau = T - t$ is the time to maturity whereas the coefficients d_1 and d_2 are defined as follows:

$$\begin{aligned} d_1 &= \frac{1}{\Sigma_t\sqrt{\tau}}\left(\ln\frac{S_t}{K} + b_t\tau\right) + \frac{1}{2}\Sigma_t\sqrt{\tau} \\ d_2 &= d_1 - \Sigma_t\sqrt{\tau} \end{aligned}$$

We can then write the option price as follows:

$$\boldsymbol{\mathcal{C}}_t = f_{\mathrm{BS}}\left(\theta_{\mathrm{contract}}; \theta\right)$$

where $\theta_{\mathrm{contract}}$ are the parameters of the contract (strike K and maturity T) and θ are the other parameters than can be objective as the underlying price S_t or subjective as the volatility Σ_t. Any one of these parameters θ may serve as risk factors:

- S_t is obviously a risk factor;
- if Σ_t is not constant, the option price may be sensitive to the volatility risk;
- the option may be impacted by changes in the interest rate or the cost-of-carry.

[60]The cost-of-carry depends on the underlying asset. We have $b_t = r_t$ for non-dividend stocks and total return indices, $b_t = r_t - d_t$ for stocks paying a continuous dividend yield d_t, $b_t = 0$ for forward and futures contracts and $b_t = r_t - r_t^\star$ for foreign exchange options where $r_t^\star$ is the foreign interest rate.

The risk manager faces here a big issue, because the risk measure will depend on the choice of the risk factors[61]. A typical example is the volatility parameter. We observe a difference between the historical volatility $\hat{\sigma}_t$ and the Black-Scholes volatility Σ_t. Because this implied volatility is not a market price, its value will depend on the option model and the assumptions, which are required to calibrate it. For instance, it will be different if we use a stochastic volatility model or a local volatility model. Even if two banks use the same model, they will certainly obtain two different values of the implied volatility, because there is little possibility that they exactly follows the same calibration procedure.

With the underlying asset S_t, the implied volatility Σ_t is the most important risk factor, but other risk factors may be determinant. They concern the dividend risk for equity options, the yield curve risk for interest rate options, the term structure for commodity options or the correlation risk for basket options. In fact, the choice of risk factors is not always obvious because it is driven by the pricing model and the characteristics of the option. We will take a closer look at this point in Chapter 12.

2.2.4.2 Methods to calculate the value-at-risk

The method of full pricing To calculate the value-at-risk of option portfolios, we use the same approaches as previously. The difference with primary financial assets comes from the pricing function which is non-linear and more complex. In the case of historical and Monte Carlo methods, the P&L of the s^{th} scenario has the following expression:

$$\Pi_s(w) = g(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s}; w) - P_t(w)$$

In the case of the introducing example, the P&L becomes then:

$$\Pi_s(w) = \begin{cases} w_S S_t R_s + w_C \left(f_C \left(S_t \left(1 + R_s\right); \Sigma_t \right) - \boldsymbol{\mathcal{C}}_t \right) & \text{with one risk factor} \\ w_S S_t R_s + w_C \left(f_C \left(S_t \left(1 + R_s\right), \Sigma_s \right) - \boldsymbol{\mathcal{C}}_t \right) & \text{with two risk factors} \end{cases}$$

where R_s and Σ_s are the asset return and the implied volatility generated by the s^{th} scenario. If we assume that the interest rate and the cost-of-carry are constant, the pricing function is:

$$f_C(S; \Sigma) = C_{\text{BS}}(S, K, \Sigma, T - h, b_t, r_t)$$

and we notice that the remaining maturity of the option decreases by h days. In the model with two risk factors, we have to simulate the underlying price and the implied volatility. For the single factor model, we use the current implied volatility Σ_t instead of the simulated value Σ_s.

Example 18 *We consider a long position on* 100 *call options with strike* $K =$

[61] We encounter the same difficulties for pricing and hedging purposes.

100. *The value of the call option is* \$4.14, *the residual maturity*[62] *is* 52 *days and the current price of the underlying asset is* \$100. *We assume that* $\Sigma_t = 20\%$ *and* $b_t = r_t = 5\%$. *The objective is to calculate the daily value-at-risk with a* 99% *confidence level. For that, we consider* 250 *historical scenarios, whose the first nine values are the following:*

s	1	2	3	4	5	6	7	8	9
R_s	−1.93	−0.69	−0.71	−0.73	1.22	1.01	1.04	1.08	−1.61
$\Delta\Sigma_s$	−4.42	−1.32	−3.04	2.88	−0.13	−0.08	1.29	2.93	0.85

TABLE 2.8: Daily P&L of the long position on the call option when the risk factor is the underlying price

s	R_s (in %)	S_{t+h}	$\mathcal{C}_{t+h}$	Π_s
1	−1.93	98.07	3.09	−104.69
2	−0.69	99.31	3.72	−42.16
3	−0.71	99.29	3.71	−43.22
4	−0.73	99.27	3.70	−44.28
5	1.22	101.22	4.81	67.46
6	1.01	101.01	4.68	54.64
7	1.04	101.04	4.70	56.46
8	1.08	101.08	4.73	58.89
9	−1.61	98.39	3.25	−89.22

Using the price and the characteristics of the call option, we can show that the implied volatility Σ_t is equal to 19.99% (rounded to 20%). We first consider the case of the single risk factor. In Table 2.8, we show the values of the P&L for the first nine scenarios. As an illustration, we provide the detailed calculation for the first scenario. The asset return R_s is equal to −1.93%, thus implying that the asset price S_{t+h} is equal to $100 \times (1 - 1.93\%) = 98.07$. The residual maturity τ is equal to 51/252 years. It follows that:

$$\begin{aligned} d_1 &= \frac{1}{20\% \times \sqrt{51/252}} \left(\ln \frac{98.07}{100} + 5\% \times \frac{51}{252} \right) + \frac{1}{2} \times 20\% \times \sqrt{\frac{51}{252}} \\ &= -0.0592 \end{aligned}$$

and:

$$d_2 = -0.0592 - 20\% \times \sqrt{\frac{51}{252}} = -0.1491$$

We deduce that:

$$\begin{aligned} \mathcal{C}_{t+h} &= 98.07 \times e^{(5\%-5\%)\frac{51}{252}} \times \Phi(-0.0592) - 100 \times e^{5\% \times \frac{51}{252}} \times \Phi(-0.1491) \\ &= 98.07 \times 1.00 \times 0.4764 - 100 \times 1.01 \times 0.4407 \\ &= 3.093 \end{aligned}$$

[62] We assume that there is 252 trading days per year.

The simulated P&L for the first historical scenario is then equal to:

$$\Pi_s = 100 \times (3.093 - 4.14) = -104.69$$

Based on the 250 historical scenarios, the value-at-risk is equal to \$154.79.

Remark 16 *In Figure 2.16, we illustrate that the option return R_C is not a new risk factor. We plot R_S against R_C for the 250 historical scenarios. The points are on the curve of the Black-Scholes formula. The correlation between the two returns is equal to 99.78%, which indicates that R_S and R_C are highly dependent. However, this dependence is non-linear for large positive or negative asset returns. The figure shows also the leverage effect of the call option, because R_C is not of the same order of magnitude as R_S. This illustrates the non-linear characteristic of options. A linear position with a volatility equal to 20% implies a daily VaR around 3%. In our example, the VaR is equal to 37.4% of the portfolio value, which corresponds to a linear exposure in a stock with a volatility of 259%!*

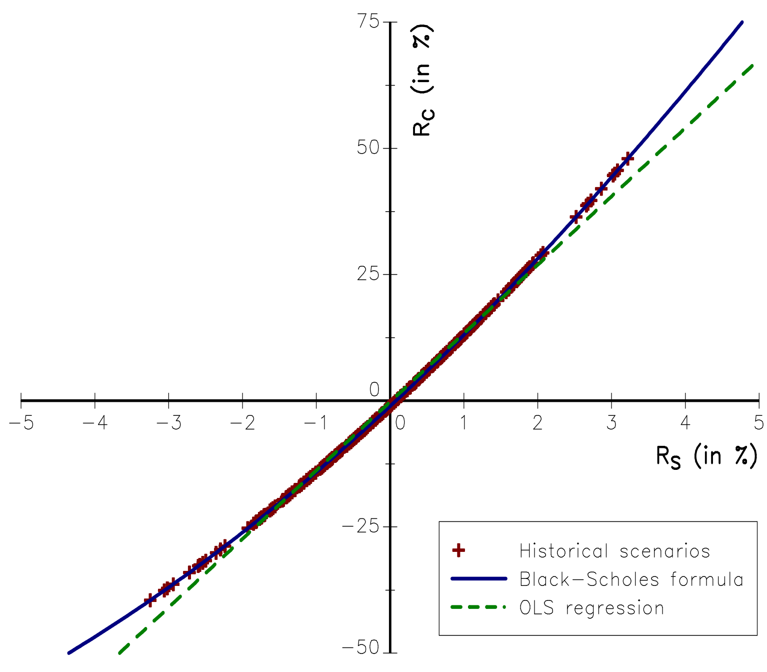

FIGURE 2.16: Relationship between the asset return R_S and the option return R_C

Let us consider the case with two risk factors when the implied volatility changes from t to $t + h$. We assume that the absolute variation of the implied volatility is the right risk factor to model the future implied volatility. It follows that:

$$\Sigma_{t+h} = \Sigma_t + \Delta\Sigma_s$$

In Table 2.9, we indicate the value taken by Σ_{t+h} for the first nine scenarios. This allows us to price the call option and deduce the P&L. For instance, the call option becomes[63] \$2.32 instead of \$3.09 for $s = 1$ because the implied volatility has decreased. Finally, the value-at-risk is equal to \$181.70 and is larger than the previous one due to the second risk factor.

TABLE 2.9: Daily P&L of the long position on the call option when the risk factors are the underlying price and the implied volatility

s	R_s (in %)	S_{t+h}	$\Delta\Sigma_s$ (in %)	Σ_{t+h}	$\boldsymbol{\mathcal{C}}_{t+h}$	Π_s
1	−1.93	98.07	−4.42	15.58	2.32	−182.25
2	−0.69	99.31	−1.32	18.68	3.48	−65.61
3	−0.71	99.29	−3.04	16.96	3.17	−97.23
4	−0.73	99.27	2.88	22.88	4.21	6.87
5	1.22	101.22	−0.13	19.87	4.79	65.20
6	1.01	101.01	−0.08	19.92	4.67	53.24
7	1.04	101.04	1.29	21.29	4.93	79.03
8	1.08	101.08	2.93	22.93	5.24	110.21
9	−1.61	98.39	0.85	20.85	3.40	−74.21

The method of sensitivities The previous approach is called *full pricing*, because it consists in re-pricing the option. In the method based on the Greek coefficients, the idea is to approximate the change in the option price by Taylor expansion. For instance, we define the delta approach as follows[64]:

$$\boldsymbol{\mathcal{C}}_{t+h} - \boldsymbol{\mathcal{C}}_t \simeq \boldsymbol{\Delta}_t \left(S_{t+h} - S_t\right)$$

where $\boldsymbol{\Delta}_t$ is the option delta:

$$\boldsymbol{\Delta}_t = \frac{\partial\, C_t\left(S_t, \Sigma_t, T\right)}{\partial\, S_t}$$

This approximation consists in replacing the non-linear exposure by a linear exposure with respect to the underlying price. As noted by Duffie and Pan (1997), this approach is not satisfactory because it is not accurate for large changes in the underlying price that are the most useful scenarios for calculating the VaR. The delta approach may be implemented for the three VaR methods. For instance, the Gaussian VaR of the call option is:

$$\mathrm{VaR}_\alpha\left(w; h\right) = \Phi^{-1}\left(\alpha\right) \times \left|\boldsymbol{\Delta}_t\right| \times S_t \times \sigma\left(R_{S,t+h}\right)$$

[63] We have $d_1 = -0.0986$, $d_2 = -0.1687$, $\Phi\left(d_1\right) = 0.4607$, $\Phi\left(d_2\right) = 0.4330$ and $\boldsymbol{\mathcal{C}}_{t+h} = 2.318$.

[64] We write the call price as the function $C_t\left(S_t, \Sigma_t, T\right)$.

If we consider the introductory example, we have:

$$\begin{aligned}\Pi(w) &= w_S(S_{t+h}-S_t)+w_C(\mathcal{C}_{t+h}-\mathcal{C}_t)\\ &\simeq (w_S+w_C\boldsymbol{\Delta}_t)(S_{t+h}-S_t)\\ &= (w_S+w_C\boldsymbol{\Delta}_t)S_tR_{S,t+h}\end{aligned}$$

With the delta approach, we aggregate the risk by netting the different delta exposures[65]. In particular, the portfolio is delta neutral if the net exposure is zero:

$$w_S+w_C\boldsymbol{\Delta}_t=0\Leftrightarrow w_S=-w_C\boldsymbol{\Delta}_t$$

With the delta approach, the VaR of delta neutral portfolios is then equal to zero.

To overcome this drawback, we can use the second-order approximation or the delta-gamma approach:

$$\mathcal{C}_{t+h}-\mathcal{C}_t\simeq\boldsymbol{\Delta}_t(S_{t+h}-S_t)+\frac{1}{2}\boldsymbol{\Gamma}_t(S_{t+h}-S_t)^2$$

where $\boldsymbol{\Gamma}_t$ is the option gamma:

$$\boldsymbol{\Gamma}_t=\frac{\partial^2 C_t(S_t,\Sigma_t,T)}{\partial S_t^2}$$

In Figure 2.17, we compare the two Taylor expansions with the re-pricing method when h is equal to one trading day. We observe that the delta approach provides a bad approximation if the future price S_{t+h} is far from the current price S_t. The inclusion of the gamma helps to correct the pricing error. However, if the time period h is high, the two approximations may be inaccurate even in the neighborhood de S_t (see the case $h=30$ days in Figure 2.17). It is therefore important to take into account the time or maturity effect:

$$\mathcal{C}_{t+h}-\mathcal{C}_t\simeq\boldsymbol{\Delta}_t(S_{t+h}-S_t)+\frac{1}{2}\boldsymbol{\Gamma}_t(S_{t+h}-S_t)^2+\boldsymbol{\Theta}_t h$$

where $\boldsymbol{\Theta}_t=\partial_t C_t(S_t,\Sigma_t,T)$ is the option theta[66].

The Taylor expansion can be generalized to a set of risk factors $\mathcal{F}_t=(\mathcal{F}_{1,t},\ldots,\mathcal{F}_{m,t})$:

$$\begin{aligned}\mathcal{C}_{t+h}-\mathcal{C}_t &\simeq \sum_{j=1}^m\frac{\partial\mathcal{C}_t}{\partial\mathcal{F}_{j,t}}(\mathcal{F}_{j,t+h}-\mathcal{F}_{j,t})+\\ &\quad \frac{1}{2}\sum_{j=1}^m\sum_{k=1}^m\frac{\partial^2\mathcal{C}_t}{\partial\mathcal{F}_{j,t}\,\partial\mathcal{F}_{k,t}}(\mathcal{F}_{j,t+h}-\mathcal{F}_{j,t})(\mathcal{F}_{k,t+h}-\mathcal{F}_{k,t})\end{aligned}$$

[65] A long (or short) position in the underlying asset is equivalent to $\boldsymbol{\Delta}_t=1$ (or $\boldsymbol{\Delta}_t=-1$).

[66] An equivalent formula is $\boldsymbol{\Theta}_t=-\partial_T C_t(S_t,\Sigma_t,T)=-\partial_\tau C_t(S_t,\Sigma_t,T)$ because the maturity T (or the time to maturity τ) is moving in the opposite way with respect to the time t.

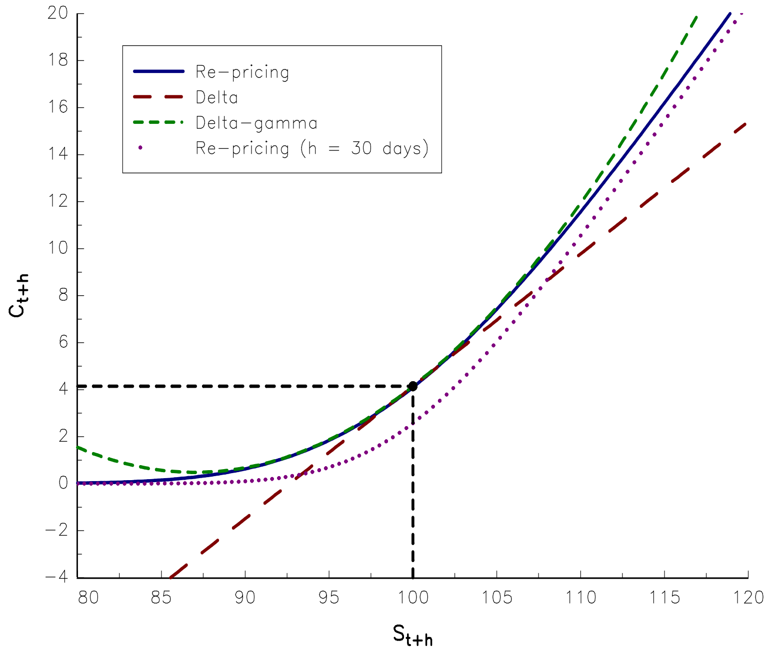

FIGURE 2.17: Approximation of the option price with the Greek coefficients

The delta-gamma-theta approach consists in considering the underlying price and the maturity as risk factors. If we add the implied volatility as a new risk factor, we obtain:

$$\begin{aligned}\boldsymbol{\mathcal{C}}_{t+h} - \boldsymbol{\mathcal{C}}_t &\simeq \boldsymbol{\Delta}_t \left(S_{t+h} - S_t\right) + \frac{1}{2}\boldsymbol{\Gamma}_t \left(S_{t+h} - S_t\right)^2 + \boldsymbol{\Theta}_t h + \\ &\quad \boldsymbol{\upsilon}_t \left(\Sigma_{t+h} - \Sigma_t\right)\end{aligned}$$

where $\boldsymbol{\upsilon}_t = \partial_{\Sigma_t} C_t \left(S_t, \Sigma_t, T\right)$ is the option vega. Here, we have considered that only the second derivative of $\boldsymbol{\mathcal{C}}_t$ with respect to S_t is significant, but we could also include the vanna or volga effect[67].

In the case of the call option, the Black-Scholes sensitivities are equal to:

$$\begin{aligned}\boldsymbol{\Delta}_t &= e^{(b_t - r_t)\tau}\Phi\left(d_1\right) \\ \boldsymbol{\Gamma}_t &= \frac{e^{(b_t - r_t)\tau}\phi\left(d_1\right)}{S_t \Sigma_t \sqrt{\tau}}\end{aligned}$$

[67]The vanna coefficient corresponds to the cross-derivative of $\boldsymbol{\mathcal{C}}_t$ with respect to S_t and Σ_t whereas the volga effect is the second derivative of $\boldsymbol{\mathcal{C}}_t$ with respect to Σ_t.

$$\begin{aligned}
\boldsymbol{\Theta}_t &= -r_t K e^{-r_t \tau} \Phi(d_2) - \frac{1}{2\sqrt{\tau}} S_t \Sigma_t e^{(b_t - r_t)\tau} \phi(d_1) - \\
&\quad (b_t - r_t) S_t e^{(b_t - r_t)\tau} \Phi(d_1) \\
\boldsymbol{\upsilon}_t &= e^{(b_t - r_t)\tau} S_t \sqrt{\tau} \phi(d_1)
\end{aligned}$$

If we consider again Example 17, we obtain[68] $\boldsymbol{\Delta}_t = 0.5632$, $\boldsymbol{\Gamma}_t = 0.0434$, $\boldsymbol{\Theta}_t = -11.2808$ and $\boldsymbol{\upsilon}_t = 17.8946$. In Table 2.10, we have reported the approximated P&Ls for the first nine scenarios and the one-factor model. The fourth column indicates the P&L obtained by the full pricing method, which were already reported in Table 2.8. $\Pi_s^{\boldsymbol{\Delta}}(w)$, $\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}}(w)$ and $\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}}(w)$ correspond respectively to delta, delta-gamma, delta-gamma-theta approaches. For example, we have $\Pi_1^{\boldsymbol{\Delta}}(w) = 100 \times 0.5632 \times (98.07 - 100) = -108.69$, $\Pi_1^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}}(w) = -108.69 + 100 \times \frac{1}{2} \times 0.0434 \times (98.07 - 100)^2 = -100.61$ and $\Pi_1^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}}(w) = -100.61 - 11.2808 \times 1/252 = -105.09$. We notice that we obtain a good approximation with the delta, but it is more accurate to combine delta, gamma and theta sensibilities. Finally, the 99% VaRs for a one-day holding period are \$171.20 and \$151.16 and \$155.64. This is the delta-gamma-theta approach which gives the closest result[69]. If the set of risk factors includes the implied volatility, we obtain the figures in Table 2.11. We notice that the vega effect is very significant (fifth column). As an illustration, we have $\Pi_1^{\boldsymbol{\upsilon}}(w) = 100 \times 17.8946 \times (15.58\% - 20\%) = -79.09$, implying that the volatility risk explains 43.4% of the loss of \$182.25 for the first scenario. Finally, the VaR is equal to \$183.76 with the delta-gamma-theta-vega approach whereas we found previously that it was equal to \$181.70 with the full pricing method.

TABLE 2.10: Calculation of the P&L based on the Greek sensitivities

s	R_s (in %)	S_{t+h}	Π_s	$\Pi_s^{\boldsymbol{\Delta}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}}$
1	−1.93	98.07	−104.69	−108.69	−100.61	−105.09
2	−0.69	99.31	−42.16	−38.86	−37.83	42.30
3	−0.71	99.29	−43.22	−39.98	−38.89	−43.37
4	−0.73	99.27	−44.28	−41.11	−39.96	−44.43
5	1.22	101.22	67.46	68.71	71.93	67.46
6	1.01	101.01	54.64	56.88	59.09	54.61
7	1.04	101.04	56.46	58.57	60.91	56.44
8	1.08	101.08	58.89	60.82	63.35	58.87
9	−1.61	98.39	−89.22	−90.67	−85.05	−89.53

Remark 17 *We do not present here the non-linear quadratic VaR, which*

[68] We have $d_1 = 0.1590$, $\Phi(d_1) = 0.5632$, $\phi(d_1) = 0.3939$, $d_2 = 0.0681$ and $\Phi(d_2) = 0.5272$.

[69] We found previously that the VaR was equal to \$154.79 with the full pricing method.

TABLE 2.11: Calculation of the P&L using the vega coefficient

s	S_{t+h}	Σ_{t+h}	Π_s	$\Pi_s^{\boldsymbol{\upsilon}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\upsilon}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\upsilon}}$	$\Pi_s^{\boldsymbol{\Delta}+\boldsymbol{\Gamma}+\boldsymbol{\Theta}+\boldsymbol{\upsilon}}$
1	98.07	15.58	−182.25	−79.09	−187.78	−179.71	−184.19
2	99.31	18.68	−65.61	−23.62	−62.48	−61.45	−65.92
3	99.29	16.96	−97.23	−54.40	−94.38	−93.29	−97.77
4	99.27	22.88	6.87	51.54	10.43	11.58	7.10
5	101.22	19.87	65.20	−2.33	66.38	69.61	65.13
6	101.01	19.92	53.24	−1.43	55.45	57.66	53.18
7	101.04	21.29	79.03	23.08	81.65	84.00	79.52
8	101.08	22.93	110.21	52.43	113.25	115.78	111.30
9	98.39	20.85	−74.21	15.21	−75.46	−69.84	−74.32

consists in computing the VaR of option portfolios with the Cornish-Fisher expansion (Zangari, 1996; Britten-Jones and Schaefer, 1999). It is called 'quadratic' because it uses the delta-gamma approximation and requires to calculate the moments of the quadratic form $(S_{t+h} - S_t)^2$. The treatment of this approach is left as an exercise (Section 2.4.9 in page 135).

The hybrid method On the one hand, the full pricing method has the advantage to be accurate, but also the drawback to be time-consuming because it performs a complete revaluation of the portfolio for each scenario. On the other hand, the method based on the sensitivities is less accurate, but also faster than the re-pricing approach. Indeed, the Greek coefficients are calculated once and for all, and their values do not depend on the scenario. The hybrid method consists of combining the two approaches:

1. we first calculate the P&L for each (historical or simulated) scenario with the method based on the sensitivities;

2. we then identify the worst scenarios;

3. we finally revalue these worst scenarios by using the full pricing method.

The underlying idea is to consider the faster approach to locate the value-at-risk, and then to use the most accurate approach to calculate the right value.

In Table 2.12, we consider the previous example with the implied volatility as a risk factor. We have reported the worst scenarios corresponding to the order statistic $i : n_S$ with $i \leq 10$. In the case of the full pricing method, the five worst scenarios are the 100^{th}, 1^{st}, 134^{th}, 27^{th} and 169^{th}. This implies that the hybrid method will give the right result if it is able to select the 100^{th}, 1^{st} and 134^{th} scenarios to compute the value-at-risk which corresponds to the average of the second and third order statistics. If we consider the $\boldsymbol{\Delta} - \boldsymbol{\Gamma} - \boldsymbol{\Theta} - \boldsymbol{\upsilon}$ approximation, we identify the same ten worst scenarios. It is perfectly normal,

TABLE 2.12: The 10 worst scenarios identified by the hybrid method

i	Full pricing		Greeks					
			$\mathbf{\Delta}-\mathbf{\Gamma}-\mathbf{\Theta}-\boldsymbol{\upsilon}$		$\mathbf{\Delta}-\mathbf{\Theta}$		$\mathbf{\Delta}-\mathbf{\Theta}-\boldsymbol{\upsilon}$	
	s	Π_s	s	Π_s	s	Π_s	s	Π_s
1	100	−183.86	100	−186.15	182	−187.50	134	−202.08
2	1	−182.25	1	−184.19	169	−176.80	100	−198.22
3	134	−181.15	134	−183.34	27	−174.55	1	−192.26
4	27	−163.01	27	−164.26	134	−170.05	169	−184.32
5	169	−162.82	169	−164.02	69	−157.66	27	−184.04
6	194	−159.46	194	−160.93	108	−150.90	194	−175.36
7	49	−150.25	49	−151.43	194	−149.77	49	−165.41
8	245	−145.43	245	−146.57	49	−147.52	182	−164.96
9	182	−142.21	182	−142.06	186	−145.27	245	−153.37
10	79	−135.55	79	−136.52	100	−137.38	69	−150.68

as it is easy to price an European call option. It will not be the case with exotic options, because the approximation may not be accurate. For instance, if we consider our example with the $\mathbf{\Delta}-\mathbf{\Theta}$ approximation, the five worst scenarios becomes the 182^{th}, 169^{st}, 27^{th}, 134^{th} and 69^{th}. If we revaluate these 5 worst scenarios, the 99% value-at-risk is equal to:

$$\mathrm{VaR}_{99\%}\left(w;\text{one day}\right)=\frac{1}{2}\left(163.01+162.82\right)=\$162.92$$

which is a result far from the value of \$180.70 found with the full pricing method. With the 10 worst scenarios, we obtain:

$$\mathrm{VaR}_{99\%}\left(w;\text{one day}\right)=\frac{1}{2}\left(181.15+163.01\right)=\$172.08$$

Once again, we do not find the exact value, because the $\mathbf{\Delta}-\mathbf{\Theta}$ approximation fails to identify the 1^{st} among the 10 worst scenarios. This problem vanishes with the $\mathbf{\Delta}-\mathbf{\Theta}-\boldsymbol{\upsilon}$ approximation, even if it gives a ranking different than this obtained with the full pricing method. In practice, the hybrid approach is widespread and professionals generally use the identification method with 10 worst scenarios[70].

2.2.4.3 Backtesting

When we consider a model to price a product, the valuation is known as '*mark-to-model*' and requires more attention than the mark-to-market approach. In this last case, the simulated P&L is the difference between the

[70] Its application is less frequent than in the past because computational times have dramatically decreased with the evolution of technology, in particular the development of parallel computing.

mark-to-model value at time $t+1$ and the current mark-to-market value:

$$\Pi_s(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-market}}$$

At time $t+1$, the realized P&L is the difference between two mark-to-market values:

$$\Pi(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-market}} - \underbrace{P_t(w)}_{\text{mark-to-market}}$$

For exotic options and OTC derivatives, we don't have market prices and the portfolio is valuated using the mark-to-model approach. This means that the simulated P&L is the difference between two mark-to-model values:

$$\Pi_s(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-model}}$$

and the realized P&L is also the difference between two mark-to-model values:

$$\Pi(w) = \underbrace{P_{t+1}(w)}_{\text{mark-to-model}} - \underbrace{P_t(w)}_{\text{mark-to-model}}$$

In the case of the mark-to-model valuation, we see the relevance of the pricing model in terms of risk management. Indeed, if the pricing model is wrong, the value-at-risk is wrong too and this cannot be detected by the backtesting procedure, which has little signification. This is why the supervisory authority places great importance on model risk.

2.2.4.4 Model risk

Model risk can not be summarized in a unique definition due to its complexity. For instance, Derman (1996, 2001) considers six types of model risk (inapplicability of modeling, incorrect model, incorrect solutions, badly approximated solution, bugs and unstable data). Rebonato (2001) defines model risk as "*the risk of a significant difference between the mark-to-model value of an instrument, and the price at which the same instrument is revealed to have traded in the market*". According to Morini (2001), these two approaches are different. For Riccardo Rebonato, there is not a true value of an instrument before it will be traded on the market. Model risk can therefore be measured by selling the instrument in the market. For Emanuel Derman, an instrument has an intrinsic true value, but it is unknown. The proposition of Rebonato is certainly the right way to define model risk, but it does not help to measure model risk from an ex-ante point of view. Moreover, this approach does not distinguish between model risk and liquidity risk. The conception of Derman is more adapted to manage model risk and calibrate the associated provisions. This is the approach that has been adopted by banks and regulators.

Nevertheless, the multifaceted nature of this approach induces very different implementations across banks, because it appears as a catalogue with an infinite number of rules.

We consider a classification with four main types of model risk:

1. The operational risk
This is the risk associated to the implementation of the pricer. It concerns programming mistakes or bugs, but also mathematical errors in closed-form formulas, approximations or numerical methods. A typical example is the use of a numerical scheme for solving a partial differential equation. The accuracy of the option price and the Greeks coefficients will depend on the specification of the numerical algorithm (explicit, implicit or mixed schemes) and the discretization parameters (time and space steps). Another example is the choice of the Monte Carlo method and the number of simulations.

2. The parameter risk
This is the risk associated to the input parameters, in particular those which are difficult to estimate. A wrong value of one parameter can lead to a mis-pricing, even though the model is right and well implemented. In this context, the question of available and reliable data is a key issue. It is particular true when the parameters are unobservable and are based on an expert's opinion. A typical example concerns the value of correlations in multi-asset options. Even if there is no problem with data, some parameters are indirectly related to market data via a calibration set. In this case, they may change with the specification of the calibration set. A typical example is the pricing of exotic interest rate options, which are based on parameters calibrated from prices of plain vanilla instruments (caplets and swaptions). The analysis of parameter risk consists then of measuring the impact of parameter changes on the price and the hedging portfolio of the exotic option.

3. The risk of mis-specification
This is the risk associated to the mathematical model, because it may not include all the risk factors, the dynamics of the risk factors is not adequate or the dependence between them is not well defined. It is generally easy to highlight this risk, because various models calibrated with the same set of instruments can produce different prices for the same exotic option. The big issue is to define what is the least bad model. For example, in the case of equity options, we have the choice between many models: Black-Scholes, local volatility, Heston model, other stochastic volatility models, jump diffusion, etc. In practice, the frontier between the risk of parameters and the risk of mis-specification may be unclear as shown by the seminal work of uncertainty on pricing and hedging by Avellaneda *et al.* (1995). Moreover, a model which appears to be good for pricing may not be well adapted for risk management.

4. The hedging risk
   This is the risk associated to the trading management of the option portfolio. The sales margin corresponds to the difference between the transaction price and the mark-to-model price. The sales margin is calculated at the inception date of the transaction. To freeze the margin, we have to hedge the option. The mark-to-model value is then transferred to the option trader and represents the hedging cost. We face here the risk that the realized hedging cost is larger than the mark-to-model price. A typical example is a put option, which has a negative delta. The hedging portfolio corresponds then to a short selling on the underlying asset. Sometimes, this short position may be difficult to implement (e.g. a ban on short selling) or may be very costly (e.g. due to a change in the bank funding condition). Some events may also generate a rebalancing risk. The most famous example is certainly the hedge funds crisis in October 2008, which has imposed redemption restrictions or gates. This caused difficulties to traders, who managed call options on hedge funds and were unable to reduce their deltas at this time. The hedging risk does not only concerns the feasibility of the hedging implementation, but also its adequacy with the model. As an illustration, we suppose that we use a stochastic volatility model for an option, which is sensitive to the vanna coefficient. The risk manager can then decide to use this model for measuring the value-at-risk, but the trader can also prefer to implement a Black-Scholes hedging portfolio[71]. This is not a problem that the risk manager uses a different model than the trader if the model risk only includes the first three categories. However, it will be a problem if it also concerns hedging risk.

Model risk justifies that model validation is an integral part of the risk management process for exotic options. The tasks of a model validation team are multiple and concerns reviewing the programming code, checking mathematical formulas and numerical approximations, validating market data, testing the calibration stability, challenging the pricer with alternative models, proposing provision buffers, etc. These teams generally operate at the earliest stages of the pricer development, whereas the risk manager is involved to follow the product on a daily basis.

Remark 18 *It is a mistake to think that model risk is an operational risk. Model risk is intrinsically a market risk. Indeed, it exists because exotic options are difficult to price and hedge, implying that commercial risk is high. This explains that sales margin are larger than for vanilla options and implicitly include model risk, which is therefore inherent to the business of exotic derivatives.*

[71] There may be many reasons for implementing more simple hedging portfolios: the trader may be more confident in the robustness, there is no market instrument to replicate the vanna position, etc.

2.3 Risk allocation

Measuring the risk of a portfolio is a first step to manage it. In particular, a risk measure is a single number that is not very helpful for understanding the sources of the portfolio risk. To go further, we must define precisely the notion of risk contribution in order to propose risk allocation principles. Until now, we have used the value-at-risk without defining more precisely what a risk measure is. This allowed to manipulate risk measures from a practical point of view. However, we need now to focus on the theory if we want to understand the principles of capital budgeting.

2.3.1 Properties of a risk measure

Let $\mathcal{R}(w)$ be the risk measure of portfolio w. In this section, we define the different properties that should satisfy risk measure $\mathcal{R}(w)$ in order to be acceptable in terms of capital allocation.

2.3.1.1 Coherency and convexity of risk measures

Following Artzner *et al.* (1999), $\mathcal{R}$ is said to be '*coherent*' if it satisfies the following properties:

1. Subadditivity
$$\mathcal{R}(w_1 + w_2) \leq \mathcal{R}(w_1) + \mathcal{R}(w_2)$$
The risk of two portfolios should be less than adding the risk of the two separate portfolios.

2. Homogeneity
$$\mathcal{R}(\lambda w) = \lambda \mathcal{R}(w) \quad \text{if } \lambda \geq 0$$
Leveraging or deleveraging of the portfolio increases or decreases the risk measure in the same magnitude.

3. Monotonicity
$$\text{if } w_1 \prec w_2, \text{ then } \mathcal{R}(w_1) \geq \mathcal{R}(w_2)$$
If portfolio w_2 has a better return than portfolio w_1 under all scenarios, risk measure $\mathcal{R}(w_1)$ should be higher than risk measure $\mathcal{R}(w_2)$.

4. Translation invariance
$$\text{if } m \in \mathbb{R}, \text{ then } \mathcal{R}(w + m) = \mathcal{R}(w) - m$$
Adding a cash position of amount m to the portfolio reduces the risk by m. This implies that we can hedge the risk of the portfolio by considering a capital that is equal to the risk measure:
$$\mathcal{R}(w + \mathcal{R}(w)) = \mathcal{R}(w) - \mathcal{R}(w) = 0$$

The definition of coherent risk measures led to a considerable interest in the quantitative management of risk. Thus, Föllmer and Schied (2002) propose to replace the homogeneity and subadditivity conditions by a weaker condition called the convexity property:

$$\mathcal{R}\left(\lambda w_1 + (1-\lambda) w_2\right) \leq \lambda \mathcal{R}\left(w_1\right) + (1-\lambda) \mathcal{R}\left(w_2\right)$$

This condition means that diversification should not increase the risk.

We have seen that the loss of the portfolio is $L(w) = -P_t(w) R_{t+h}(w)$ where $P_t(w)$ and $R_{t+h}(w)$ are the current value and the future return of the portfolio. Without loss of generality[72], we assume that $P_t(w)$ is equal to 1. We consider then different risk measures:

- Volatility of the loss

$$\mathcal{R}(w) = \sigma(L(w)) = \sigma(w)$$

  The volatility of the loss is the standard deviation of the portfolio's loss.

- Standard deviation-based risk measure

$$\mathcal{R}(w) = \mathrm{SD}_c(w) = \mathbb{E}\left[L(w)\right] + c \cdot \sigma(L(w)) = -\mu(w) + c \cdot \sigma(w)$$

  To obtain this measure, we scale the volatility by factor $c > 0$ and subtract the expected return of the portfolio.

- Value-at-risk

$$\mathcal{R}(w) = \mathrm{VaR}_\alpha(w) = \inf\left\{\ell : \Pr\left\{L(w) \leq \ell\right\} \geq \alpha\right\}$$

  The value-at-risk is the α-quantile of the loss distribution $\mathbf{F}$ and we note it $\mathbf{F}^{-1}(\alpha)$.

- Expected shortfall

$$\mathcal{R}(w) = \mathrm{ES}_\alpha(w) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(w)\,\mathrm{d}u$$

  The expected shortfall is the average of the VaRs at level α and higher (Acerbi and Tasche, 2002). We note that it is also equal to the expected loss given that the loss is beyond the value-at-risk:

$$\mathrm{ES}_\alpha(w) = \mathbb{E}\left[L(w) \mid L(w) \geq \mathrm{VaR}_\alpha(w)\right]$$

[72] The homogeneity property implies that:

$$\mathcal{R}\left(\frac{w}{P_t(w)}\right) = \frac{\mathcal{R}(w)}{P_t(w)}$$

We can therefore calculate the risk measure using the absolute loss (expressed in $) or the relative loss (expressed in %). The two approaches are perfectly equivalent.

We can show that the standard deviation-based risk measure and the expected shortfall satisfy the previous coherency and convexity conditions. For the value-at-risk, the subadditivity property does not hold in general. This is a problem because the portfolio's risk may have be meaningful in this case. More curiously, the volatility is not a coherent risk measure because it does not verify the translation invariance axiom.

Example 19 *We consider a defaultable* \$100 *zero-coupon bond, whose the default probability is equal to* 200 *bps. We assume that the recovery rate* R *is a binary random variable with* $\Pr\{R = 0.25\} = \Pr\{R = 0.75\} = 50\%$.

Let L be the loss of the zero-coupon bond. We have $\mathbf{F}(0) = \Pr\{L \leq 0\} = 98\%$, $\mathbf{F}(25) = \Pr\{L_i \leq 25\} = 99\%$ and $\mathbf{F}(75) = \Pr\{L_i \leq 75\} = 100\%$. We deduce that the 99% value-at-risk is equal to \$25. We have then:

$$\begin{aligned} \mathrm{ES}_{99\%}(L_i) &= \mathbb{E}[L_i \mid L_i \geq 25] \\ &= \frac{25+75}{2} \\ &= \$50 \end{aligned}$$

We assume now that the portfolio contains two zero-coupon bonds, whose default times are independent. The probability density function of (L_1, L_2) is given below:

	$L_1 = 0$	$L_1 = 25$	$L_1 = 75$	
$L_2 = 0$	96.04%	0.98%	0.98%	98.00%
$L_2 = 25$	0.98%	0.01%	0.01%	1.00%
$L_2 = 75$	0.98%	0.01%	0.01%	1.00%
	98.00%	1.00%	1.00%	

We deduce that the probability distribution function of $L = L_1 + L_2$ is:

ℓ	0	25	50	75	100	150
$\Pr\{L \leq \ell\}$	96.04%	1.96%	0.01%	1.96%	0.02%	0.01%
$\Pr\{L = \ell\}$	96.04%	98%	98.01%	99.97%	99.99%	100%

It follows that $\mathrm{VaR}_{99\%}(L) = 75$ and:

$$\begin{aligned} \mathrm{ES}_{99\%}(L) &= \frac{75 \times 1.96\% + 100 \times 0.02\% + 150 * 0.01\%}{1.96\% + 0.02\% + 0.01\%} \\ &= \$75.63 \end{aligned}$$

For this example, the value-at-risk does not satisfy the subadditivity property, which is not the case of the expected shortfall[73].

[73] We have $\mathrm{VaR}_{99\%}(L_1) + \mathrm{VaR}_{99\%}(L_2) = 50$, $\mathrm{VaR}_{99\%}(L_1 + L_2) > \mathrm{VaR}_{99\%}(L_1) + \mathrm{VaR}_{99\%}(L_2)$, $\mathrm{ES}_{99\%}(L_1) + \mathrm{ES}_{99\%}(L_2) = 100$ and $\mathrm{ES}_{99\%}(L_1 + L_2) < \mathrm{ES}_{99\%}(L_1) + \mathrm{ES}_{99\%}(L_2)$.

For this reason, the value-at-risk has been frequently criticized by academics. They also pointed out that it does not capture the tail risk of the portfolio. This led the Basel Committee to replace the 99% value-at-risk by the 97.5% expected shortfall for the internal model-based approach in Basel IV (BCBS, 2013b). In practice, it is not sure that it will improve significantly the predictive power of the risk measure. As an illustration, we assume that the portfolio loss is normally distributed: $L(w) \sim \mathcal{N}\left(\mu(L), \sigma^2(L)\right)$. The expression of the value-at-risk is:

$$\mathrm{VaR}_\alpha(w) = \mu(L) + \Phi^{-1}(\alpha)\,\sigma(L)$$

whereas the expected shortfall is equal to:

$$\mathrm{ES}_\alpha(w) = \frac{1}{1-\alpha}\int_{\mu(L)+\Phi^{-1}(\alpha)\sigma(L)}^{\infty} \frac{x}{\sigma(L)\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{x-\mu(L)}{\sigma(L)}\right)^2\right) \mathrm{d}x$$

With the variable change $t = \sigma(L)^{-1}(x - \mu(L))$, we obtain:

$$\begin{aligned}
\mathrm{ES}_\alpha(w) &= \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{\infty} (\mu(L) + \sigma(L)\,t)\,\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\
&= \frac{\mu(L)}{1-\alpha}\left[\Phi(t)\right]_{\Phi^{-1}(\alpha)}^{\infty} + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\int_{\Phi^{-1}(\alpha)}^{\infty} t\exp\left(-\frac{1}{2}t^2\right)\mathrm{d}t \\
&= \mu(L) + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\left[-\exp\left(-\frac{1}{2}t^2\right)\right]_{\Phi^{-1}(\alpha)}^{\infty} \\
&= \mu(L) + \frac{\sigma(L)}{(1-\alpha)\sqrt{2\pi}}\exp\left(-\frac{1}{2}\left[\Phi^{-1}(\alpha)\right]^2\right)
\end{aligned}$$

The expected shortfall of the portfolio w is then:

$$\mathrm{ES}_\alpha(w) = \mu(L) + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)}\sigma(L)$$

When the portfolio loss is Gaussian, the value-at-risk and the expected shortfall are both a standard deviation-based risk measure. They coincide when the the scaling parameters $c_{\mathrm{VaR}} = \Phi^{-1}(\alpha_{\mathrm{VaR}})$ and $c_{\mathrm{ES}} = \phi\left(\Phi^{-1}(\alpha_{\mathrm{ES}})\right)/(1-\alpha_{\mathrm{ES}})$ are equal. In Table 2.13, we report the values taken by c_{VaR} and c_{ES}. We notice that the 97.5% expected shortfall gives the same risk measure than a 99% value-at-risk in the Gaussian case.

TABLE 2.13: Scaling factors c_{VaR} and c_{ES}

α (in %)	95.0	96.0	97.0	97.5	98.0	98.5	99.0	99.5
c_{VaR}	1.64	1.75	1.88	1.96	2.05	2.17	**2.33**	2.58
c_{ES}	2.06	2.15	2.27	**2.34**	2.42	2.52	2.67	2.89

2.3.1.2 Euler allocation principle

According to Litterman (1996), risk allocation consists in decomposing the risk portfolio into a sum of risk contributions by sub-portfolios (assets, trading desks, etc.). The concept of risk contribution is key in identifying concentrations and understanding the risk profile of the portfolio, and there are different methods for defining them. As illustrated by Denault (2001), some methods are more pertinent than others and the Euler principle is certainly the most used and accepted one.

We decompose the P&L as follows:

$$\Pi = \sum_{i=1}^{n} \Pi_i$$

where Π_i is the P&L of the i^{th} sub-portfolio. We note $\mathcal{R}(\Pi)$ the risk measure associated with the P&L[74]. Let us consider the risk-adjusted performance measure (RAPM) defined by[75]:

$$\text{RAPM}(\Pi) = \frac{\mathbb{E}[\Pi]}{\mathcal{R}(\Pi)}$$

Tasche (2008) considers the portfolio-related RAPM of the i^{th} sub-portfolio defined by:

$$\text{RAPM}(\Pi_i \mid \Pi) = \frac{\mathbb{E}[\Pi_i]}{\mathcal{R}(\Pi_i \mid \Pi)}$$

Based on the notion of RAPM, Tasche (2008) states two properties of risk contributions that are desirable from an economic point of view:

1. Risk contributions $\mathcal{R}(\Pi_i \mid \Pi)$ to portfolio-wide risk $\mathcal{R}(\Pi)$ satisfy the full allocation property if:

$$\sum_{i=1}^{n} \mathcal{R}(\Pi_i \mid \Pi) = \mathcal{R}(\Pi) \tag{2.8}$$

2. Risk contributions $\mathcal{R}(\Pi_i \mid \Pi)$ are RAPM compatible if there are some $\varepsilon_i > 0$ such that[76]:

$$\text{RAPM}(\Pi_i \mid \Pi) > \text{RAPM}(\Pi) \Rightarrow \text{RAPM}(\Pi + h\Pi_i) > \text{RAPM}(\Pi) \tag{2.9}$$

for all $0 < h < \varepsilon_i$.

[74]We remind that $\mathcal{R}(\Pi) = \mathcal{R}(-L)$.

[75]This concept is close to the RAROC measure introduced by Banker Trust (see page 2).

[76]This property means that assets with a better risk-adjusted performance than the portfolio continue to have a better RAPM if their allocation increases in a small proportion.

Tasche (2008) shows therefore that if there are risk contributions that are RAPM compatible in the sense of the two previous properties (2.8) and (2.9), then $\mathcal{R}(\Pi_i \mid \Pi)$ is uniquely determined as:

$$\mathcal{R}(\Pi_i \mid \Pi) = \frac{\mathrm{d}}{\mathrm{d}h}\mathcal{R}(\Pi + h\Pi_i)\bigg|_{h=0} \tag{2.10}$$

and the risk measure is homogeneous of degree 1. In the case of a subadditive risk measure, one can also show that:

$$\mathcal{R}(\Pi_i \mid \Pi) \leq \mathcal{R}(\Pi_i) \tag{2.11}$$

This means that the risk contribution of asset i is always smaller than its stand-alone risk measure. The difference is related to the risk diversification.

Let us return to risk measure $\mathcal{R}(w)$ defined in terms of weights. The previous framework implies that the risk contribution of sub-portfolio i is uniquely defined as:

$$\mathcal{RC}_i = w_i \frac{\partial\,\mathcal{R}(w)}{\partial\, w_i} \tag{2.12}$$

and the risk measure satisfies the Euler decomposition:

$$\mathcal{R}(w) = \sum_{i=1}^{n} w_i \frac{\partial\,\mathcal{R}(w)}{\partial\, w_i} = \sum_{i=1}^{n} \mathcal{RC}_i \tag{2.13}$$

This relationship is also called the Euler allocation principle.

Remark 19 *We can always define the risk contributions of a risk measure by using Equation (2.12). However, this does not mean that it satisfies the Euler decomposition (2.13).*

Remark 20 *Kalkbrener (2005) develops an axiomatic approach to risk contributions. In particular, he shows that the Euler allocation principle is the only risk allocation method compatible with diversification principle (2.11) if the risk measure is subadditive.*

If we assume that the portfolio return $R(w)$ is a linear function of the weights w, the expression of the standard deviation-based risk measure becomes:

$$\begin{aligned} \mathcal{R}(w) &= -\mu(w) + c \times \sigma(w) \\ &= -w^\top \mu + c\sqrt{w^\top \Sigma w} \end{aligned}$$

where μ and Σ are the mean vector and the covariance matrix of sub-portfolios. It follows that the vector of marginal risks is:

$$\begin{aligned} \frac{\partial\,\mathcal{R}(w)}{\partial\, w} &= -\mu + c\frac{1}{2}\left(w^\top \Sigma w\right)^{-1}(2\Sigma w) \\ &= -\mu + c\frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \end{aligned}$$

The risk contribution of the i^{th} sub-portfolio is then:

$$\mathcal{RC}_i = w_i \times \left(-\mu_i + c \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right)$$

We verify that the standard deviation-based risk measure satisfies the full allocation property:

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{RC}_i &= \sum_{i=1}^{n} w_i \times \left(-\mu_i + c \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right) \\
&= w^\top \left(-\mu + c \frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \right) \\
&= -w^\top \mu + c\sqrt{w^\top \Sigma w} \\
&= \mathcal{R}(w)
\end{aligned}$$

Because Gaussian value-at-risk and expected shortfall are two special cases of the standard deviation-based risk measure, we conclude that they also satisfy the Euler allocation principle. In the case of the value-at-risk, the risk contribution becomes:

$$\mathcal{RC}_i = w_i \times \left(-\mu_i + \Phi^{-1}(\alpha) \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right) \tag{2.14}$$

whereas in the case of the expected shortfall, it is:

$$\mathcal{RC}_i = w_i \times \left(-\mu_i + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)} \times \frac{(\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \right) \tag{2.15}$$

Remark 21 *Even if the risk measure is coherent and convex, it does not necessarily satisfy the Euler allocation principle. The most famous example is the variance of the portfolio return. We have* $\operatorname{var}(w) = w^\top \Sigma w$ *and* $\partial_w \operatorname{var}(w) = 2\Sigma w$*. It follows that* $\sum_{i=1}^n w_i \times \partial_{w_i} \operatorname{var}(w) = \sum_{i=1}^n w_i \times (2\Sigma w)_i = 2w^\top \Sigma w = 2\operatorname{var}(w) > \operatorname{var}(w)$*. In the case of the variance, the sum of the risk contributions is then always larger than the risk measure itself.*

Example 20 *We consider the Apple/Coca-Cola portfolio that have been used for calculating the Gaussian VaR in page 76. We remind that the nominal exposures were* \$1 093.3 *(Apple) and* \$842.8 *(Coca-Cola), the estimated standard deviation of daily returns was equal to* 1.3611% *for Apple and* 0.9468% *for Coca-Cola and the cross-correlation of stock returns was equal to* 12.0787%*.*

In the two-asset case, the expression of the value-at-risk or the expected shortfall is:

$$\mathcal{R}(w) = -w_1\mu_1 - w_2\mu_2 + c\sqrt{w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2}$$

It follows that the marginal risk of the first asset is:

$$\mathcal{MR}_1 = -\mu_1 + c\frac{w_1\sigma_1^2 + w_2\rho\sigma_1\sigma_2}{\sqrt{w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2}}$$

We then deduce that the risk contribution of the first asset is:

$$\mathcal{RC}_1 = -w_1\mu_1 + c\frac{w_1^2\sigma_1^2 + w_1w_2\rho\sigma_1\sigma_2}{\sqrt{w_1^2\sigma_1^2 + 2w_1w_2\rho\sigma_1\sigma_2 + w_2^2\sigma_2^2}}$$

By using the numerical values[77] of Example 20, we obtain the results given in Tables 2.14 and 2.14. We verify that the sum of risk contributions is equal to the risk measure. We notice that the stock Apple explains 75.14% of the risk whereas it represents 56.47% of the allocation.

TABLE 2.14: Risk decomposition of the 99% Gaussian value-at-risk

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	1093.3	2.83%	30.96	75.14%
Coca-Cola	842.8	1.22%	10.25	24.86%
$\mathcal{R}(w)$			41.21	

TABLE 2.15: Risk decomposition of the 99% Gaussian expected shortfall

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	1093.3	3.24%	35.47	75.14%
Coca-Cola	842.8	1.39%	11.74	24.86%
$\mathcal{R}(w)$			47.21	

2.3.2 Application to non-normal risk measures

2.3.2.1 Main results

In the previous section, we provided formulas for when asset returns are normally distributed. However, the previous expressions can be extended in the general case. For the value-at-risk, Gouriéroux *et al.* (2000) show that the risk contribution is equal to[78]:

$$\begin{aligned}
\mathcal{RC}_i &= \mathcal{R}(\Pi_i \mid \Pi) \\
&= -\mathbb{E}[\Pi_i \mid \Pi = -\operatorname{VaR}_\alpha(\Pi)] \\
&= \mathbb{E}[L_i \mid L(w) = \operatorname{VaR}_\alpha(L)]
\end{aligned} \tag{2.16}$$

[77] We set $\mu_1 = \mu_2 = 0$.

[78] See also Hallerbach (2003).

Formula (2.16) is more general than Equation (2.14) obtained in the Gaussian case. Indeed, we can retrieve the latter if we assume that the returns are Gaussian. We recall that the portfolio return is $R(w) = \sum_{i=1}^{n} w_i R_i = w^\top R$. The portfolio loss is defined by $L(w) = -R(w)$. We deduce that:

$$\begin{aligned} \mathcal{RC}_i &= \mathbb{E}\left[-w_i R_i \mid -R(w) = \mathrm{VaR}_\alpha(w)\right] \\ &= -w_i \mathbb{E}\left[R_i \mid R(w) = -\mathrm{VaR}_\alpha(w)\right] \end{aligned}$$

Because $R(w)$ is a linear combination of R, the random vector $(R, R(w))$ is Gaussian and we have:

$$\begin{pmatrix} R \\ R(w) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu \\ w^\top \mu \end{pmatrix}, \begin{pmatrix} \Sigma & \Sigma w \\ w^\top \Sigma & w^\top \Sigma w \end{pmatrix}\right)$$

We know that $\mathrm{VaR}_\alpha(w) = -w^\top \mu + \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w}$. It follows that[79]:

$$\begin{aligned} \mathbb{E}\left[R \mid R(w) = -\mathrm{VaR}_\alpha(w)\right] &= \mathbb{E}\left[R \mid R(w) = w^\top \mu - \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w}\right] \\ &= \mu + \Sigma w \left(w^\top \Sigma w\right)^{-1} \times \\ &\quad \left(w^\top \mu - \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w} - w^\top \mu\right) \\ &= \mu - \Phi^{-1}(\alpha)\Sigma w \frac{\sqrt{w^\top \Sigma w}}{\left(w^\top \Sigma w\right)^{-1}} \\ &= \mu - \Phi^{-1}(\alpha)\frac{\Sigma w}{\sqrt{w^\top \Sigma w}} \end{aligned}$$

We obtain finally the same expression as Equation (2.14):

$$\begin{aligned} \mathcal{RC}_i &= -w_i \left(\mu - \Phi^{-1}(\alpha)\frac{\Sigma w}{\sqrt{w^\top \Sigma w}}\right)_i \\ &= -w_i \mu_i + \Phi^{-1}(\alpha)\frac{w_i \cdot (\Sigma w)_i}{\sqrt{w^\top \Sigma w}} \end{aligned}$$

In the same way, Tasche (2002) shows that the general expression of the risk contributions for the expected shortfall is:

$$\begin{aligned} \mathcal{RC}_i &= \mathcal{R}(\Pi_i \mid \Pi) \\ &= -\mathbb{E}\left[\Pi_i \mid \Pi \leq -\mathrm{VaR}_\alpha(\Pi)\right] \\ &= \mathbb{E}\left[L_i \mid L(w) \geq \mathrm{VaR}_\alpha(L)\right] \end{aligned} \tag{2.17}$$

Using Bayes' theorem, it follows that:

$$\mathcal{RC}_i = \frac{\mathbb{E}\left[L_i \cdot \mathbb{1}\{L(w) \geq \mathrm{VaR}_\alpha(L)\}\right]}{1-\alpha}$$

[79]We use the formula of the conditional expectation presented in Appendix A.2.2.3 in page 456.

If we apply the previous formula to the Gaussian case, we obtain:

$$\mathcal{RC}_i = -\frac{w_i}{1-\alpha}\mathbb{E}\left[R_i \cdot \mathbb{1}\left\{R(w) \leq -\operatorname{VaR}_\alpha(L)\right\}\right]$$

After some tedious computations, we retrieve the same expression as found previously[80].

2.3.2.2 Calculating risk contributions with historical and simulated scenarios

The case of value-at-risk When using historical or simulated scenarios, the VaR is calculated as follows:

$$\operatorname{VaR}_\alpha(w;h) = -\Pi_{((1-\alpha)n_S:n_S)} = L_{(\alpha n_S:n_S)}$$

Let $\mathfrak{R}_\Pi(s)$ be the rank of the P&L associated to the s^{th} observation meaning that:

$$\mathfrak{R}_\Pi(s) = \sum_{j=1}^{n_S} \mathbf{1}\left\{\Pi_j \leq \Pi_s\right\}$$

We deduce that:

$$\Pi_s = \Pi_{(\mathfrak{R}_\Pi(s):n_S)}$$

Formula (2.16) is then equivalent to decompose $\Pi_{((1-\alpha)n_S:n_S)}$ into individual P&Ls. We have $\Pi_s = \sum_{i=1}^n \Pi_{i,s}$ where $\Pi_{i,s}$ is the P&L of the i^{th} sub-portfolio for the s^{th} scenario. It follows that:

$$\begin{aligned}
\operatorname{VaR}_\alpha(w;h) &= -\Pi_{((1-\alpha)n_S:n_S)} \\
&= -\Pi_{\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)} \\
&= -\sum_{i=1}^n \Pi_{i,\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)}
\end{aligned}$$

where $\mathfrak{R}_\Pi^{-1}$ is the inverse function of the rank. We finally deduce that:

$$\begin{aligned}
\mathcal{RC}_i &= -\Pi_{i,\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)} \\
&= L_{i,\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)}
\end{aligned}$$

The risk contribution of the i^{th} sub-portfolio is the loss of the i^{th} sub-portfolio corresponding to the scenario $\mathfrak{R}_\Pi^{-1}((1-\alpha)n_S)$. If $(1-\alpha)n_S$ is not an integer, we have:

$$\mathcal{RC}_i = -\left(\Pi_{i,\mathfrak{R}_\Pi^{-1}(q)} + ((1-\alpha)n_S - q)\left(\Pi_{i,\mathfrak{R}_\Pi^{-1}(q+1)} - \Pi_{i,\mathfrak{R}_\Pi^{-1}(q)}\right)\right)$$

where $q = q_{\bar{\alpha}}(n_S)$ is the integer part of $(1-\alpha)n_S$.

[80] The derivation of the formula is left as an exercise (Section 2.4.9 in page 135).

Let us consider Example 13 in page 68. We have found that the historical value-at-risk is \$47.39. It corresponds to the linear interpolation between the second and third largest loss. Using results in Table 2.4, we notice that $\mathfrak{R}_{\Pi}^{-1}(1) = 236$, $\mathfrak{R}_{\Pi}^{-1}(2) = 69$, $\mathfrak{R}_{\Pi}^{-1}(3) = 85$, $\mathfrak{R}_{\Pi}^{-1}(4) = 23$ and $\mathfrak{R}_{\Pi}^{-1}(5) = 242$. We deduce that the second and third order statistics correspond to the 69^{th} and 85^{th} historical scenarios. The risk decomposition is reported in Table 2.16. Therefore, we calculate the risk contribution of the Apple stock as follows:

$$\begin{aligned}\mathcal{RC}_i &= -\frac{1}{2}\left(\Pi_{1,69} + \Pi_{1,85}\right) \\ &= -\frac{1}{2}\left(10 \times (105.16 - 109.33) + 10 \times (104.72 - 109.33)\right) \\ &= \$43.9\end{aligned}$$

For the Coca-Cola stock, we obtain:

$$\begin{aligned}\mathcal{RC}_i &= -\frac{1}{2}\left(\Pi_{2,69} + \Pi_{2,85}\right) \\ &= -\frac{1}{2}\left(20 \times (41.65 - 42.14) + 20 \times (42.28 - 42.14)\right) \\ &= \$3.5\end{aligned}$$

If we compare these results with those obtained with the Gaussian VaR, we observe that the risk decomposition is more concentrated for the historical VaR. Indeed, the exposure on Apple represents 96.68% whereas it was previously equal to 75.14%. The problem is that the estimator of the risk contribution only uses two observations, implying that its variance is very high.

TABLE 2.16: Risk decomposition of the 99% historical value-at-risk

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	56.47%	77.77	43.92	92.68%
Coca-Cola	43.53%	7.97	3.47	7.32%
$\mathcal{R}(w)$			47.39	

We can consider three techniques to improve the efficiency of the estimator $\mathcal{RC}_i = L_{i,\mathfrak{R}_{\Pi}^{-1}(n_S(1-a))}$. The first approach is to use a regularization method (Scaillet, 2004). The idea is to estimate the value-at-risk by weighting the order statistics:

$$\begin{aligned}\mathrm{VaR}_\alpha(w;h) &= -\sum_{s=1}^{n_S} \varpi_\alpha(s;n_S)\,\Pi_{(s:n_S)} \\ &= -\sum_{s=1}^{n_S} \varpi_\alpha(s;n_S)\,\Pi_{\mathfrak{R}_{\Pi}^{-1}(s)}\end{aligned}$$

where $\varpi_\alpha(s;n_S)$ is a weight function dependent on the confidence level α.

The expression of the risk contribution then becomes:

$$\mathcal{RC}_i = -\sum_{s=1}^{n_S} \varpi_\alpha(s; n_S)\, \Pi_{i,\mathfrak{R}_\Pi^{-1}(s)}$$

Of course, this naive method can be improved by using more sophisticated approaches such as importance sampling (Glasserman, 2005).

In the second approach, asset returns are assumed to be elliptically distributed. In this case, Carroll *et al.* (2001) shows that[81]:

$$\mathcal{RC}_i = \mathbb{E}[L_i] + \frac{\operatorname{cov}(L, L_i)}{\sigma^2(L)} (\mathrm{VaR}_\alpha(L) - \mathbb{E}[L])$$

Estimating the risk contributions with the scenarios is then straightforward. It suffices to apply Formula (2.16) by replacing the statistical moments by their sample statistics:

$$\mathcal{RC}_i = \bar{L}_i + \frac{\sum_{s=1}^{n_S} \left(L_s - \bar{L}\right)\left(L_{i,s} - \bar{L}_i\right)}{\sum_{s=1}^{n_S} \left(L_s - \bar{L}\right)^2} \left(\mathrm{VaR}_\alpha(L) - \bar{L}\right)$$

where $\bar{L}_i = n_S^{-1} \sum_{s=1}^{n_S} L_{i,s}$ and $\bar{L} = n_S^{-1} \sum_{s=1}^{n_S} L_s$. Result (2.16) can be viewed as the estimation of the conditional expectation $\mathbb{E}[L_i \mid L = \mathrm{VaR}_\alpha(L)]$ in a linear regression framework:

$$L_i = \beta L + \varepsilon_i$$

Because the least squared estimator is $\hat{\beta} = \operatorname{cov}(L, L_i)/\sigma^2(L)$, we deduce that:

$$\begin{aligned}
\mathbb{E}[L_i \mid L = \mathrm{VaR}_\alpha(L)] &= \hat{\beta}\,\mathrm{VaR}_\alpha(L) + \mathbb{E}[\varepsilon_i] \\
&= \hat{\beta}\,\mathrm{VaR}_\alpha(L) + \left(\mathbb{E}[L_i] - \hat{\beta}\mathbb{E}[L]\right) \\
&= \mathbb{E}[L_i] + \hat{\beta}(\mathrm{VaR}_\alpha(L) - \mathbb{E}[L])
\end{aligned}$$

Epperlein and Smillie (2006) extend Formula (2.16) in the case of non-elliptical distributions. If we consider the generalized conditional expectation $\mathbb{E}[L_i \mid L = x] = f(x)$ where the function f is unknown, the estimator is given by the kernel regression[82]:

$$\hat{f}(x) = \frac{\sum_{s=1}^{n_S} \mathcal{K}(L_s - x)\, L_{i,s}}{\sum_{s=1}^{n_S} \mathcal{K}(L_s - x)}$$

[81] We verify that the sum of the risk contributions is equal to the value-at-risk:

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{RC}_i &= \sum_{i=1}^{n} \mathbb{E}[L_i] + (\mathrm{VaR}_\alpha(L) - \mathbb{E}[L]) \sum_{i=1}^{n} \frac{\operatorname{cov}(L, L_i)}{\sigma^2(L)} \\
&= \mathbb{E}[L] + (\mathrm{VaR}_\alpha(L) - \mathbb{E}[L]) \\
&= \mathrm{VaR}_\alpha(L)
\end{aligned}$$

[82] $\hat{f}(x)$ is called the Nadaraya-Watson estimator.

where $\mathcal{K}(u)$ is the kernel function. We deduce that:

$$\mathcal{RC}_i = \hat{f}\left(\mathrm{VaR}_\alpha(L)\right)$$

Epperlein and Smillie (2006) note however that this risk decomposition does not satisfy the Euler principle. This is why they propose the following correction:

$$\begin{aligned}\mathcal{RC}_i &= \frac{\mathrm{VaR}_\alpha(L)}{\sum_{i=1}^n \mathcal{RC}_i}\hat{f}\left(\mathrm{VaR}_\alpha(L)\right) \\ &= \mathrm{VaR}_\alpha(L)\frac{\sum_{s=1}^{n_S}\mathcal{K}\left(L_s - \mathrm{VaR}_\alpha(L)\right)L_{i,s}}{\sum_{i=1}^n\sum_{s=1}^{n_S}\mathcal{K}\left(L_s - \mathrm{VaR}_\alpha(L)\right)L_{i,s}} \\ &= \mathrm{VaR}_\alpha(L)\frac{\sum_{s=1}^{n_S}\mathcal{K}\left(L_s - \mathrm{VaR}_\alpha(L)\right)L_{i,s}}{\sum_{s=1}^{n_S}\mathcal{K}\left(L_s - \mathrm{VaR}_\alpha(L)\right)L_s}\end{aligned}$$

In Table 2.17, we have reported the risk contributions of the 99% value-at-risk for Apple and Coca-Cola stocks. The case **G** corresponds to the Gaussian value-at-risk whereas all the other cases correspond to the historical value-at-risk. For the case **R1**, the regularization weights are $\varpi_{99\%}(2;250) = \varpi_{99\%}(3;250) = \frac{1}{2}$ and $\varpi_{99\%}(s;250) = 0$ when $s \neq 2$ or $s \neq 3$. It corresponds to the classical interpolation method between the second and third order statistics. For the case **R2**, we have $\varpi_{99\%}(s;250) = \frac{1}{4}$ when $s \leq 4$ and $\varpi_{99\%}(s;250) = 0$ when $s > 4$. The value-at-risk is therefore estimated by averaging the first four order statistics. The cases **E** and **K** correspond to the methods based on the elliptical and kernel approaches. For these two cases, we obtain a risk decomposition, which is closer to this obtained with the Gaussian method. This is quite logical as the Gaussian distribution is a special case of elliptical distributions and the kernel function is also Gaussian.

TABLE 2.17: Risk contributions calculated with regularization techniques

Asset	**G**	**R1**	**R2**	**E**	**K**
Apple	30.97	43.92	52.68	35.35	39.21
Coca-Cola	10.25	3.47	2.29	12.03	8.17
$\mathcal{R}(w)$	41.21	47.39	54.96	47.39	47.39

Example 21 *Let $L = L_1 + L_2$ be the portfolio loss, where L_i $(i = 1, 2)$ is defined as follows:*

$$L_i = w_i\left(\mu_i + \sigma_i T_i\right)$$

and T_i is a Student t distribution with the number of degrees of freedom ν_i. The dependence function between the losses (L_1, L_2) is given by the Clayton copula:

$$\mathbf{C}(u_1, u_2) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$$

For the numerical illustration, we consider the following values: $w_1 = 100$, $\mu_1 = 10\%$, $\sigma_1 = 20\%$, $\nu_1 = 6$, $w_2 = 200$, $\mu_2 = 10\%$, $\sigma_2 = 25\%$, $\nu_2 = 4$ *and* $\theta = 2$. *The confidence level* α *of the value-at-risk is set to* 90%.

In Figure 2.18, we compare the different statistical estimators of the risk contribution $\mathcal{RC}_1$ when we use $n_S = 5\,000$ simulations. Concerning the regularization method, we consider the following weight function applied to the order statistics of losses[83]:

$$\varpi_\alpha^L(s; n_S) = \frac{1}{2hn_S + 1} \times \mathbb{1}\left\{\frac{|s - q_\alpha(n_S)|}{n_S} \leq h\right\}$$

It corresponds to a uniform kernel on the range $[q_\alpha(n_S) - hn_S, q_\alpha + q_\alpha(n_S)\, n_S]$. In the first panel, we report the probability density function of $\mathcal{RC}_1$ when h is equal to 0% and 2.5%. The case $h = 0\%$ is the estimator based on only one observation. We verify that the variance of this estimator is larger for $h = 0\%$ than for $h = 2.5\%$. However, we notice that this last estimator is a little biased, because we estimate the quantile 90% by averaging the order statistics corresponding to the range $[87.5\%, 92.5\%]$. In the second panel, we compare the weighting method with the elliptical and kernel approaches. These two estimators have a smaller variance, but a larger bias because they assume that the loss distribution is elliptical or may be estimated using a Gaussian kernel. Finally, the third panel shows the probability density function of $\mathcal{RC}_1$ estimated with the Gaussian value-at-risk.

The case of expected shortfall The expected shortfall is estimated as follows[84]:

$$\begin{aligned}
\mathrm{ES}_\alpha(L) &= -\frac{1}{q_{\bar{\alpha}}(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{\Pi_s \leq \mathrm{VaR}_\alpha(L)\} \times \Pi_s \\
&= \frac{1}{q_{\bar{\alpha}}(n_S)} \sum_{s=1}^{n_S} \mathbb{1}\{L_s \geq \mathrm{VaR}_\alpha(L)\} \times L_s
\end{aligned}$$

[83]This is equivalent to use this weight function applied to the order statistics of P&Ls:

$$\varpi_\alpha(s; n_S) = \frac{1}{2hn_S + 1} \times \mathbb{1}\left\{\frac{|s - q_{\bar{\alpha}}(n_S)|}{n_S} \leq h\right\}$$

[84]Because we have:

$$\sum_{s=1}^{n_S} \mathbb{1}\{\Pi_s \leq \mathrm{VaR}_\alpha(L)\} = q_{\bar{\alpha}}(n_S)$$

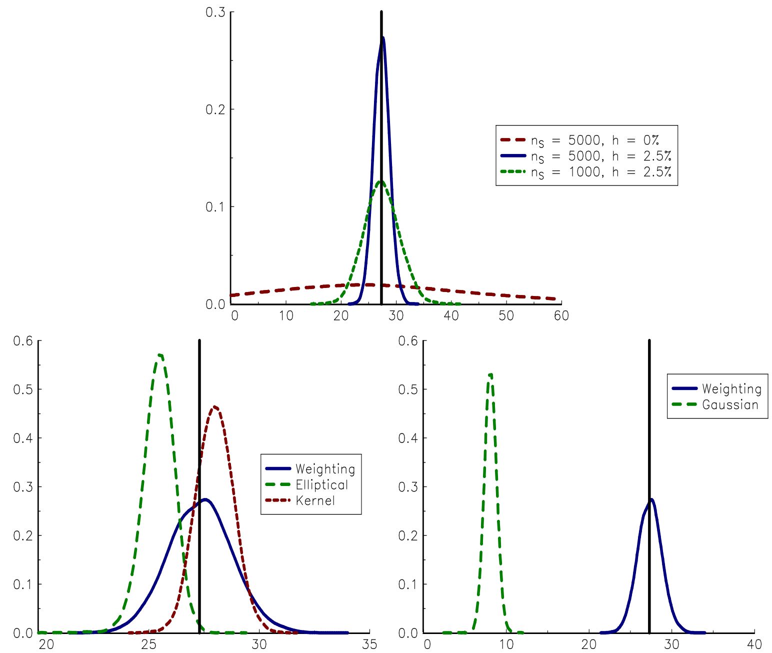

FIGURE 2.18: Probability density function of the different risk contribution estimators

It corresponds to the average of the losses larger or equal than the value-at-risk. It follows that:

$$\begin{aligned}
\mathrm{ES}_\alpha(L) &= -\frac{1}{q_{\bar{\alpha}}(n_S)} \sum_{s=1}^{q_{\bar{\alpha}}(n_S)} \Pi_{(s:n_S)} \\
&= -\frac{1}{q_{\bar{\alpha}}(n_S)} \sum_{s=1}^{q_{\bar{\alpha}}(n_S)} \Pi_{\mathfrak{R}_\Pi^{-1}(s)} \\
&= -\frac{1}{q_{\bar{\alpha}}(n_S)} \sum_{s=1}^{q_{\bar{\alpha}}(n_S)} \sum_{i=1}^{n} \Pi_{i,\mathfrak{R}_\Pi^{-1}(s)}
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathcal{RC}_i &= -\frac{1}{q_{\bar{\alpha}}(n_S)} \sum_{s=1}^{q_{\bar{\alpha}}(n_S)} \Pi_{i,\mathfrak{R}_\Pi^{-1}(s)} \\
&= \frac{1}{q_{\bar{\alpha}}(n_S)} \sum_{s=1}^{q_{\bar{\alpha}}(n_S)} L_{i,\mathfrak{R}_\Pi^{-1}(s)}
\end{aligned}$$

In the Apple/Coca-Cola example, we remind that the 99% daily value-at-risk is equal to \$47.39. The corresponding expected shortfall is then the average of the two largest losses:

$$\mathrm{ES}_\alpha\left(w;\text{one day}\right)=\frac{84.34+51.46}{2}=\$67.90$$

For the risk contributions, we obtain[85]:

$$\mathcal{RC}_1=\frac{87.39+41.69}{2}=\$64.54$$

and:

$$\mathcal{RC}_2=\frac{-3.05+9.77}{2}=\$3.36$$

The corresponding risk decomposition is given in Tables 2.18 and 2.19 for $\alpha = 99\%$ and $\alpha = 97.5\%$. With the new rules of Basel IV, the capital is higher for this example.

TABLE 2.18: Risk decomposition of the 99% historical expected shortfall

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	56.47%	114.29	64.54	95.05%
Coca-Cola	43.53%	7.72	3.36	4.95%
$\mathcal{R}(w)$			67.90	

TABLE 2.19: Risk decomposition of the 97.5% historical expected shortfall

Asset	w_i	$\mathcal{MR}_i$	$\mathcal{RC}_i$	$\mathcal{RC}_i^\star$
Apple	56.47%	78.48	44.32	91.31%
Coca-Cola	43.53%	9.69	4.22	8.69%
$\mathcal{R}(w)$			48.53	

In Figure 2.19, we report the probability density function of the $\mathcal{RC}_1$ estimator in the case of Example 21. We consider the 99% value-at-risk and the 97.5% expected shortfall with $n_S = 5\,000$ simulated scenarios. For the VaR risk measure, the risk contribution is estimated using respectively only one single observation and a weighting function corresponding a uniform window[86]. We notice that the estimator has a smaller variance with the expected shortfall risk measure.

[85] Because we have:

$$\Pi_{(1:250)}=-87.39+3.05=-84.34$$

and:

$$\Pi_{(2:250)}=-41.69-9.77=-51.46$$

[86] We set $h = 0.5\%$ meaning that the risk contribution is estimated with 51 observations for the VaR with a 99% confidence level.

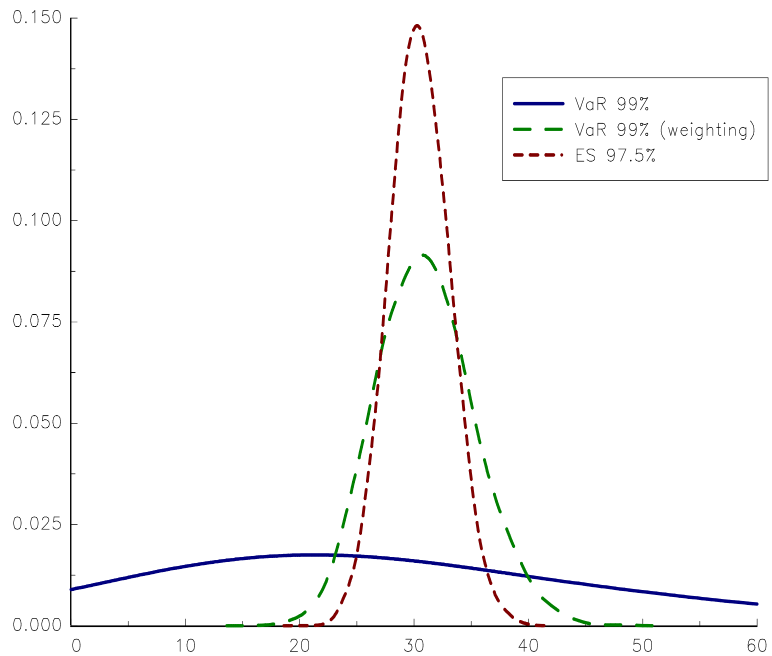

FIGURE 2.19: Probability density function of the $\mathcal{RC}_1$ estimator for the VaR 99% and ES 99.5%

2.4 Exercises

2.4.1 Calculating regulatory capital with the standardized measurement method

1. We consider the following portfolio of stocks:

Stock	3M	Exxon	IBM	Pfizer	AT&T	Cisco	Oracle
$\mathcal{L}_i$	100	100	10	50	60	90	
$\mathcal{S}_i$		50					80

where $\mathcal{L}_i$ and $\mathcal{S}_i$ indicate the long and short exposures on stock i expressed in \$ mn.

(a) Calculate the capital charge for the specific risk.

(b) Calculate the capital charge for the general market risk.

(c) How can the investor hedge the market risk of his portfolio by using S&P 500 futures contracts? What is the corresponding capital charge? Verify that the investor minimizes the total capital charge in this case.

2. We consider a net exposure $\mathcal{N}_w$ on an equity portfolio w. We note $\sigma(w)$ the annualized volatility of the portfolio return.

   (a) Calculate the required capital under the standardized measurement method.

   (b) Calculate the required capital under the internal model method if we assume that the bank uses a Gaussian value-at-risk[87].

   (c) Deduce an upper bound $\sigma(w) \leq \sigma^+$ under which the required capital under SMM is higher than the required capital under IMA.

   (d) Comment on these results.

3. We consider the portfolio with the following long and short positions expressed in \$ mn:

Asset	EUR	JPY	CAD	Gold	Sugar	Corn	Cocoa
$\mathcal{L}_i$	100	50		50	50	60	90
$\mathcal{S}_i$	100	100	50			80	110

   (a) How do you explain that some assets present both long and short positions?

   (b) Calculate the required capital under the simplified approach.

2.4.2 Covariance matrix

We consider a universe of there stocks A, B and C.

1. The covariance matrix of stock returns is:

$$\Sigma = \begin{pmatrix} 4\% & & \\ 3\% & 5\% & \\ 2\% & -1\% & 6\% \end{pmatrix}$$

   (a) Calculate the volatility of stock returns.

   (b) Deduce the correlation matrix.

2. We assume that the volatilities are 10%, 20% and 30%. whereas the correlation matrix is equal to:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 25\% & 0\% & 100\% \end{pmatrix}$$

   (a) Write the covariance matrix.

   (b) Calculate the volatility of the portfolio $(50\%, 50\%, 0)$.

[87]We consider the Basel II capital requirement rules.

(c) Calculate the volatility of the portfolio $(60\%, -40\%, 0)$. Comment on this result.

(d) We assume that the portfolio is long \$150 in stock A, long \$500 in stock B and short \$200 in stock C. Find the volatility of this long/short portfolio.

3. We consider that the vector of stock returns follows a one-factor model:

$$R = \beta \mathcal{F} + \varepsilon$$

We assume that $\mathcal{F}$ and ε are independent. We note $\sigma_{\mathcal{F}}^2$ the variance of $\mathcal{F}$ and $D = \operatorname{diag}\left(\tilde{\sigma}_1^2, \tilde{\sigma}_2^2, \tilde{\sigma}_3^2\right)$ the covariance matrix of idiosyncratic risks ε_t. We use the following numerical values: $\sigma_{\mathcal{F}} = 50\%$, $\beta_1 = 0.9$, $\beta_2 = 1.3$, $\beta_3 = 0.1$, $\tilde{\sigma}_1 = 5\%$, $\tilde{\sigma}_2 = 5\%$ and $\tilde{\sigma}_3 = 15\%$.

(a) Calculate the volatility of stock returns.

(b) Calculate the correlation between stock returns.

4. Let X and Y be two independent random vectors. We note $\mu(X)$ and $\mu(Y)$ the vector of means and $\Sigma(X)$ and $\Sigma(Y)$ the covariance matrices. We define the random vector $Z = (Z_1, Z_2, Z_3)$ where Z_i is equal to the product $X_i Y_i$.

(a) Calculate $\mu(Z)$ and $\operatorname{cov}(Z)$.

(b) We consider that $\mu(X)$ is equal to zero and $\Sigma(X)$ corresponds to the covariance matrix of Question 2. We assume that Y_1, Y_2 and Y_3 are three independent uniform random variables $\mathcal{U}(0,1)$. Calculate the 99% Gaussian value-at-risk of the portfolio corresponding to Question 2(d) when Z is the random vector of asset returns. Compare this value with the Monte Carlo VaR.

2.4.3 Risk measure

1. We denote $\mathbf{F}$ the cumulative probability distribution of the loss L.

(a) Give the mathematical definition of the value-at-risk and expected shortfall risk measures.

(b) Show that:

$$\mathrm{ES}_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 \mathbf{F}^{-1}(t)\,\mathrm{d}t$$

(c) We assume that L follows a Pareto distribution $\mathcal{P}(\theta; x_-)$ defined by:

$$\Pr\{L \leq x\} = 1 - \left(\frac{x}{x_-}\right)^{-\theta}$$

where $x \geq x_-$ and $\theta > 1$. Calculate the moments of order one and two. Interpret the parameters x_- and θ. Calculate ES_α and show that:

$$\mathrm{ES}_\alpha > \mathrm{VaR}_\alpha$$

(d) Calculate the expected shortfall when L is a Gaussian random variable $\mathcal{N}\left(\mu, \sigma^2\right)$. Show that:

$$\Phi\left(x\right) = -\frac{\phi\left(x\right)}{x^1} + \frac{\phi\left(x\right)}{x^3} + \ldots$$

Deduce that:

$$\mathrm{ES}_\alpha \rightarrow \mathrm{VaR}_\alpha \text{ when } \alpha \rightarrow 1$$

(e) Comment on these results in a risk management perspective.

2. Let $\mathcal{R}\left(L\right)$ be a risk measure of the loss L.

(a) Is $\mathcal{R}\left(L\right) = \mathbb{E}\left[L\right]$ a coherent risk measure?

(b) Same question if $\mathcal{R}\left(L\right) = \mathbb{E}\left[L\right] + \sigma\left(L\right)$.

3. We assume that the probability distribution $\mathbf{F}$ of the loss L is defined by:

$$\Pr\left\{L = \ell_i\right\} = \left\{\begin{array}{ll} 20\% & \text{if} \quad \ell_i = 0 \\ 10\% & \text{if} \quad \ell_i \in \{1, 2, 3, 4, 5, 6, 7, 8\} \end{array}\right.$$

(a) Calculate ES_α for $\alpha = 50\%$, $\alpha = 75\%$ and $\alpha = 90\%$.

(b) Let us consider two losses L_1 and L_2 with the same distribution $\mathbf{F}$. Build a joint distribution of (L_1, L_2) which does not satisfy the subadditivity property when the risk measure is the value-at-risk.

2.4.4 Value-at-risk of a long/short portfolio

We consider a long/short portfolio composed of a long (buying) position in asset A and a short (selling) position in asset B. The long exposure is \$2 mn whereas the short exposure is \$1 mn. Using the historical prices of the last 250 trading days of assets A and B, we estimate that the asset volatilities σ_A and σ_B are both equal to 20% per year and that the correlation $\rho_{A,B}$ between asset returns is equal to 50%. In what follows, we ignore the mean effect.

1. Calculate the Gaussian VaR of the long/short portfolio for a one-day holding period and a 99% confidence level.

2. How do you calculate the historical VaR? Using the historical returns of the last 250 trading days, the five worst scenarios of the 250 simulated daily P&L of the portfolio are $-58\,700$, $-56\,850$, $-54\,270$, $-52\,170$ and $-49\,231$. Calculate the historical VaR for a one-day holding period and a 99% confidence level.

3. We assume that the multiplication factor m_c is 3. Deduce the required capital if the bank uses an internal model based on the Gaussian value-at-risk. Same question when the bank uses the historical VaR. Compare these figures with those calculated with the standardized measurement method.

4. Show that the Gaussian VaR is multiplied by a factor equal to $\sqrt{7/3}$ if the correlation $\rho_{A,B}$ is equal to -50%. How do you explain this result?

5. The portfolio manager sells a call option on the stock A. The delta of the option is equal to 50%. What does the Gaussian value-at-risk of the long/short portfolio become if the nominal of the option is equal to \$2 mn? Same question when the nominal of the option is equal to \$4 mn. How do you explain this result?

6. The portfolio manager replaces the short position on the stock B by selling a call option on the stock B. The delta of the option is equal to 50%. Show that the Gaussian value-at-risk is minimum when the nominal is equal to four times the correlation $\rho_{A,B}$. Deduce then an expression of the lowest Gaussian VaR. Comment on these results.

2.4.5 Value-at-risk of an equity portfolio hedged with put options

We consider two stocks A and B and an equity index I. We assume that the risk model corresponds to the CAPM and we have:

$$R_j = \beta_j R_I + \varepsilon_j$$

where R_j and R_I are the return of stock j and the index. We assume that R_I and ε_j are independent. The covariance matrix of idiosyncratic risks is diagonal and we note $\tilde{\sigma}_j$ the volatility of ε_j.

1. The parameters are the following: $\sigma^2(R_I) = 4\%$, $\beta_A = 0.5$, $\beta_B = 1.5$, $\tilde{\sigma}_A^2 = 3\%$ and $\tilde{\sigma}_B^2 = 7\%$.

   (a) Calculate the volatility of stocks A and B and the cross-correlation.

   (b) Find the correlation between the stocks and the index.

   (c) Deduce the covariance matrix.

2. The current price of stocks A and B is equal to \$100 and \$50 whereas the value of the index is equal to \$50. The composition of the portfolio is 4 shares of A, 10 shares of B and 5 shares of I.

   (a) Determine the Gaussian value-at-risk for a confidence level of 99% and a 10-day holding period.

(b) Using the historical returns of the last 260 trading days, the five lowest simulated daily P&Ls of the portfolio are -62.39, -55.23, -52.06, -51.52 and -42.83. Calculate the historical VaR for a confidence level of 99% and a 10-day holding period.

(c) What is the regulatory capital[88] if the bank uses an internal model based on the Gaussian value-at-risk? Same question when the bank uses the historical value-at-risk. Compare these figures with those calculated with the standardized measurement method.

3. The portfolio manager would like to hedge the directional risk of the portfolio. For that, he purchases put options on the index I at a strike price of \$45 with a delta equal to -25%. Write the expression of the P&L using the delta approach.

   (a) How many options should the portfolio manager purchase for hedging 50% of the index exposure? Deduce the Gaussian value-at-risk of the corresponding portfolio?

   (b) The portfolio manager believes that the purchase of 96 put options minimizes the value-at-risk. What is the basis for his reasoning? Do you think that it is justified? Calculate then the Gaussian VaR of this new portfolio.

2.4.6 Risk management of exotic options

Let us consider a short position on an exotic option, whose its current value $\mathcal{C}_t$ is equal to \$6.78. We assume that the price S_t of the underlying asset is \$100 and the implied volatility Σ_t is equal to 20%.

1. At time $t+1$, the value of the underlying asset is \$97 and the implied volatility remains constant. We find that the P&L of the trader between t and $t+1$ is equal to \$1.37. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_t$, gamma $\boldsymbol{\Gamma}_t$ and vega[89] $\boldsymbol{\upsilon}_t$ are respectively equal to 49%, 2% and 40%?

2. At time $t+2$, the price of the underlying asset is \$97 while the implied volatility increases from 20% to 22%. The value of the option $\mathcal{C}_{t+2}$ is now equal to \$6.17. Can we explain the P&L by the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+1}$, gamma $\boldsymbol{\Gamma}_{t+1}$ and vega $\boldsymbol{\upsilon}_{t+1}$ are respectively equal to 43%, 2% and 38%?

3. At time $t+3$, the price of the underlying asset is \$95 and the value of the implied volatility is 19%. We find that the P&L of the trader between $t+2$ and $t+3$ is equal to \$0.58. Can we explain the P&L by

[88] We assume that the multiplication factor m_c is equal to 3.

[89] measured in volatility points.

the sensitivities knowing that the estimates of delta $\boldsymbol{\Delta}_{t+2}$, gamma $\boldsymbol{\Gamma}_{t+2}$ and vega $\boldsymbol{v}_{t+2}$ are respectively equal to 44%, 1.8% and 38%.

4. What can we conclude in terms of model risk?

2.4.7 P&L approximation with Greek sensitivities

1. Let $\mathcal{C}_t$ be the value of an option at time t. Define the delta, gamma, theta and vega coefficients of the option.

2. We consider an European call option with strike K. Give the value of option in the case of the Black-Scholes model. Deduce then the Greek coefficients.

3. We assume that the underlying asset is a non-dividend stock, the residual maturity of the call option is equal to one year, the current price of the stock is equal to \$100 and the interest rate is equal to 5%. We also assume that the implied volatility is constant and equal to 20%. In the table below, we give the value of the call option $\mathcal{C}_0$ and the Greek coefficients $\boldsymbol{\Delta}_0$, $\boldsymbol{\Gamma}_0$ and $\boldsymbol{\Theta}_0$ for different values of K:

K	80	95	100	105	120
$\mathcal{C}_0$	24.589	13.346	10.451	8.021	3.247
$\boldsymbol{\Delta}_0$	0.929	0.728	0.637	0.542	0.287
$\boldsymbol{\Gamma}_0$	0.007	0.017	0.019	0.020	0.017
$\boldsymbol{\Theta}_0$	-4.776	-6.291	-6.414	-6.277	-4.681

   (a) Explain how these values have been calculated. Comment on these numerical results.

   (b) One day later, the value of the underlying asset is \$102. Using the Black-Scholes formula, we obtain:

K	80	95	100	105	120
$\mathcal{C}_1$	26.441	14.810	11.736	9.120	3.837

   Explain how the option premium $\mathcal{C}_1$ is calculated. Deduce then the P&L of a long position on this option for each strike K.

   (c) For each strike price, calculate an approximation of the P&L considering the sensitivities $\boldsymbol{\Delta}$, $\boldsymbol{\Delta}-\boldsymbol{\Gamma}$, $\boldsymbol{\Delta}-\boldsymbol{\Theta}$ and $\boldsymbol{\Delta}-\boldsymbol{\Gamma}-\boldsymbol{\Theta}$. Comment on these results.

   (d) Six months later, the value of the underlying asset is \$148. Repeat Questions 3(b) and 3(c) with these new parameters. Comment on these results.

2.4.8 Calculating the non-linear quadratic value-at-risk

1. Let $X \sim \mathcal{N}(0,1)$. Show that the even moments of X are given by the following relationship:

$$\mathbb{E}\left[X^{2n}\right] = (2n-1)\,\mathbb{E}\left[X^{2n-2}\right]$$

with $n \in \mathbb{N}$. Calculate the odd moments of X.

2. We consider a long position on a call option. The actual price S_t of the underlying asset is equal to \$100, whereas the delta and the gamma of the option are respectively equal to 50% and 2%. We assume that the annual return of the asset is a Gaussian distribution with an annual volatility equal to 32.25%.

   (a) Calculate the Gaussian daily value-at-risk using the delta approximation with a 99% confidence level.

   (b) Calculate the Gaussian daily value-at-risk by considering the delta-gamma approximation.

   (c) Deduce the Cornish-Fisher daily value-at-risk.

3. Let $X \sim \mathcal{N}(\mu, I)$ and $Y = X^\top A X$ with A a symmetric square matrix.

   (a) We recall that:

$$\begin{aligned}
\mathbb{E}[Y] &= \mu^\top A \mu + \operatorname{tr}(A) \\
\mathbb{E}\left[Y^2\right] &= \mathbb{E}^2[Y] + 4\mu^\top A^2 \mu + 2\operatorname{tr}\left(A^2\right)
\end{aligned}$$

   Deduce the moments of $Y = X^\top A X$ when $X \sim \mathcal{N}(\mu, \Sigma)$.

   (b) We suppose that $\mu = \mathbf{0}$. We recall that:

$$\begin{aligned}
\mathbb{E}\left[Y^3\right] &= (\operatorname{tr}(A))^3 + 6\operatorname{tr}(A)\operatorname{tr}\left(A^2\right) + 8\operatorname{tr}\left(A^3\right) \\
\mathbb{E}\left[Y^4\right] &= (\operatorname{tr}(A))^4 + 32\operatorname{tr}(A)\operatorname{tr}\left(A^3\right) + 12\left(\operatorname{tr}\left(A^2\right)\right)^2 + \\
&\quad 12(\operatorname{tr}(A))^2\operatorname{tr}\left(A^2\right) + 48\operatorname{tr}\left(A^4\right)
\end{aligned}$$

   Compute the moments, the skewness and the excess kurtosis of $Y = X^\top A X$ when $X \sim \mathcal{N}(\mathbf{0}, \Sigma)$.

4. We consider a portfolio $w = (w_1, \ldots, w_n)$ of options. We assume that the vector of daily asset returns is distributed according to the Gaussian distribution $\mathcal{N}(\mathbf{0}, \Sigma)$. We note $\boldsymbol{\Delta}$ and $\boldsymbol{\Gamma}$ the vector of deltas and the matrix of gammas.

   (a) Calculate the Gaussian daily value-at-risk using the delta approximation. Define the analytical expression of the risk contributions.

(b) Calculate the Gaussian daily value-at-risk by considering the delta-gamma approximation.

(c) Calculate the Cornish-Fisher daily value-at-risk when assuming that the portfolio is delta neutral.

(d) Calculate the Cornish-Fisher daily value-at-risk in the general case by only considering the skewness.

5. We consider a portfolio composed of 50 options in a first asset, 20 options in a second asset and 20 options in a third asset. We assume that the gamma matrix is:

$$\mathbf{\Gamma} = \begin{pmatrix} 4.0\% & & \\ 1.0\% & 1.0\% & \\ 0.0\% & -0.5\% & 1.0\% \end{pmatrix}$$

The actual price of the assets is normalized and is equal to 100. The daily volatility levels of the assets are respectively 1%, 1.5% and 2% whereas the correlation matrix of asset returns is:

$$\rho = \begin{pmatrix} 100\% & & \\ 50\% & 100\% & \\ 25\% & 15\% & 100\% \end{pmatrix}$$

(a) Compare the different methods to compute the daily value-at-risk with a 99% confidence level if the portfolio is delta neutral.

(b) Same question if we now consider that the deltas are equal to 50%, 40% and 60%. Compute the risk decomposition in the case of the delta and delta-gamma approximations. What do you notice?

2.4.9 Risk decomposition of the expected shortfall

We consider a portfolio composed of n assets. We assume that asset returns $R = (R_1, \ldots, R_n)$ are normally distributed with $R \sim \mathcal{N}(\mu, \Sigma)$. We note $L(w)$ the loss of the portfolio.

1. Find the distribution of $L(w)$.

2. Define the expected shortfall $\mathrm{ES}_\alpha(w)$. Calculate its expression in the present case.

3. Calculate the risk contribution $\mathcal{RC}_i$ of asset i. Deduce that the expected shortfall verifies the Euler allocation principle.

4. Give the expression of $\mathcal{RC}_i$ in terms of conditional loss. Retrieve the formula of $\mathcal{RC}_i$ found in Question 3. What is the interest of the conditional representation?

2.4.10 Expected shortfall of an equity portfolio

We consider an investment universe, which is composed of two stocks A and B. The current prices of the two stocks are respectively equal to \$100 and \$200. Their volatilities are equal to 25% and 20% whereas the cross-correlation is equal to −20%. The portfolio is long of 4 stocks A and 3 stocks B.

1. Calculate the Gaussian expected shortfall at the 97.5% confidence level for a ten-day time horizon.

2. The eight worst scenarios of daily stock returns among the last 250 historical scenarios are the following:

s	1	2	3	4	5	6	7	8
R_A	−3%	−4%	−3%	−5%	−6%	+3%	+1%	−1%
R_B	−4%	+1%	−2%	−1%	+2%	−7%	−3%	−2%

   Calculate then the historical expected shortfall at the 97.5% confidence level for a ten-day time horizon.

Chapter 3

Credit Risk

In this chapter, we give an overview of the credit market. It concerns loans and bonds, but also credit derivatives whose development was impressive during the 2000s. A thorough knowledge of the products is necessary to understand the regulatory framework for computing the calculation of capital requirements for credit risk. In this second section, we will therefore compare Basel I and Basel II and present the new propositions to reform the standardized approach. However, the case of counterparty credit risk will be treated in the next chapter, which focuses on collateral risk. Finally, the last section is dedicated to the modeling of credit risk. We will develop the statistical methods for representing and estimating the main parameters (probability of default, loss given default and default correlations) and we will show the tools of credit risk management.

3.1 The market of credit risk

3.1.1 The loan market

In this section, we present the traditional debt market of loans based on banking intermediation, as opposed to the financial market of debt securities (money market instruments, bonds and notes). We generally distinguish this credit market along two main lines: counterparties and products.

Counterparties are divided into 4 main categories: sovereign, financial, corporate and retail. Banking groups have adopted this customer-oriented approach by differentiating retail banking and corporate and investment banking (CIB) businesses. Retail banking refers to individuals. It may also include micro-sized firms and small and medium-sized enterprises (SME). CIBs concern middle market firms, corporates, financial institutions and public entities. In retail banking, the bank pursues a client segmentation, meaning that all the clients that belongs to the same segment have the same conditions in terms of financing and financial investment. This also implies that the pricing of the loan is the same for two individuals of the same segment. The issue for the bank is then to propose or not a loan offer to his client. For that, the bank uses statistical decision-making methods, which are called credit scoring models.

Contrary to this binary approach (yes or no), CIBs have a personalized approach to their clients. They estimate their probability of default and changes the pricing condition of the loan on the basis of the results. A client with a low default probability will have a lower rate or credit spread than a client with a higher default probability for the same loan.

The household credit market is organized as follows: mortgage and housing debt, consumer credit and student loans. A mortgage is a debt instrument secured by the collateral of a real estate property. In the case which the borrower defaults on the loan, the lender can take possession and sell the secured property. For instance, the home buyer pledges his house to the bank in a residential mortgage. This type of credit is very frequent in English-speaking countries, notably England and the United States. In continental Europe, home loans are generally not collateralized for a primary home. This is not always the case for buy-to-let investments and second-home loans. Consumer credit is used for equipment financing or leasing. We usually make the distinction between auto loans, credit cards, revolving credit and other loans (personal loans and sales financing). Auto loans are personal loans to purchase a car. Credit cards and revolving credit are two forms of personal lines of credit. Revolving credit facilities for individuals are very popular in the US. It can be secured, as in the case of a home equity line of credit (HELOC). Student loans are used to finance educational expenses, for instance post-graduate studies at the university. The corporate credit market is organized differently, because large corporates have access to the financial market for long-term financing. This explains that revolving credit facilities are essential to provide liquidity for the firm's day-to-day operations. The average maturity is then lower for corporates than for individuals.

Credit statistics for the private non-financial sector (households and non-financial corporations) are reported in Figures 3.1 and 3.2. These statistics include loan instruments, but also debt securities. In the case of the United States[1], we notice that the credit amount for households[2] is close to the figure for non-financial business. We also observe the significant share of consumer credit and the strong growth of student loans. Figure 3.2 illustrates the evolution of debt outstanding[3] for different countries: China, United Kingdom, Japan, United States and the Euro area. In China, the annual growth rate is larger than 15% these last five years. Even if credit for households develops much faster than credit for corporations, it only represent 18.7% of the total credit market of the private non-financial sector. The Chinese market con-

[1] Data are from the statistical release Z.1 "*Financial Accounts of the United States*". They are available from the website of the Federal Reserve System: `http://www.federalreserve.gov/releases/z1/` or more easily with the database of the Federal Reserve Bank of St. Louis: `https://research.stlouisfed.org/fred2`.

[2] Data for households include non-profit institutions serving households (NPISH).

[3] Data are collected by the Bank for International Settlement and are available in the website of the BIS: `http://www.bis.org/statistics`. The series are adjusted for breaks (Dembiermont *et al.* (2013)) and we use the average exchange rate from 2000 to 2014 in order to obtain credit amounts in USD.

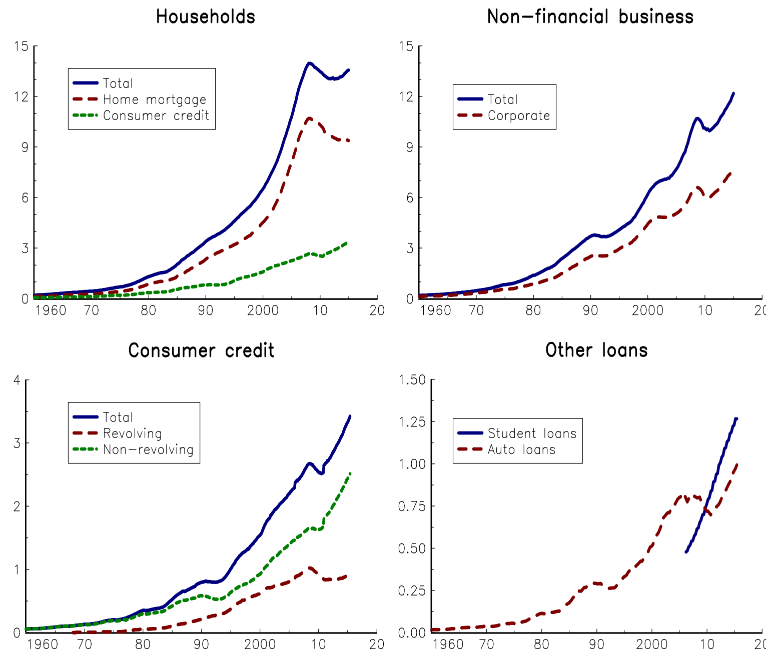

FIGURE 3.1: Notional outstanding of credit in the United States (in $ tn)

Source: Board of Governors of the Federal Reserve System (2015).

trasts with developed markets where the share of household credit is larger[4] and growth rates are almost flat since the 2008 financial crisis. The Japanese case is also very specific, because this country experienced a strong financial crisis after the bursting of a bubble in the 1990s. At that time, the Japanese market was the world's leading market followed by the United States.

3.1.2 The bond market

Contrary to loan instruments, bonds are debt securities that are traded in a financial market. The primary market concerns the issuance of bonds whereas bond trading is organized through the secondary market. The bond issuance market is dominated by two sectors: central and local governments (including public entities) and corporates. This is the principal financing source for government projects and public budget deficit. Large corporates also use extensively the bond market for investments, business expansions and external growth. The distinction government bonds/corporate bonds was crucial before the 2008 financial crisis. Indeed, it was traditionally believed that government

[4]This is especially true in the UK and the US.

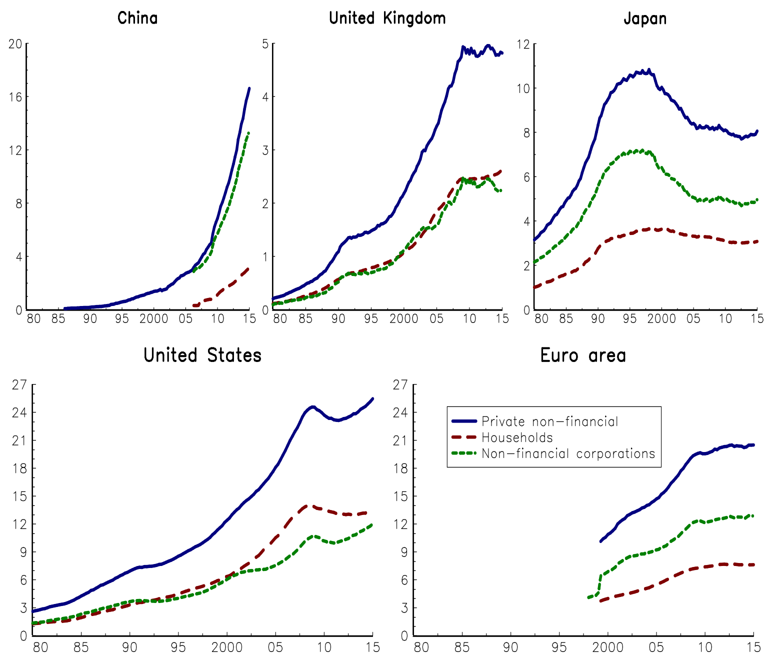

FIGURE 3.2: Credit to the private non-financial sector (in $ tn)

Source: Bank for International Settlement (2015).

bonds (in developed countries) were not risky because the probability of default was very low. In this case, the main risk was the interest rate risk, which is a market risk. Conversely, corporate bonds were supposed to be risky because the probability of default was higher. Beside the interest rate risk, it was important to take into account the credit risk. Bonds issued from the financial and banking sector were considered as low risk investments. Since 2008, this difference between non-risky and risky bonds has disappeared, meaning that all issuers are risky. The 2008 financial crisis had also another important consequence on the bond market. It is today less liquid even for sovereign bonds. Liquidity risk is then a concern when measuring and managing the risk of a bond portfolio. This point will be developed in Chapter 6.

3.1.2.1 Statistics of the bond market

In Table 3.1, we indicate the amounts outstanding of debt securities by residence of issuer[5]. The total is split into three sectors: general government (Gov.), financial corporations (Fin.) and non-financial corporations (Cor.). In

[5]The data are available in the website of the BIS: `http://www.bis.org/statistics`.

TABLE 3.1: Debt securities by residence of issuer (in $ bn)

		Dec. 2004	Dec. 2007	Dec. 2010	Dec. 2014
Canada	Gov.	683	835	1 144	1 202
	Fin.	279	423	369	524
	Cor.	208	238	310	411
	Total	1 171	1 497	1 822	2 137
France	Gov.	1 236	1 512	1 837	2 084
	Fin.	968	1 621	1 819	1 597
	Cor.	373	382	483	628
	Total	2 577	3 515	4 139	4 310
Germany	Gov.	1 380	1 717	2 040	2 000
	Fin.	2 296	2 766	2 283	1 625
	Cor.	133	174	168	155
	Total	3 809	4 657	4 491	3 780
Italy	Gov.	1 637	1 928	2 069	2 182
	Fin.	772	1 156	1 403	1 128
	Cor.	68	95	121	159
	Total	2 478	3 178	3 593	3 469
Japan	Gov.	6 240	6 162	10 064	8 226
	Fin.	2 509	2 721	3 442	2 188
	Cor.	996	758	989	658
	Total	9 745	9 642	14 496	11 072
Spain	Gov.	462	498	796	1 057
	Fin.	434	1 385	1 442	942
	Cor.	15	19	19	26
	Total	910	1 901	2 256	2 024
UK	Gov.	770	1 026	1 648	2 630
	Fin.	1 775	3 121	3 087	2 874
	Cor.	480	549	519	615
	Total	3 026	4 700	5 255	6 122
US	Gov.	6 422	7 385	11 911	15 454
	Fin.	12 576	17 410	15 356	14 995
	Cor.	2 987	3 329	3 936	5 109
	Total	22 183	28 374	31 466	35 781

Source: Bank for International Settlement (2015).

most countries, debt securities issued by general government largely dominate, except in the UK and US where debt securities issued by financial corporations (banks and other financial institutions) are more important. The share of non-financial business varies considerably from one country to another. For instance, it represents less than 10% in Germany, Italy, Japan and Spain, whereas it is equal to 20% in Canada. The total amount of debt securities tends to rise, with the notable exception of Germany, Japan and Spain.

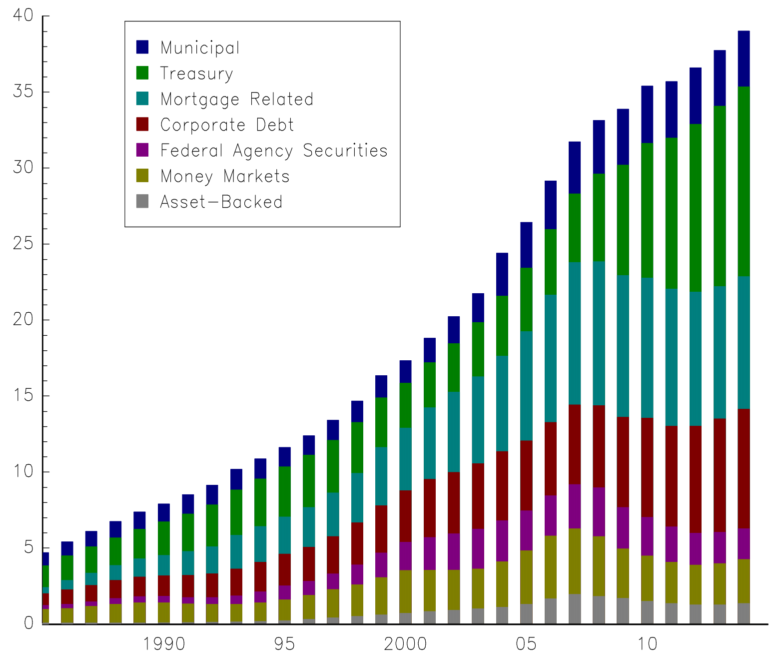

FIGURE 3.3: Amounts outstanding US bond market debt (in $ tn)

Source: Securities Industry and Financial Markets Association (2015).

The analysis of the US market is particularly interesting and relevant. Using the data collected by the Securities Industry and Financial Markets Association[6] (SIFMA), we have reported in Figure 3.3 the evolution of amounts outstanding for the following sectors: municipal bonds, treasury bonds, mortgage-related bonds, corporate related debt, federal agency securities, money markets and asset-backed securities. We notice an important growth during the beginning of the 2000s (see also Figure 3.4), followed by a slowdown after 2008. However, the debt outstanding continues to grow because the average maturity of new issuance increases. Another remarkable fact is the fall of the

[6]Data are available in the website of the SIFMA: `http://www.sifma.org/research/statistics.aspx`.

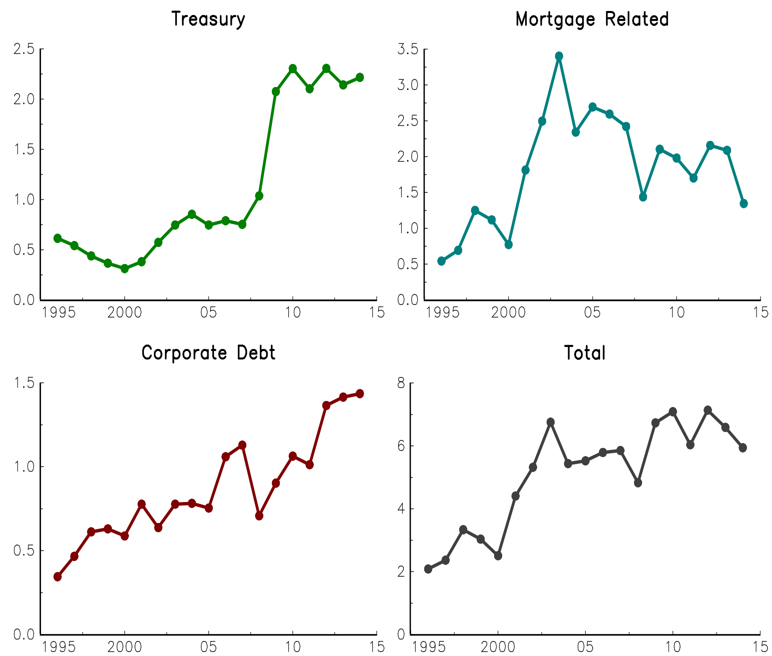

FIGURE 3.4: Issuance in the US bond markets (in \$ tn)

Source: Securities Industry and Financial Markets Association (2015).

liquidity, which can be measured by the average daily volume (ADV). Figure 3.5 shows that the ADV of treasury bonds remains constant since 2000 whereas the outstanding has been multiplied by four during the same period. We also notice that the turnover of US bonds mainly concerns treasury and agency MBS bonds. The liquidity on the other sectors is very poor. For instance, according to SIFMA (2015), the ADV of US corporate bonds is less than \$30 bn in 2014, which is 22 times lower than the ADV for treasury bonds[7].

3.1.2.2 Pricing of bonds

Without default risk We consider that the bond pays coupons $C(t_m)$ with fixing dates t_m and the notional N (or the par value) at the maturity date T. We have reported an example of a cash flows scheme in Figure 3.6. Knowing the yield curve[8], the price of the bond at the inception date t_0 is the

[7]However, the ratio between their amounts outstanding is only 1.6.

[8]A convenient way to define the yield curve is to use a parametric model for the zero-coupon rates $R_t(T)$. The most famous model is the parsimonious functional form proposed

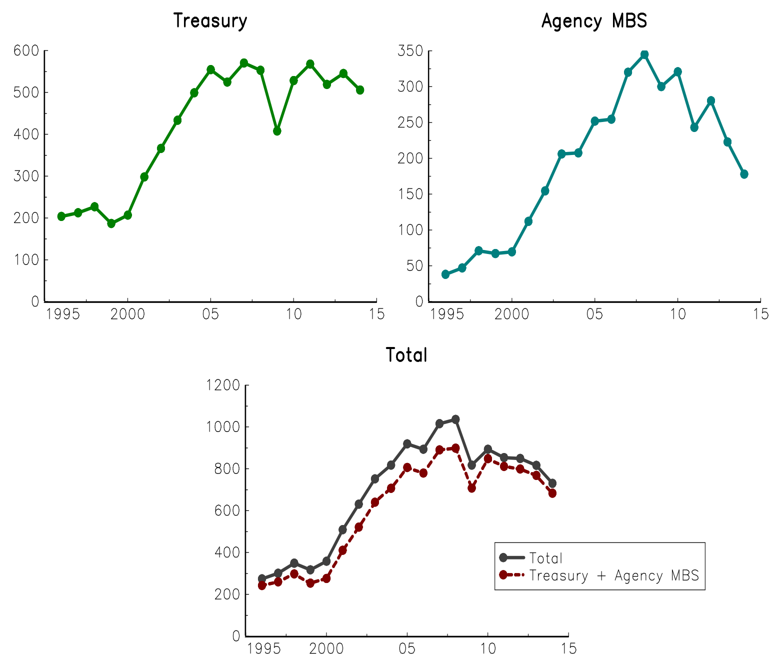

FIGURE 3.5: Average daily trading volume in US bond markets (in $ bn)

Source: Securities Industry and Financial Markets Association (2015).

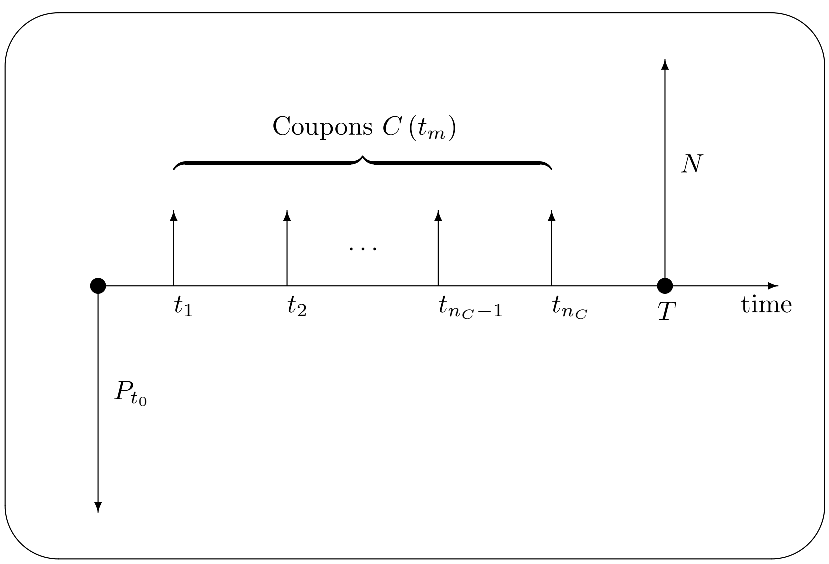

FIGURE 3.6: Cash flows of a bond with a fixed coupon rate

sum of the present values of all expected coupon payments and the par value:

$$P_{t_0} = \sum_{m=1}^{n_C} C(t_m) B_{t_0}(t_m) + N B_{t_0}(T)$$

where $B_t(t')$ is the discount factor at time t for the maturity date t'. When the valuation date is not the issuance date, the previous formula remains valid if we take into account the accrued interests. In this case, the buyer of the bond has the benefit of the next coupon. The price of the bond then satisfies:

$$P_t + AC_t = \sum_{t_m \geq t} C(t_m) B_t(t_m) + N B_t(T) \tag{3.2}$$

Here, AC_t is the accrued coupon:

$$AC_t = C(t_c) \times \frac{t - t_c}{365}$$

and t_c is the last coupon payment date with $c = \{m : t_{m+1} > t, t_m \leq t\}$. $P_t + AC_t$ is called the *dirty price* whereas P_t refers to the *clean price*. The term structure of interest rates impacts the bond price. We generally distinguish three movements:

1. The movement of level corresponds to a parallel shift of interest rates.
2. A twist in the slope of the yield curve indicates how the spread between long and short interest rates moves.
3. A change in the curvature of the yield curve affects the convexity of the term structure.

All these movements are illustrated in Figure 3.7.

The yield to maturity y of a bond is the constant discount rate which returns its market price:

$$\sum_{t_m \geq t} C(t_m) e^{-(t_m - t)y} + N e^{-(T-t)y} = P_t + AC_t$$

---

by Nelson and Siegel (1987):

$$\begin{aligned} R_t(T) &= \theta_1 + \theta_2 \left(\frac{1 - \exp(-(T-t)/\theta_4)}{(T-t)/\theta_4} \right) + \\ & \quad \theta_3 \left(\frac{1 - \exp(-(T-t)/\theta_4)}{(T-t)/\theta_4} - \exp(-(T-t)/\theta_4) \right) \end{aligned} \tag{3.1}$$

This is a model with four parameters: θ_1 is a parameter of level, θ_2 is a parameter of rotation, θ_3 controls the shape of the curve and θ_4 permits to localize the break of the curve. We also note that the short-term and long-term interest rates $R_t(t)$ and $R_t(\infty)$ are respectively equal to $\theta_1 + \theta_2$ and θ_1.

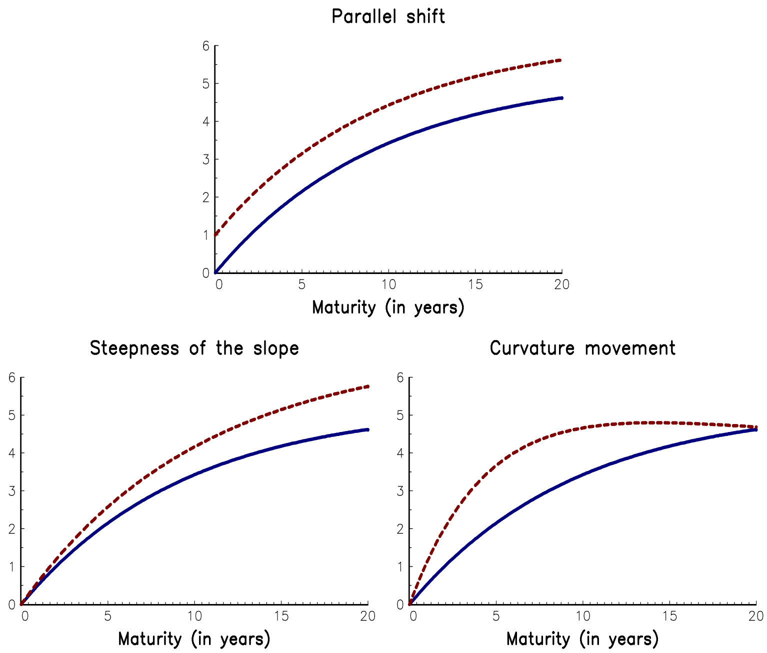

FIGURE 3.7: Movements of the yield curve

We also define the sensitivity[9] S of the bond price as the derivative of the clean price P_t with respect to the yield to maturity $\boldsymbol{y}$:

$$\begin{aligned} S &= \frac{\partial P_t}{\partial \boldsymbol{y}} \\ &= -\sum_{t_m \geq t} (t_m - t)\, C(t_m)\, e^{-(t_m - t)\boldsymbol{y}} - (T - t)\, N e^{-(T-t)\boldsymbol{y}} \end{aligned}$$

It indicates how the P&L of a long position in the bond moves when the yield to maturity changes:

$$\Pi \approx S \times \Delta \boldsymbol{y}$$

Because $S < 0$, the bond price is a decreasing function with respect to interest rates. This implies that an increase of interest rates reduces the value of the bond portfolio.

Example 22 *We assume that the term structure of interest rates is generated by the Nelson-Siegel model with $\theta_1 = 5\%$, $\theta_2 = -5\%$, $\theta_3 = 6\%$ and $\theta_4 = 10$. We consider a bond with a constant 5% annual coupon. The nominal of the bond is \$100. We would like to price the bond when the maturity T ranges from 1 to 5 years.*

TABLE 3.2: Price, yield to maturity and sensitivity of bonds

T	$R_t(T)$	$B_t(T)$	P_t	y	S
1	0.52%	99.48	104.45	0.52%	−104.45
2	0.99%	98.03	107.91	0.98%	−210.86
3	1.42%	95.83	110.50	1.39%	−316.77
4	1.80%	93.04	112.36	1.76%	−420.32
5	2.15%	89.82	113.63	2.08%	−520.16

TABLE 3.3: Impact of a parallel shift of the yield curve on the bond with five-year maturity

ΔR (in bps)	$\breve{P}_t$	ΔP_t	$\hat{P}_t$	ΔP_t	$S \times \Delta y$
−50	116.26	2.63	116.26	2.63	2.60
−30	115.20	1.57	115.20	1.57	1.56
−10	114.15	0.52	114.15	0.52	0.52
0	113.63	0.00	113.63	0.00	0.00
10	113.11	−0.52	113.11	−0.52	−0.52
30	112.08	−1.55	112.08	−1.55	−1.56
50	111.06	−2.57	111.06	−2.57	−2.60

Using the Nelson-Siegel yield curve, we report in Table 3.2 the price of the bond with maturity T (expressed in years) with a 5% annual coupon. For instance, the price of the four-year bond is calculated in the following way:

$$P_t = \frac{5}{(1+0.52\%)} + \frac{5}{(1+0.99\%)^2} + \frac{5}{(1+1.42\%)^3} + \frac{105}{(1+1.80\%)^4} = \$112.36$$

We also indicate the yield to maturity y (in %) and the corresponding sensitivity S. Let $\breve{P}_t$ (resp. $\hat{P}_t$) be the bond price by taking into account a parallel shift ΔR (in bps) directly on the zero-coupon rates (resp. on the yield to maturity). The results are given in Table 3.3 in the case of the bond with a five-year maturity[10]. We verify that the computation based on the sensitivity provides a good approximation. This method has been already used in the previous chapter (page 78) to calculate the value-at-risk of bonds.

[9] This sensitivity is also called the \$-duration or DV01.

[10] We have:

$$\breve{P}_t = \sum_{t_m \geq t} C(t_m)\, e^{-(t_m - t)(R_t(t_m)+\Delta R)} + N e^{-(T-t)(R_t(T)+\Delta R)}$$

$$\hat{P}_t = \sum_{t_m \geq t} C(t_m)\, e^{-(t_m - t)(y+\Delta R)} + N e^{-(T-t)(y+\Delta R)}$$

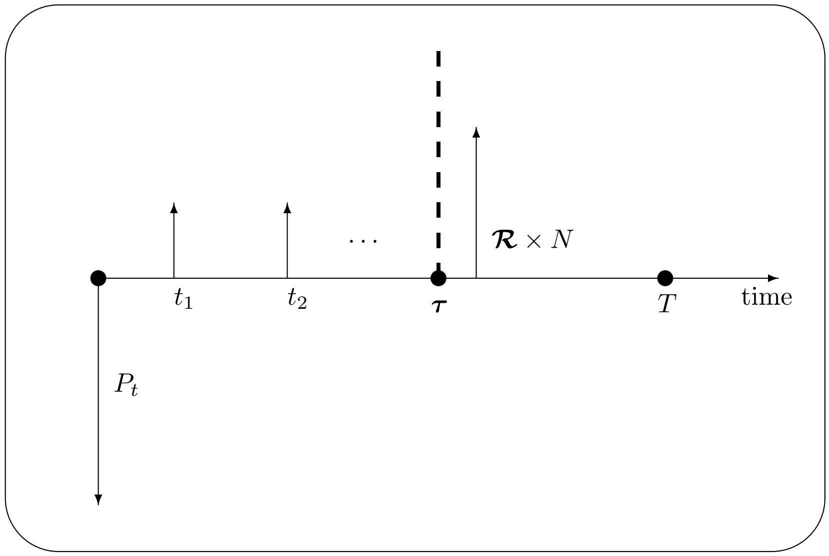

FIGURE 3.8: Cash flows of a bond with default risk

With default risk In the previous paragraph, we implicitly assumed that there is no default risk. If the issuer defaults at time $\boldsymbol{\tau}$ before the bond maturity T, some coupons and the notional are not paid. In this case, the buyer of the bond recovers part of the notional after the default time. An illustration is given in Figure 3.8. In terms of cash flows, we have therefore:

- the coupons $C(t_m)$ if the bond issuer does not default before the coupon date t_m:
$$\sum_{t_m \geq t} C(t_m) \times \mathbb{1}\{\boldsymbol{\tau} > t_m\}$$

- the notional if the bond issuer does not default before the maturity date:
$$N \times \mathbb{1}\{\boldsymbol{\tau} > T\}$$

- the recovery part if the bond issuer defaults before the maturity date:
$$\boldsymbol{\mathcal{R}} \times N \times \mathbb{1}\{\boldsymbol{\tau} \leq T\}$$
where $\boldsymbol{\mathcal{R}}$ is the corresponding recovery rate.

If we assume that the recovery part is exactly paid at the default time $\boldsymbol{\tau}$, we deduce that the stochastic discounted value of the cash flow leg is:

$$\begin{aligned} SV_t &= \sum_{t_m \geq t} C(t_m) \times e^{-\int_t^{t_m} r_s \, \mathrm{d}s} \times \mathbb{1}\{\boldsymbol{\tau} > t_m\} + \\ & \quad N \times e^{-\int_t^{T} r_s \, \mathrm{d}s} \times \mathbb{1}\{\boldsymbol{\tau} > T\} + \boldsymbol{\mathcal{R}} \times N \times e^{-\int_t^{\boldsymbol{\tau}} r_s \, \mathrm{d}s} \times \mathbb{1}\{\boldsymbol{\tau} \leq T\} \end{aligned}$$

The price of the bond is the expected value of the stochastic discounted value[11]: $P_t + AC_t = \mathbb{E}\left[SV_t \mid \mathcal{F}_t\right]$. If we assume that $(\mathcal{H}_1)$ the default time and the interest rates are independent and $(\mathcal{H}_2)$ the recovery rate is known and not stochastic, we obtain the following closed-form formula:

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) B_t(t_m) \mathbf{S}_t(t_m) + N B_t(T) \mathbf{S}_t(T) + \\ &\quad \boldsymbol{\mathcal{R}} N \int_t^T B_t(u) f_t(u) \, \mathrm{d}u \end{aligned} \tag{3.3}$$

where $\mathbf{S}_t(u)$ is the survival function at time u and $f_t(u)$ the associated density function[12].

Remark 22 *If the issuer is not risky, we have $\mathbf{S}_t(u) = 1$ and $f_t(u) = 0$. In this case, Equation (3.3) reduces to Equation (3.2).*

Remark 23 *If we consider an exponential default time with parameter λ – $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$, we have $\mathbf{S}_t(u) = e^{-\lambda(u-t)}$, $f_t(u) = \lambda e^{-\lambda(u-t)}$ and:*

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) B_t(t_m) e^{-\lambda(t_m - t)} + N B_t(T) e^{-\lambda(T-t)} + \\ &\quad \lambda \boldsymbol{\mathcal{R}} N \int_t^T B_t(u) e^{-\lambda(u-t)} \, \mathrm{d}u \end{aligned}$$

If we assume a flat yield curve – $R_t(u) = r$, we obtain:

$$\begin{aligned} P_t + AC_t &= \sum_{t_m \geq t} C(t_m) e^{-(r+\lambda)(t_m - t)} + N e^{-(r+\lambda)(T-t)} + \\ &\quad \lambda \boldsymbol{\mathcal{R}} N \left(\frac{1 - e^{-(r+\lambda)(T-t)}}{r + \lambda} \right) \end{aligned}$$

Example 23 *We consider a bond with ten-year maturity. The notional is* \$100 *whereas the annual coupon rate is equal to 4.5%.*

If we consider that $r = 0$, the price of the non-risky bond is \$145. With $r = 5\%$, the price becomes \$95.19. Let us now take into account the default risk. We assume that the recovery rate $\boldsymbol{\mathcal{R}}$ is 40%. If $\lambda = 2\%$ (resp. 10%), the price of the risky bond is \$86.65 (resp. \$64.63). If the yield curve is not flat, we must use the general formula (3.3) to compute the price of the bond. In this case, the integral is evaluated with a numerical integration procedure, typically a Gauss-Legendre quadrature. For instance, if we consider the yield

[11] It is also called the present value.

[12] We have:

$$\mathbf{S}_t(u) = \mathbb{E}\left[\mathbb{1}\{\boldsymbol{\tau} > u \mid \boldsymbol{\tau} > t\}\right] = \Pr\{\boldsymbol{\tau} > u \mid \boldsymbol{\tau} > t\}$$

The density function is then given by $f_t(u) = -\partial_u \mathbf{S}_t(u)$.

curve defined in Example 22, the bond price is equal to \$110.13 if there is no default risk, \$99.91 if $\lambda = 2\%$ and \$73.34 if $\lambda = 10\%$.

The yield to maturity of the defaultable bond is computed exactly in the same way as without default risk. The credit spread $\mathcal{s}$ is then defined as the difference of the yield to maturity with default risk $\mathcal{y}$ and the yield to maturity without default risk $\mathcal{y}^\star$:

$$\mathcal{s} = \mathcal{y} - \mathcal{y}^\star \tag{3.4}$$

This spread is a credit risk measure and is an increasing function of the default risk. Reconsider the simple model with a flat yield curve and an exponential default time. If the recovery rate $\mathcal{R}$ is equal to zero, we deduce that the yield to maturity of the defaultable bond is $\mathcal{y} = r + \lambda$. It follows that the credit spread is equal to the parameter λ of the exponential distribution. Moreover, if λ is relatively small (less than 20%), the annual default probability is:

$$\mathrm{PD} = \mathbf{S}_t(t+1) = 1 - e^{-\lambda} \approx \lambda$$

In this case, the credit spread is approximately equal to the annual default probability ($\mathcal{s} \simeq \mathrm{PD}$).

If we reuse our previous example with the yield curve specified in Example 22, we obtain the results reported in Table 3.3. For instance, the yield to maturity of the bond is equal to 3.24% without default risk. If λ and $\mathcal{R}$ are set to 200 bps and 0%, the yield to maturity becomes 5.22% which implies a credit spread of 198.1 bps. If the recovery rate is higher, the credit spread decreases. Indeed, with λ equal to 200 bps, the credit spread is equal to 117.1 bps if $\mathcal{R} = 40\%$ and only 41.7 bps if $\mathcal{R} = 80\%$.

TABLE 3.4: Computation of the credit spread $\mathcal{s}$

$\mathcal{R}$ (in %)	λ (in bps)	PD (in bps)	P_t (in \$)	$\mathcal{y}$ (in %)	$\mathcal{s}$ (in bps)
0	0	0.0	110.1	3.24	0.0
	10	10.0	109.2	3.34	9.9
	200	198.0	93.5	5.22	198.1
	1000	951.6	50.4	13.13	988.9
40	0	0.0	110.1	3.24	0.0
	10	10.0	109.6	3.30	6.0
	200	198.0	99.9	4.41	117.1
	1000	951.6	73.3	8.23	498.8
80	0	0.0	110.1	3.24	0.0
	10	10.0	109.9	3.26	2.2
	200	198.0	106.4	3.66	41.7
	1000	951.6	96.3	4.85	161.4

Remark 24 *In the case of loans, we do not calculate a capital requirement for market risk, only a capital requirement for credit risk. The reason is that there is no market price of the loan, because it can not be traded in an exchange. For bonds, we calculate a capital requirement for both market and credit risks. In the case of the market risk, risk factors are the yield curve, but also the parameters associated to the credit risk, for instance the default probabilities and the recovery rate. In this context, market risk has a credit component. To illustrate this property, we consider the previous example and we assume that* λ_t *varies across time whereas the recovery rate* $\boldsymbol{\mathcal{R}}_t$ *is equal to* 40%. *In Figure 3.9, we show the evolution of the process* λ_t *for the next 10 years (top panel) and the clean price*[13] P_t *(bottom/left panel). If we suppose now that the issuer defaults suddenly at time* $t = 6.25$, *we observe a jump in the clean price (bottom/right panel). It is obvious that the market risk takes into account the short-term evolution of the credit component (or the smooth part), but does not incorporate the price jump risk (or the discontinuous part) and also the large uncertainty on the recovery price. This is why these risks are covered by credit risk capital requirements.*

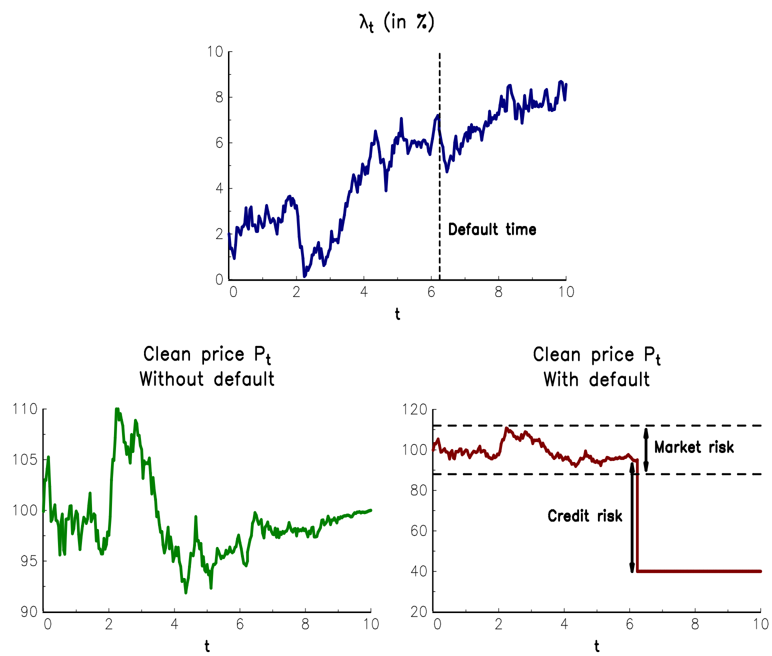

FIGURE 3.9: Difference between market and credit risk for a bond

[13]We assume that the yield curve remains constant.

3.1.3 Securitization and credit derivatives

Since the 1990s, banks have developed credit transfer instruments in two directions: credit securitization and credit derivatives. The term securitization refers to the process of transforming illiquid and non-tradable assets into tradable securities. Credit derivatives are financial instruments whose the payoff explicitly depends on credit events like the default of an issuer. These two topics are highly connected because credit securities can be used as underlying assets of credit derivatives.

3.1.3.1 Credit securitization

According to AFME (2015), amounts outstanding of securitization is close to €9 tn. Figure 3.10 shows the evolution of issuance in Europe and US since 2000. We observe that the financial crisis had a negative impact of the growth of credit securitization, especially in Europe that represents less than 20% of this market.

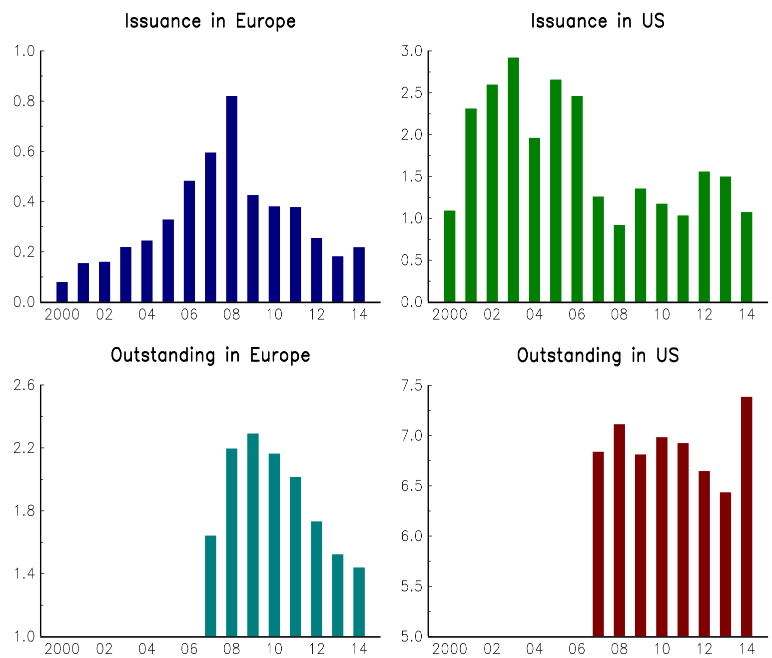

FIGURE 3.10: Securitization in Europe and US (in €tn)

Source: Association for Financial Markets in Europe (2015).

Credit securities are better known as asset-backed securities (ABS), even if this term is generally reserved to assets that are not mortgage, loans or corporate bonds. In its simplest form, an ABS is a bond whose coupons are derived

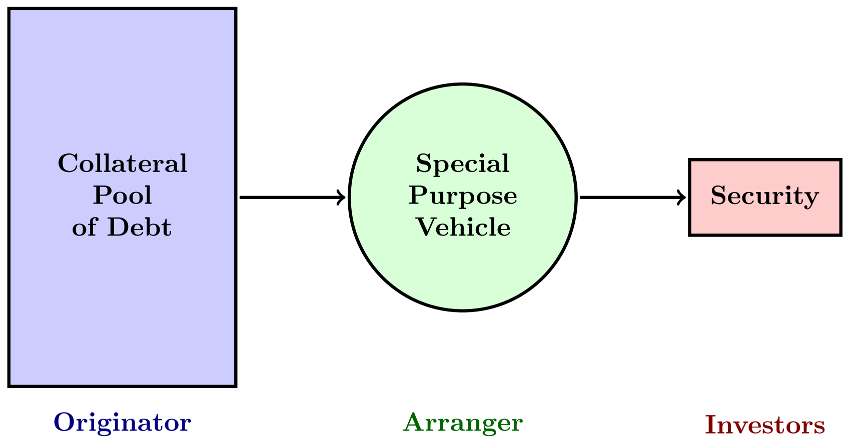

FIGURE 3.11: Structure of pass-through securities

by a collateral pool of assets. We generally make the following distinction with respect to the type of collateral assets:

- Mortgage-backed securities (MBS)
  - Residential mortgage-backed securities (RMBS)
  - Commercial mortgage-backed securities (CMBS)
- Collateralized debt obligations (CDO)
  - Collateralized loan obligations (CLO)
  - Collateralized bond obligations (CBO)
- Asset-backed securities (ABS)
  - Auto loans
  - Credit cards and revolving credit
  - Student loans

MBS are securities that are backed by residential and commercial mortgage loans. The most basic structure is a pass-through security, where the coupons are the same for all the investors and are proportional to the revenue of the collateral pool. Such structure is showed in Figure 3.11. The originator (e.g. a bank) sells a pool of debt to a special purpose vehicle (SPV). The SPV is an ad-hoc legal entity[14] whose sole function is to hold the loans as assets and issue the securities for investors. In the pass-through structure, the securities

[14]It may be a subsidiary of the originator.

are all the same and the cash flows paid to investors are directly proportional to interests and principals of collateral assets. More complex structures are possible with several classes of bonds (see Figure 3.12). In this case, the cash flows differ from one type of securities to another one. The most famous example is the collateralized debt obligation, where the securities are divided into tranches. This category includes also collateralized mortgage obligations (CMO), which are both MBS and CDO. The two other categories of CDO are CLO, which are backed by corporate bank debt (e.g. SME loans) and CBO, which are backed by bonds (e.g. high yield bonds). Finally, pure ABS principally concerns consumer credit such as auto loans, credit cards and student loans.

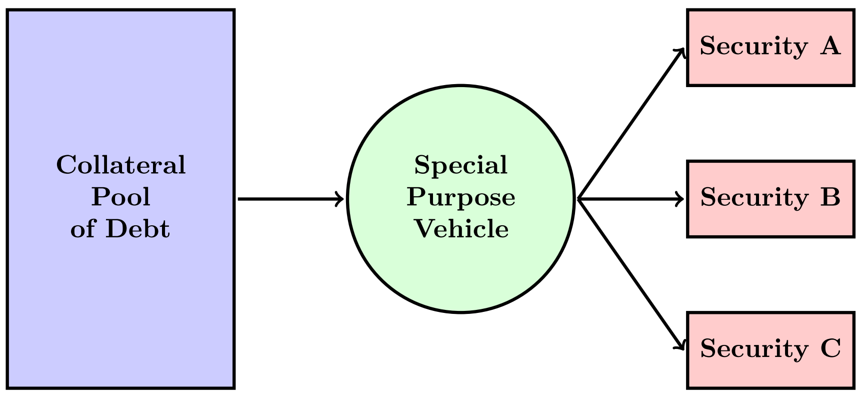

FIGURE 3.12: Structure of pay-through securities

In Table 3.5, we report some statistics about US mortgage-backed securities. SIFMA (2015) makes the distinction between agency MBS and non-agency MBS. After the Great Depression, the US government created three public entities to promote home ownership and provide insurance of mortgage loans: the Federal National Mortgage Association (FNMA or Fannie Mae), the Federal Home Loan Mortgage Corporation (FHLMC or Freddie Mac) and the Government National Mortgage Association (GNMA or Ginnie Mae). Agency MBS refer to securities guaranteed by these three public entities and represent the main part of the US MBS market. This is especially true since the 2008 financial crisis. Indeed, non-agency MBS represent 52.1% of the issuance in 2007 and only 11.4% in 2014. Because agency MBS are principally based on home mortgage loans, the RMBS market is ten times more larger than the CMS market. CDO and ABS markets are smaller and represent together about \$2 bn (see Tables 3.6 and 3.7). The CDO market strongly suffered from the subprime crisis[15]. During the same period, the structure of the ABS market changed with an increasing proportion of ABS backed by auto loans and a fall of ABS backed by credit cards and student loans.

[15]For instance, the issuance of US CDO was less than 2bn in 2010.

TABLE 3.5: US mortgage-backed securities

Year	Agency		Non-agency		Total
	MBS	CMO	CMBS	RMBS	(in $ bn)
Issuance					
2002	58.0%	24.0%	2.0%	16.0%	2 493
2006	35.6%	12.2%	7.1%	45.1%	2 593
2010	71.8%	25.3%	1.2%	1.7%	1 978
2014	72.8%	15.7%	7.5%	3.9%	1 346
Outstanding					
2002	59.7%	17.5%	5.6%	17.2%	5 286
2006	45.8%	15.0%	8.3%	30.9%	8 376
2010	59.4%	14.7%	8.1%	17.8%	9 221
2014	68.8%	13.0%	7.2%	11.0%	8 728

TABLE 3.6: Global collateralized debt obligations

Year	2002	2006	2008	2010	2014
Issuance					
USD	70.0%	79.1%	40.0%	43.8%	76.2%
Total (in $ bn)	83	521	62	9	141
Outstanding					
USD	75.3%	76.2%	70.5%	68.9%	75.5%
Total (in $ bn)	339	1 058	1 356	1 117	811

TABLE 3.7: US asset-backed securities

Year	Auto Loans	Credit Cards	Student Loans	Other	Total (in $ bn)
Issuance					
2002	36.8%	28.9%	18.6%	15.7%	231
2006	28.1%	34.4%	21.2%	16.2%	289
2010	54.9%	13.0%	11.3%	20.8%	124
2014	50.7%	13.6%	7.1%	28.6%	148
Outstanding					
2002	20.8%	32.5%	6.5%	40.2%	902
2006	11.9%	17.7%	12.2%	58.1%	1 643
2010	7.8%	14.7%	16.3%	61.2%	1 476
2014	13.4%	10.2%	16.2%	60.2%	1 336

Source: Securities Industry and Financial Markets Association (2015) & author's calculations.

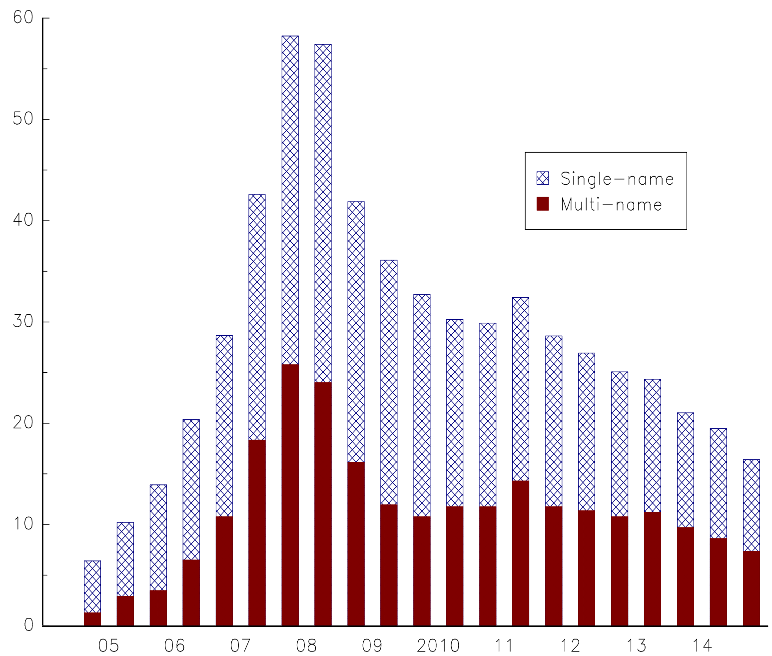

FIGURE 3.13: Amounts outstanding of credit default swaps (in \$ tn)
Source: Bank of International Settlement (2015).

Remark 25 *Even if credit securities may be viewed as a bond, their pricing is not straightforward. Indeed, the measure of the default probability and the recovery depends on the characteristics of the collateral assets (individual default probabilities and recovery rates), but also on the correlation between these risk factors. Measuring credit risk of such securities is then a challenge. Another issue concerns design and liquidity problems faced when packaging and investing in these assets*[16] *(Duffie and Rahi, 1995; DeMarzo and Duffie, 1999). This explains that credit securities suffered a lot during the 2008 financial crisis, even if some were not linked to subprime mortgages. In fact, securitization markets pose a potential risk to financial stability (Segoviano et al., 2013). This is a topic we will return to in the chapter 12, which deals with systemic risk.*

3.1.3.2 Credit default swap

A credit default swap (CDS) may be defined as an insurance derivative, whose the goal is to transfer the credit risk from one party to another. In a standard contract, the protection buyer makes periodic payments, known as the premium leg, to the protection seller. In return, the protection seller

[16]The liquidity issue will be treated in Chapter 6.

pays a compensation, known as the default leg, to the protection buyer in the case of a credit event, which can be a bankruptcy, a failure to pay or a debt restructuring. In its most basic form, the credit event refers to an issuer (sovereign or corporate) and this corresponds to single-name CDS. If the credit event relates to a universe of different entities, we speak about multi-name CDS. In Figure 3.13, we report the evolution of amounts outstanding of CDS since 2007. The growth of this market was very strong before 2008 with a peak close to \$60 tn. The situation today is different, because the market of single-name CDS stabilized whereas the market of basket default swaps continues to fall significantly. Nevertheless, it remains an important OTC market with a total outstanding around \$20 tn.

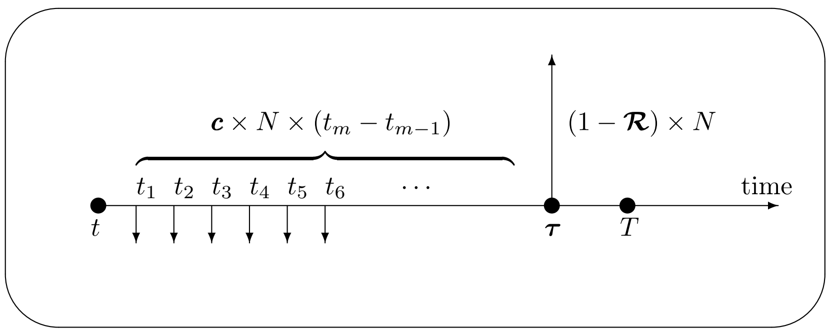

FIGURE 3.14: Cash flows of a single-name credit default swap

In Figure 3.14, we report the mechanisms of a single-name CDS. The contract is defined by a reference entity (the name), a notional principal N, a maturity or tenor T, a payment frequency, a recovery rate $\boldsymbol{\mathcal{R}}$ and a coupon rate[17] $\boldsymbol{c}$. From the inception date t to the maturity date T or the default time $\boldsymbol{\tau}$, the protection buyer pays a fixed payment, which is equal to $\boldsymbol{c} \times N \times \Delta t_m$ at the fixing date t_m with $\Delta t_m = t_m - t_{m-1}$. This means that the annual premium leg is equal to $\boldsymbol{c} \times N$. If there is no credit even, the protection buyer will also pay a total of $\boldsymbol{c} \times N \times (T - t)$. In case of credit even before the maturity, the protection seller will compensate the protection seller and will pay $(1 - \boldsymbol{\mathcal{R}}) \times N$.

Example 24 *We consider a credit default swap, whose the notional principal is* \$10 *mn, the maturity is* 5 *years and the payment frequency is quarterly. The credit even is the bankruptcy of a corporate entity A. We assume that the recovery rate is set to* 40% *and the coupon rate is equal to* 2%.

Because the payment frequency is quarterly, there are 20 fixing dates, which are 3M, 6M, 9M, 1Y,...,5Y. Each quarter, if the corporate A does not default. the protection buyer pays a premium, which is approximately

[17]We will see that the coupon rate $\boldsymbol{c}$ is in fact the CDS spread $\boldsymbol{s}$ for par swaps.

equal to \$10mn $\times$ 2% $\times$ 0.25 = \$50 000. If there is no default during the next five years, the protection buyer will pay a total of \$50 000 $\times$ 20 = \$1 mn whereas the protection seller will pay nothing. Suppose now that the corporate defaults two years and four months after the CDS inception date. In this case, the protection buyer will pay \$50 000 during 9 quarters and will receive the protection leg from the protection seller at the default time. This protection leg is equal to $(1 - 40\%) \times \$10\text{mn} = \$6$ mn.

To compute the mark-to-market value of a CDS, we use the reduced-form approach as in the case of bond pricing. If we assume that the premium is not paid after the default time $\boldsymbol{\tau}$, the stochastic discounted value of the premium leg is[18]:

$$SV_t(\mathcal{PL}) = \sum_{t_m \geq t} \boldsymbol{c} \times N \times (t_m - t_{m-1}) \times \mathbb{1}\{\boldsymbol{\tau} > t_m\} \times e^{-\int_t^{t_m} r_s \, \mathrm{d}s}$$

Using the standard assumptions that the default time is independent of interest rates and the recovery rate, we deduce the present value of the premium leg as follows:

$$\begin{aligned}
PV_t(\mathcal{PL}) &= \mathbb{E}\left[\sum_{t_m \geq t} \boldsymbol{c} \times N \times \Delta t_m \times \mathbb{1}\{\boldsymbol{\tau} > t_m\} \times e^{-\int_t^{t_m} r_s \, \mathrm{d}s} \,\middle|\, \mathcal{F}_t\right] \\
&= \sum_{t_m \geq t} \boldsymbol{c} \times N \times \Delta t_m \times \mathbb{E}\left[\mathbb{1}\{\boldsymbol{\tau} > t_m\}\right] \times \mathbb{E}\left[e^{-\int_t^{t_m} r_s \, \mathrm{d}s}\right] \\
&= \boldsymbol{c} \times N \times \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m) B_t(t_m)
\end{aligned}$$

where $\mathbf{S}_t(u)$ is the survival function at time u. If we assume that the default leg is exactly paid at the default time $\boldsymbol{\tau}$, the stochastic discount value of the default (or protection) leg is[19]:

$$SV_t(\mathcal{DL}) = (1 - \boldsymbol{\mathcal{R}}) \times N \times \mathbb{1}\{\boldsymbol{\tau} \leq T\} \times e^{-\int_t^{\boldsymbol{\tau}} r(s) \, \mathrm{d}s}$$

It follows that its present value is:

$$\begin{aligned}
PV_t(\mathcal{DL}) &= \mathbb{E}\left[(1 - \boldsymbol{\mathcal{R}}) \times N \times \mathbb{1}\{\boldsymbol{\tau} \leq T\} \times e^{-\int_t^{\boldsymbol{\tau}} r_s \, \mathrm{d}s} \,\middle|\, \mathcal{F}_t\right] \\
&= (1 - \boldsymbol{\mathcal{R}}) \times N \times \mathbb{E}\left[\mathbb{1}\{\boldsymbol{\tau} \leq T\} \times B_t(\boldsymbol{\tau})\right] \\
&= (1 - \boldsymbol{\mathcal{R}}) \times N \times \int_t^T B_t(u) f_t(u) \, \mathrm{d}u
\end{aligned}$$

where $f_t(u)$ is the density associated to the survival function $\mathbf{S}_t(u)$. We deduce

[18] In order to obtain a simple formula, we do not deal with the premium accrued (see Remark 28 in page 166).

[19] Here the recovery rate $\boldsymbol{\mathcal{R}}$ is assumed to be deterministic.

that the mark-to-market of the swap is[20]:

$$\begin{aligned} P_t(T) &= PV_t(\mathcal{DL}) - PV_t(\mathcal{PL}) \\ &= (1-\boldsymbol{\mathcal{R}})\, N \int_t^T B_t(u)\, f_t(u)\, \mathrm{d}u - \boldsymbol{c} N \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m)\, B_t(t_m) \\ &= N\left((1-\boldsymbol{\mathcal{R}}) \int_t^T B_t(u)\, f_t(u)\, \mathrm{d}u - \boldsymbol{c} \times \mathrm{RPV}_{01}\right) \qquad (3.5) \end{aligned}$$

where $\mathrm{RPV}_{01} = \sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m)\, B_t(t_m)$ is called the risky PV01 and corresponds to the present value of 1 bp paid on the premium leg. The CDS price is then inversely related to the spread. At the inception date, the present value of the premium leg is equal to the present value of the default leg meaning that the CDS spread corresponds to the coupon rate such $P_t^{\mathrm{buyer}} = 0$. We obtain the following expression:

$$\boldsymbol{s} = \frac{(1-\boldsymbol{\mathcal{R}}) \int_t^T B_t(u)\, f_t(u)\, \mathrm{d}u}{\sum_{t_m \geq t} \Delta t_m \mathbf{S}_t(t_m)\, B_t(t_m)} \qquad (3.6)$$

The spread $\boldsymbol{s}$ is in fact the fair value coupon rate $\boldsymbol{c}$ in such a way that the initial value of the credit default swap is zero.

We notice that if there is no default risk, this implies that $\mathbf{S}_t(u) = 1$ and we get $\boldsymbol{s} = 0$. In the same way, the spread is also equal to zero if the recovery rate is set to to 100%. If we assume that the premium is paid continuously, the formula (3.6) becomes:

$$\boldsymbol{s} = \frac{(1-\boldsymbol{\mathcal{R}}) \int_t^T B_t(u)\, f_t(u)\, \mathrm{d}u}{\int_t^T B_t(u)\, \mathbf{S}_t(u)\, \mathrm{d}u}$$

If the interest rates are equal to zero ($B_t(u) = 1$) and the default times is exponential with parameter λ ($\mathbf{S}_t(u) = e^{-\lambda(u-t)}$ and $f_t(u) = \lambda e^{-\lambda(u-t)}$), we get:

$$\begin{aligned} \boldsymbol{s} &= \frac{(1-\boldsymbol{\mathcal{R}}) \times \lambda \int_t^T e^{-\lambda(u-t)}\, \mathrm{d}u}{\int_t^T e^{-\lambda(u-t)}\, \mathrm{d}u} \\ &= (1-\boldsymbol{\mathcal{R}}) \times \lambda \end{aligned}$$

If λ is relatively small, we also notice that this relationship can be written as follows:

$$\boldsymbol{s} \approx (1-\boldsymbol{\mathcal{R}}) \times \mathrm{PD}$$

[20] P_t is the swap price for the protection buyer. We have then $P_t^{\mathrm{buyer}}(T) = P_t(T)$ $P_t^{\mathrm{seller}}(T) = -P_t(T)$.

where PD is the one-year default probability[21]. This relationship is known as the *credit triangle* because it is a relationship between three variables where knowledge of any two is sufficient to calculate the third (O'Kane, 2008). It basically states that the CDS spread is approximatively equal to one-year loss. The spread contains also the same information than the survival function and is an increasing function of the default probability. It can then be interpreted as a credit risk measure of the reference entity.

We recall that the first CDS was traded by JP Morgan in 1994 (Augustin *et al.*, 2014). The CDS market structure has been organized since then, especially the standardization of the CDS contract. Today, CDS agreements are governed by 2003 and 2014 ISDA credit derivatives definitions. For instance, settlement of the CDS contract can be either physical or in cash. In the case of cash settlement, there is a monetary exchange from the protection seller to the protection buyer[22]. In the case of physical settlement, the protection buyer delivers a bond to the protection seller and receives the notional principal. Because the price of the defaulted bond is equal to $\boldsymbol{\mathcal{R}} \times N$, this means that the implied mark-to-market of this operation is $N - \boldsymbol{\mathcal{R}} \times N$ or equivalently $(1 - \boldsymbol{\mathcal{R}}) \times N$. Or course, physical settlement is only possible if the reference entity is a bond or if the credit event is based on the bond default. Whereas physical settlement was prevailing in the 1990s, most of the settlements are in cash today. Another standardization concerns the price of CDS. With the exception of very specific cases[23], CDS contracts are quoted in (fair) spread expressed in bps. In Figures 3.15 and 3.16, we show the evolution of some CDS spreads for a five-year maturity. We notice the increase of credit spreads since the 2008 financial turmoil and the default of Lehman Brothers' bankruptcy, the sensitivity of German and Italian spreads with respect to the Eurozone crisis and also the difference in level between the different countries. Indeed, the spread is globally lower for US than for Germany or Japan. In the case of Italy, the spread is high and has reached 600 bps in 2012. We observe that the spread of some corporate entities may be lower than the spread of many developed countries (see Figure 3.16). This is the case of Walmart, whose spread is lower than 20 bps since 2014. When a company (or a country) is in great difficulty, the CDS spread explodes as in the case of Ford in February

[21]We have:

$$
\begin{aligned}
\mathrm{PD} &= \Pr\{\boldsymbol{\tau} \leq t+1 \mid \boldsymbol{\tau} \leq t\} \\
&= 1 - \mathbf{S}_t(t+1) \\
&= 1 - e^{-\lambda} \\
&\simeq \lambda
\end{aligned}
$$

For instance, if λ is equal respectively to 5%, 10% and 20%, the one-year default probability takes the values 4.88%, 9.52% and 18.13%.

[22]This monetary exchange is equal to $(1 - \boldsymbol{\mathcal{R}}) \times N$.

[23]When the default probability is high (larger than 20%), CDS contracts can be quoted with an upfront meaning that the protection seller is asking an initial amount to enter into the swap. For instance, it was the case of CDS on Greece in spring 2013.

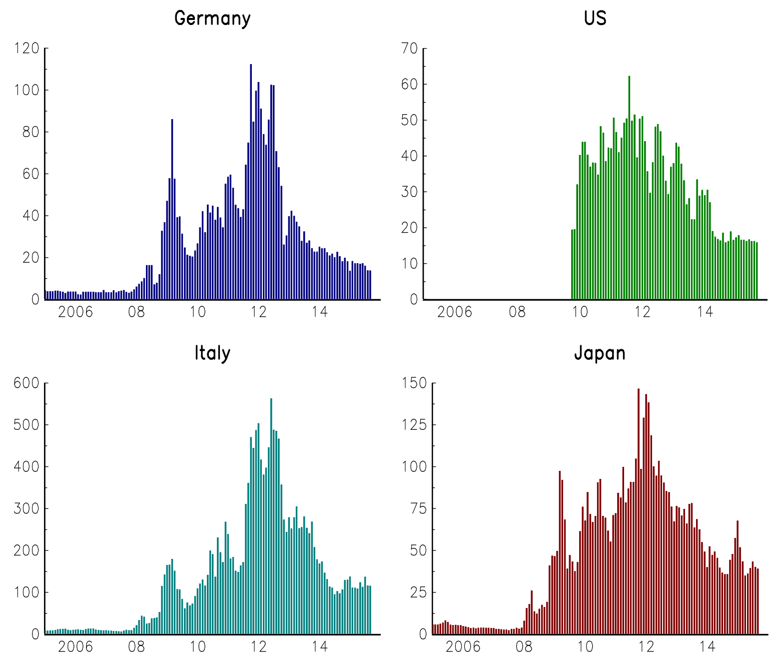

FIGURE 3.15: Evolution of some sovereign CDS spreads

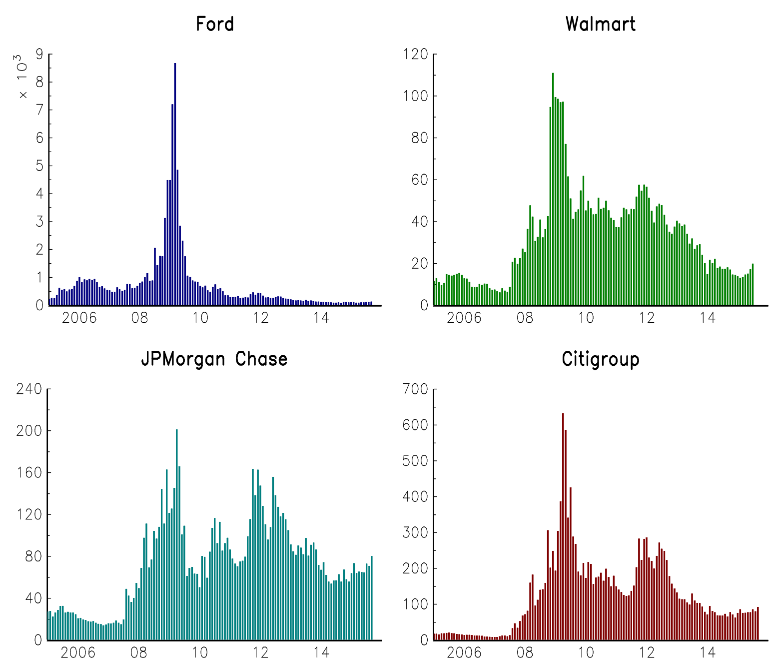

FIGURE 3.16: Evolution of some financial and corporate CDS spreads

2009. CDS spreads can be used to compare the default risk of two entities in the same sector. For instance, Figure 3.16 shows than the default risk of Citigroup is higher than this of JPMorgan Chase.

The CDS spread changes over time, but depends also on the maturity or tenor. This implies that we have a term structure of credit spreads for a given date t. This term structure is known as the credit spread curve and is noted $\mathcal{s}_t(T)$ where T is the maturity time. Figure 3.17 shows the credit curve for different entities as of 2015-09-17. We notice that the CDS spread increases with the maturity. This is the most common case for investment-grade (IG) entities, whose short-term default risk is low, but long-term default risk is higher. Nevertheless, we observe some distinguishing patterns between these credit curves. For instance, the credit risk of Germany is lower than the credit risk of US if the maturity is less than five years, but it is higher in the long run. There is a difference of 4 bps between Google and Apple on average when the time-to-maturity is less than 5 years. In the case of 10Y CDS, the spread of Apple is 90.8 bps whereas it is only 45.75 bps for Google.

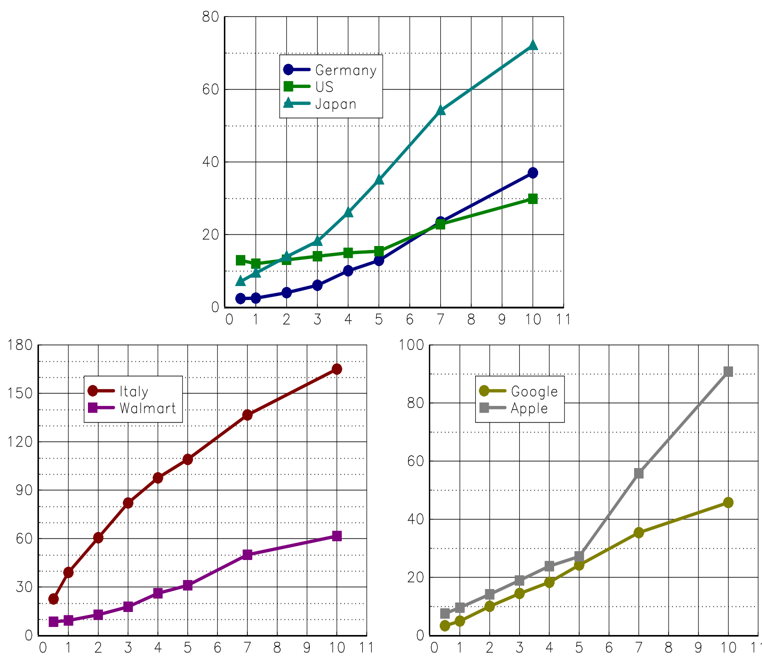

FIGURE 3.17: Examples of CDS spread curve as of 2015-09-17

Remark 26 *In other cases, the credit curve may be decreasing (for some high yield corporates) or have a complex curvature (bell-shaped or U-shaped). In fact, Longstaff* et al. *(2005) showed that the dynamics of credit default swaps also depends on the liquidity risk. For instance, the most liquid CDS contract* *is generally the 5Y CDS. The liquidity on the other maturities depends on the reference entity and other characteristics such as the bond market liquidity. For example, the liquidity may be higher for short maturities when the credit risk of the reference entity is very high.*

Initially, CDS were used to hedge the credit risk of corporate bonds by banks and insurance companies. This hedging mechanism is illustrated in Figure 3.18. We assume that the bond holder buy a protection using a CDS, whose fixing dates of the premium leg are exactly the same as the coupon dates of the bond. We also assume that the credit even is the bond default and the notional principal is equal to the notional of the bond. At each fixing date t_m, the bond holder receives the coupon $C(t_m)$ of the bond and pays to the protection seller the premium $\boldsymbol{s} \times N$. This implies that the net cash flow is $C(t_m) - \boldsymbol{s} \times N$. If the default occurs, the value of the bond becomes $\boldsymbol{\mathcal{R}} \times N$, but the protection seller pays to the bond holder the default leg $(1 - \boldsymbol{\mathcal{R}}) \times N$. In case of default, the net cash flow is then equal to $\boldsymbol{\mathcal{R}} \times N + (1 - \boldsymbol{\mathcal{R}}) \times N = N$, meaning that the exposure on the defaultable bond is perfectly hedged. We deduce that the annualized return R of this hedged portfolio is the difference between the yield to maturity $\boldsymbol{y}$ of the bond and the annual cost $\boldsymbol{s}$ of the protection:

$$R = \boldsymbol{y} - \boldsymbol{s} \tag{3.7}$$

We recognize a new formulation of Equation (3.4) in page 150. In theory, R is then equal to the yield to maturity $\boldsymbol{y}^{\star}$ of the bond without credit risk.

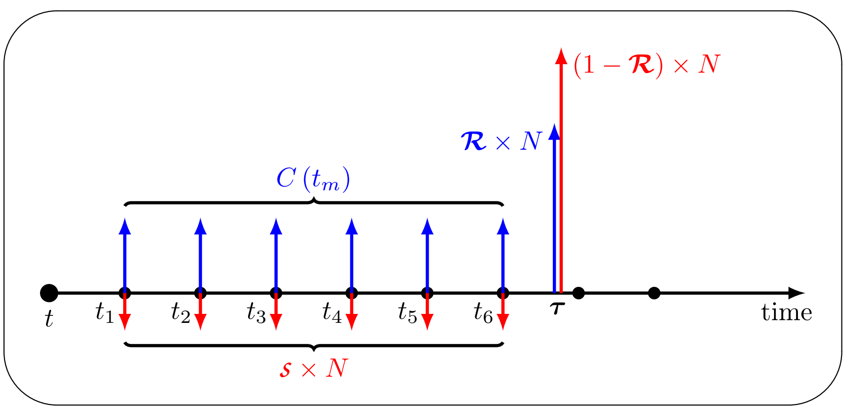

FIGURE 3.18: Hedging a defaultable bond with a credit default swap

Since the 2000s, end-users of CDS are banks and securities firms, insurance firms including pension funds, hedge funds and mutual funds. They continue to be used as hedging instruments, but they also become financial instruments to express views about credit risk. In this case, *long credit* refers to the position of the protection seller who is exposed to the credit risk, whereas *short credit* is the position of the protection buyer who sold the credit risk of the reference entity. To understand the mark-to-market of such positions, we consider the initial position at the inception date t of the CDS contract. In this case, the CDS spread $\mathcal{s}_t(T)$ verifies that the face value of the swap is equal to zero. Let us introduce the notation $P_{t,t'}(T)$, which defines the market-to-market of a CDS position whose inception date is t, valuation date is t' and maturity date is T. We have:

$$P_{t,t}^{\text{seller}}(T) = P_{t,t}^{\text{buyer}}(T) = 0$$

At date $t' > t$, the market-to-market price of the CDS is:

$$P_{t,t'}^{\text{buyer}}(T) = N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_t(T) \times \mathrm{RPV}_{01}\right)$$

whereas the value of the CDS spread satisfies the following relationship:

$$\begin{aligned} P_{t',t'}^{\text{buyer}}(T) &= N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_{t'}(T) \times \mathrm{RPV}_{01}\right) \\ &= 0 \end{aligned}$$

We deduce that the P&L of the protection buyer is:

$$\begin{aligned} \Pi^{\text{buyer}} &= P_{t,t'}^{\text{buyer}}(T) - P_{t,t}^{\text{buyer}}(T) \\ &= P_{t,t'}^{\text{buyer}}(T) \end{aligned}$$

Using Equation (3.8), we know that $P_{t',t'}^{\text{buyer}}(T) = 0$ and we obtain:

$$\begin{aligned} \Pi^{\text{buyer}} &= P_{t,t'}^{\text{buyer}}(T) - P_{t',t'}^{\text{buyer}}(T) \\ &= N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_t(T) \times \mathrm{RPV}_{01}\right) - \\ &\quad\; N\left((1-\boldsymbol{\mathcal{R}})\int_{t'}^{T} B_{t'}(u) f_{t'}(u)\,\mathrm{d}u - \mathcal{s}_{t'}(T) \times \mathrm{RPV}_{01}\right) \\ &= N \times (\mathcal{s}_{t'}(T) - \mathcal{s}_t(T)) \times \mathrm{RPV}_{01} \end{aligned} \tag{3.8}$$

This equation highlights the role of the term RPV_{01} when calculating the P&L of the CDS position. Because $\Pi^{\text{seller}} = -\Pi^{\text{buyer}}$, we distinguish two cases

- If $\mathcal{s}_{t'}(T) > \mathcal{s}_t(T)$, the protection buyer made a profit, because this short credit exposure have benefited from the increase of the default risk.
- If $\mathcal{s}_{t'}(T) < \mathcal{s}_t(T)$, the protection seller made a profit, because the default risk of the reference entity has decreased.

Suppose that we are in the first case. To realize its P&L, the protection buyer has three options (O'Kane, 2008):

1. He could unwind the CDS exposure with the protection seller if the latter agrees. This implies that the protection seller pays the mark-to-market $P_{t,t'}^{\text{buyer}}(T)$ to the protection buyer.

2. He could hedge the mark-to-market value by selling a CDS on the same reference entity and the same maturity. In this situation, he continues to pay the spread $\mathscr{s}_t(T)$, but he now receives a premium, whose spread is equal to $\mathscr{s}_{t'}(T)$.

3. He could reassign the CDS contract to another counterparty as illustrated in Figure 3.19. The new counterparty (the protection buyer C in our case) will then pay the coupon rate $\mathscr{s}_t(T)$ to the protection seller. However, the spread is $\mathscr{s}_{t'}(T)$ at time t', which is higher than $\mathscr{s}_t(T)$. This is why the new counterparty also pays the mark-to-market $P_{t,t'}^{\text{buyer}}(T)$ to the initial protection buyer.

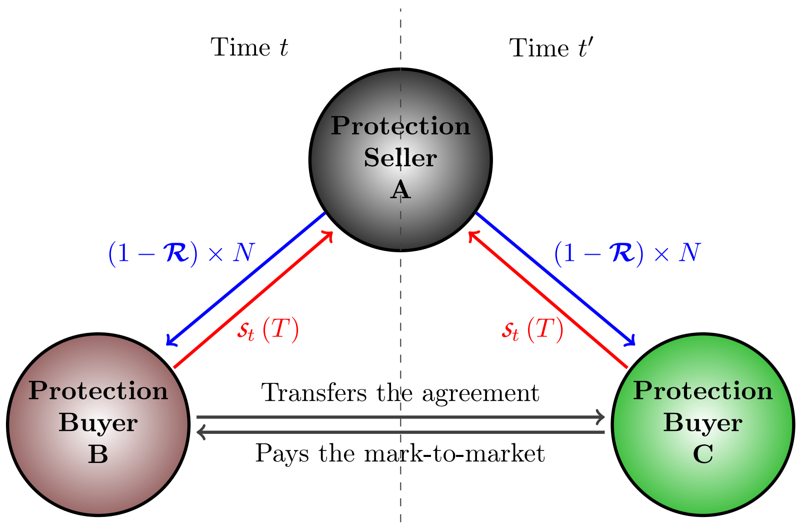

FIGURE 3.19: An example of CDS offsetting

Remark 27 *When the default risk is very high, CDS are quoted with an upfront*[24]*. In this case, the annual premium leg is equal to $\boldsymbol{c}^\star \times N$ where $\boldsymbol{c}^\star$ is a default value*[25]*, and the protection buyer has to pay an upfront UF_t to the protection seller defined as follows:*

$$UF_t = N\left((1-\boldsymbol{\mathcal{R}})\int_t^T B_t(u)\, f_t(u)\,\mathrm{d}u - \boldsymbol{c}^\star \times \mathrm{RPV}_{01}\right)$$

[24] It was the case several times for CDS on Greece

[25] For distressed names, the default coupon rate $\boldsymbol{c}^\star$ is typically equal to 500 bps.

Remark 28 *Until now, we have simplified the pricing of the premium leg in order to avoid complicating calculations. Indeed, if the default occurs between two fixing dates, the protection buyer has to pay the premium accrual. For instance, if* $\boldsymbol{\tau} \in \left]t_{m-1}, t_m\right[$*, the accrued premium is equal to* $\boldsymbol{c} \times N \times (\boldsymbol{\tau} - t_{m-1})$ *or equivalently to:*

$$\mathcal{AP} = \sum_{t_m \geq t} \boldsymbol{c} \times N \times (\boldsymbol{\tau} - t_{m-1}) \times \mathbb{1}\left\{t_{m-1} \leq \boldsymbol{\tau} \leq t_m\right\}$$

We deduce that the stochastic discount value of the accrued premium is :

$$SV_t\left(\mathcal{AP}\right) = \sum_{t_m \geq t} \boldsymbol{c} \times N \times (\boldsymbol{\tau} - t_{m-1}) \times \mathbb{1}\left\{t_{m-1} \leq \boldsymbol{\tau} \leq t_m\right\} \times e^{-\int_t^{\boldsymbol{\tau}} r_s \,\mathrm{d}s}$$

It follows that:

$$PV_t\left(\mathcal{AP}\right) = \boldsymbol{c} \times N \times \sum_{t_m \geq t} \int_{t_{m-1}}^{t_m} (u - t_{m-1})\, B_t(u)\, f_t(u)\,\mathrm{d}u$$

All the previous formulas remain valid by replacing the expression of the risky PV01 by the following term:

$$\mathrm{RPV}_{01} = \sum_{t_m \geq t} \left(\Delta t_m \mathbf{S}_t(t_m)\, B_t(t_m) + \int_{t_{m-1}}^{t_m} (u - t_{m-1})\, B_t(u)\, f_t(u)\,\mathrm{d}u \right) \tag{3.9}$$

Example 25 *We assume that the yield curve is generated by the Nelson-Siegel model with the following parameters:* $\theta_1 = 5\%$*,* $\theta_2 = -5\%$*,* $\theta_3 = 6\%$ *and* $\theta_4 = 10$*. We consider several credit default swaps on the same entity with quarterly coupons and a notional of* \$1 *mn. The recovery rate* $\boldsymbol{\mathcal{R}}$ *is set to* 40% *whereas the default time* $\boldsymbol{\tau}$ *is an exponential random variable, whose parameter* λ *is equal to* 50 *bps. We consider seven maturities (6M, 1Y, 2Y, 3Y, 5Y, 7Y and 10Y) and two coupon rates (*10 *and* 100 *bps).*

To calculate the prices of these CDS, we use Equation (3.5) with $N = 10^6$, $\boldsymbol{c} = 10 \times 10^{-4}$ or $\boldsymbol{c} = 100 \times 10^{-4}$, $\Delta t_m = 1/4$, $\boldsymbol{\mathcal{R}} = 0.40$, $\mathbf{S}_t(u) = e^{-50 \times 10^{-4} \times (u-t)}$, $f_t(u) = 50 \times 10^{-4} \times e^{-50 \times 10^{-4} \times (u-t)}$ and $B_t(u) = e^{-(u-t)R_t(u)}$ where the zero-coupon rate is given by Equation (3.1). To evaluate the integral, we consider a Gauss-Legendre quadrature of 128$^{\text{th}}$ order. By including the accrued premium[26], we obtain results reported in Table 3.8. For instance, the price of the 5Y CDS is equal to \$9 527 if $\boldsymbol{c} = 10 \times 10^{-4}$ and \$ − 33 173 if $\boldsymbol{c} = 100 \times 10^{-4}$. In the first case, the protection buyer has to pay an upfront to the protection seller, because the coupon rate is too low. In the

[26] This means that the risky PV01 corresponds to Equation (3.9). We also report results without taking into account the accrued premium in Table 3.9. We notice that its impact is limited.

second case, the protection buyer receives the upfront, because the coupon rate is too high. We also indicates the spread $\mathcal{s}$ and the risky PV01. We notice that the CDS spread is almost constant. This is normal since the default rate is constant. This is why the CDS spread is approximatively equal to $(1 - 40\%)$ or 30 bps. The difference between the several maturities is due to the yield curve. The risky PV01 is a useful statistic to compute the mark-to-market. Suppose for instance that the two parties entered in a 7Y credit default swap of 10 bps spread two years ago. Now, the residual maturity of the swap is five years, meaning that the mark-to-market of the protection buyer is equal to:

$$\begin{aligned}\Pi^{\text{buyer}} &= 10^6 \times \left(30.08 \times 10^{-4} - 10 \times 10^{-4}\right) \times 4.744 \\ &= \$9\,526\end{aligned}$$

We retrieve the 5Y CDS price (subject to rounding error).

TABLE 3.8: Price, spread and risky PV01 of CDS contracts

T	$P_t(T)$ $\boldsymbol{c} = 10$	$P_t(T)$ $\boldsymbol{c} = 100$	$\mathcal{s}$	RPV_{01}
1/2	998	−3 492	30.01	0.499
1	1 992	−6 963	30.02	0.995
2	3 956	−13 811	30.04	1.974
3	5 874	−20 488	30.05	2.929
5	9 527	−33 173	30.08	4.744
7	12 884	−44 804	30.10	6.410
10	17 314	−60 121	30.12	8.604

TABLE 3.9: Price, spread and risky PV01 of CDS contracts (without the accrued premium)

T	$P_t(T)$ $\boldsymbol{c} = 10$	$P_t(T)$ $\boldsymbol{c} = 100$	$\mathcal{s}$	RPV_{01}
1/2	999	−3 489	30.03	0.499
1	1 993	−6 957	30.04	0.994
2	3 957	−13 799	30.06	1.973
3	5 876	−20 470	30.07	2.927
5	9 530	−33 144	30.10	4.742
7	12 888	−44 764	30.12	6.406
10	17 319	−60 067	30.14	8.598

Example 26 *We consider a variant of Example 25 by assuming that the default time follows a Gompertz distribution:*

$$\mathbf{S}_t(u) = \exp\left(\phi\left(1 - e^{\gamma(u-t)}\right)\right)$$

The parameters ϕ and γ are set to 5% and 10%.

Results are reported in Table 3.10. In this example, the spread is increasing with the maturity of the CDS. Until now, we have assumed that we know the survival function $\mathbf{S}_t(u)$ in order to calculate the CDS spread. However, in practice, the CDS spread $\mathcal{s}$ is a market price and $\mathbf{S}_t(u)$ has to be determined thanks to a calibration procedure. Suppose for instance that we postulate that $\boldsymbol{\tau}$ is an exponential default time with parameter λ. We can calibrate the estimated value $\hat{\lambda}$ such that the theoretical price is equal to the market price. For instance, Table 3.10 shows the parameter $\hat{\lambda}$ for each CDS. We found that $\hat{\lambda}$ is equal to 51.28 bps for the six-month maturity and 82.92 bps for the ten-year maturity. We face here an issue, because the parameter $\hat{\lambda}$ is not constant, meaning that we cannot use an exponential distribution to represent the default time of the reference entity. This is why we generally consider a more flexible survival function to calibrate the default probabilities from a set of CDS spreads[27].

TABLE 3.10: Calibration of the CDS spread curve using the exponential model

T	$P_t(T)$ $\boldsymbol{c}=10$	$P_t(T)$ $\boldsymbol{c}=100$	$\mathcal{s}$	RPV_{01}	$\hat{\lambda}$
1/2	1 037	−3 454	30.77	0.499	51.28
1	2 146	−6 808	31.57	0.995	52.59
2	4 585	−13 175	33.24	1.973	55.34
3	7 316	−19 026	35.00	2.927	58.25
5	13 631	−28 972	38.80	4.734	64.54
7	21 034	−36 391	42.97	6.380	71.44
10	33 999	−42 691	49.90	8.521	82.92

3.1.3.3 Basket default swap

A basket default swap is similar to a credit default swap except that the underlying is a basket of reference entities rather than one single reference entity. These products are part of multi-name credit default swaps with collateralized debt obligations.

First-to-default and k^{th}-to-default credit derivatives Let us consider a credit portfolio with n reference entities, which are referenced by the index i. With a first-to-default (FtD) credit swap, the credit event occurs the first time that a reference entity of the credit portfolio defaults. We deduce that

[27] This problem will be solved later in Section 3.3.3.1 of this chapter.

the stochastic discounted values of the premium and default legs are[28]:

$$SV_t(\mathcal{PL}) = \boldsymbol{c} \times N \times \sum_{t_m \geq t} \Delta t_m \times \mathbb{1}\left\{\boldsymbol{\tau}_{1:n} > t_m\right\} \times e^{-\int_t^{t_m} r(s)\,\mathrm{d}s}$$

and:

$$SV_t(\mathcal{DL}) = X \times \mathbb{1}\left\{\boldsymbol{\tau}_{1:n} \leq T\right\} \times e^{-\int_t^{\boldsymbol{\tau}_{1:n}} r_s\,\mathrm{d}s}$$

where $\boldsymbol{\tau}_i$ is the default time of the i^{th} reference entity, $\boldsymbol{\tau}_{1:n} = \min(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ is the first default time in the portfolio and X is the payout of the protection leg:

$$\begin{aligned} X &= \sum_{i=1}^{n} \mathbb{1}\left\{\boldsymbol{\tau}_{1:n} = \tau_i\right\} \times (1 - \boldsymbol{\mathcal{R}}_i) \times N_i \\ &= (1 - \boldsymbol{\mathcal{R}}_{i^\star}) \times N_{i^\star} \end{aligned}$$

In this formula, $\boldsymbol{\mathcal{R}}_i$ and N_i are respectively the recovery and the notional of the i^{th} reference entity whereas the index $i^\star = \{i : \boldsymbol{\tau}_i = \boldsymbol{\tau}_{1:n}\}$ corresponds to the first reference entity that defaults. For instance, if the portfolio is composed by 10 names and the third name is the first default, the value of the protection leg will be $(1 - \boldsymbol{\mathcal{R}}_3) \times N_3$. Using the same assumptions than previously, we deduce that the FtD spread is:

$$\boldsymbol{s}^{\text{FtD}} = \frac{\mathbb{E}\left[X \times \mathbb{1}\left\{\boldsymbol{\tau}_{1:n} \leq T\right\} \times B_t(\boldsymbol{\tau}_{1:n})\right]}{N \sum_{t_m \geq t} \Delta t_m \times \mathbf{S}_{1:n,t}(t_m) \times B_t(t_m)}$$

where $\mathbf{S}_{1:n,t}(u)$ is the survival function of $\boldsymbol{\tau}_{1:n}$. If we assume a homogenous basket (same recovery $\boldsymbol{\mathcal{R}}_i = \boldsymbol{\mathcal{R}}$ and same notional $N_i = N$), the previous formula becomes:

$$\boldsymbol{s}^{\text{FtD}} = \frac{(1 - \boldsymbol{\mathcal{R}}) \int_t^T B_t(u)\, f_{1:n,t}(u)\,\mathrm{d}u}{\sum_{t_m \geq t} \Delta t_m \mathbf{S}_{1:n,t}(t_m)\, B_t(t_m)} \tag{3.10}$$

where $f_{1:n,t}(u)$ is the survival function of $\boldsymbol{\tau}_{1:n}$.

To compute the spread, we use Monte Carlo simulation (or numerical integration when the number of entities is small[29]). In fact, the survival function of $\boldsymbol{\tau}_{1:n}$ is related to the individual survival functions, but also to the dependence between the default times $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n$. The spread of the FtD is then a function of default correlations[30]. If we denote by $\boldsymbol{s}_i^{\text{CDS}}$ the CDS spread of the i^{th} reference, we can show that:

$$\max\left(\boldsymbol{s}_1^{\text{CDS}}, \ldots, \boldsymbol{s}_n^{\text{CDS}}\right) \leq \boldsymbol{s}^{\text{FtD}} \leq \sum_{i=1}^{n} \boldsymbol{s}_i^{\text{CDS}} \tag{3.11}$$

[28]In order to simplify the notations, we do not take into account the accrued premium.

[29]Laurent and Gregory (2005) provide semi-explicit formulas that are useful for pricing basket credit swaps.

[30]This point will be developed in Section 3.3.3 and in Chapter 15 dedicated on Copula functions.

When the default times are uncorrelated, the FtD is equivalent to buy the basket of all credit defaults swaps. In the case of perfect correlations, one default is immediately followed by the other $n-1$ defaults, implying that the FtD is equivalent to the CDS with the worst spread. In practice, the FtD spread is therefore located between these two bounds as expressed in Equation (3.11). From the viewpoint of the protection buyer, a FtD is seen as a hedging method of the credit portfolio with a lower cost than buying the protection for all the credits. For example, suppose that the protection buyer would like to be hedged to the default of the automobile sector. He can buy a FtD on the basket of the largest car manufacturers in the world, e.g. Volskswagen, Toyota, Hyundai, General Motors, Fiat Chrysler and Renault. If there is only one default, the protection buyer is hedged. However, the protection buyer keeps the risk of multiple defaults, which is a worst-case scenario.

Remark 29 *The previous analysis can be extended to k^{th}-to-default swaps. In this case, the default leg is paid if the k^{th} default occurs before the maturity date. We then obtain a similar expression as Equation (3.10) by considering $\boldsymbol{\tau}_{k:n}$ in place of $\boldsymbol{\tau}_{1:n}$.*

From a theoretical point of view, it is equivalent to buy the CDS protection for all the components of the credit basket or to buy all the k^{th}-to-default swaps. We have therefore the following relationship:

$$\sum_{i=1}^{n} \mathcal{s}_i^{\text{CDS}} = \sum_{i=1}^{n} \mathcal{s}^{i:n} \tag{3.12}$$

We see that the default correlation highly impacts the distribution of the k^{th}-to-default spreads.

Credit default indices Credit derivatives indices[31] have been first developed by J.P. Morgan, Morgan Stanley and iBoxx between 2001 and 2003. A credit default index (or CDX) is in fact a credit default swap on a basket of reference entities. As previously, we consider a portfolio with n credit entities. The protection buyer pays a premium leg with a coupon rate $\boldsymbol{c}$. Every time a reference entity defaults, the notional is reduced by a factor, which is equal to $1/n$. At the same time, the protection buyer receives the portfolio loss between two fixing dates. The expression of the outstanding notional is then given by:

$$N_t(u) = N \times \left(1 - \frac{1}{n}\sum_{i=1}^{n} \mathbb{1}\{\boldsymbol{\tau}_i \le u\}\right)$$

At the inception date, we verify that $N_t(t) = N$. After the first default, the outstanding notional is equal to $N(1 - 1/n)$. After the k^{th} default, its value is

[31] They are also known as synthetic credit indices, credit default swap indices or credit default indices.

$N\left(1-{}^{k}\!/\!{}_{n}\right)$. At time $u \geq t$, the cumulative loss of the credit portfolio is:

$$L_t(u) = \frac{1}{n}\sum_{i=1}^{n} N \times (1-\boldsymbol{\mathcal{R}}_i) \times \mathbb{1}\{\boldsymbol{\tau}_i \leq u\}$$

meaning that the incremental loss between two fixing dates is:

$$\Delta L_t(t_m) = L_t(t_m) - L_t(t_{m-1})$$

We deduce that the stochastic discounted values of the premium and default legs are:

$$SV_t(\mathcal{PL}) = \boldsymbol{c} \times \sum_{t_m \geq t} \Delta t_m \times N_t(t_m) \times e^{-\int_t^{t_m} r_s \, \mathrm{d}s}$$

and:

$$SV_t(\mathcal{DL}) = \sum_{t_m \geq t} \Delta L_t(t_m) \times e^{-\int_t^{t_m} r_s \, \mathrm{d}s}$$

We deduce that the spread of the CDX is:

$$\boldsymbol{s}^{\mathrm{CDX}} = \frac{\mathbb{E}\left[\sum_{t_m \geq t} \Delta L_t(t_m) \times B_t(t_m)\right]}{\mathbb{E}\left[\sum_{t_m \geq t} \Delta t_m \times N_t(t_m) \times B_t(t_m)\right]} \tag{3.13}$$

Remark 30 *A CDX is then equivalent to a portfolio of CDS whose each principal notional is equal to N/n. Indeed, when a default occurs, the protection buyer receives $N/n \times (1-\boldsymbol{\mathcal{R}}_i)$ and stops to pay the premium leg of the defaulted reference entity. At the inception date, the annual premium of the CDX is then equal to the annual premium of the CDS portfolio:*

$$\boldsymbol{s}^{\mathrm{CDX}} \times N = \sum_{i=1}^{n} \boldsymbol{s}_i^{\mathrm{CDS}} \times \frac{N}{n}$$

We deduce that the spread of the CDX is an average of the credit spreads that compose the portfolio[32]*:*

$$\boldsymbol{s}^{\mathrm{CDX}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{s}_i^{\mathrm{CDS}} \tag{3.14}$$

Today, credit default indices are all managed by Markit and have been standardized. For instance, coupon payments are made on a quarterly basis (March 20, June 20, September 20, December 20) whereas indices roll every six months with an updated portfolio[33]. With respect to the original credit indices, Markit continues to produces two families:

[32]In fact, this is an approximation because the payment of the default leg does not exactly coincide between the CDX and the CDS portfolio.

[33]See Markit (2014) for a detailed explanation of the indices' construction.

- Markit CDX
  It focuses on North America and Emerging Markets credit default indices. The three major sub-indices are IG (investment grade), HY (high yield) and EM (emerging markets). A more comprehensive list is provided in Table 3.11. Besides these credit default indices, Markit CDX produces also four other important indices: ABX (basket of ABS), CMBX (basket of CMBS), LCDX (portfolio of 100 US secured senior loans) and MCDX (basket of 50 municipal bonds).

- Markit iTraxx
  It focuses on Europe, Japan, Asia ex-Japan and Australia (see the list in Table 3.12). Markit iTraxx also produces LevX (portfolio of 40 European secured loans), sector indices (e.g. European Financials and Industrials) and SovX, which corresponds to a portfolio of sovereign issuers. There are 7 SovX indices: Asia Pacific, BRIC, CEEMEA[34], G7, Latin America, Western Europe and Global Liquid IG.

TABLE 3.11: List of Markit CDX main indices

Index name	Description	n	$\mathcal{R}$
CDX.NA.IG	Investment grade entities	125	40%
CDX.NA.IG.HVOL	High volatility IG entities	30	40%
CDX.NA.XO	Crossover entities	35	40%
CDX.NA.HY	High yield entities	100	30%
CDX.NA.HY.BB	High yield BB entities	37	30%
CDX.NA.HY.B	High yield B entities	46	30%
CDX.EM	EM sovereign issuers	14	25%
LCDX	Secured senior loans	100	70%
MCDX	Municipal bonds	50	80%

TABLE 3.12: List of Markit iTraxx main indices

Index name	Description	n	$\mathcal{R}$
iTraxx Europe	European IG entities	125	40%
iTraxx Europe HiVol	European HVOL IG entities	30	40%
iTraxx Europe Crossover	European XO entities	40	40%
iTraxx Asia	Asian (ex-Japan) IG entities	50	40%
iTraxx Asia HY	Asian (ex-Japan) HY entities	20	25%
iTraxx Australia	Australian IG entities	25	40%
iTraxx Japan	Japanese IG entities	50	35%
iTraxx SovX G7	G7 governments	7	40%
iTraxx LevX	European leveraged loans	40	40%

Source: Markit (2014).

[34]Central and Eastern Europe, Middle East and Africa.

In Table 3.13, we report the spread of some CDX/iTraxx indices. We note that the spread of the CDX.NA.HY index is on average four times larger than the spread of the CDX.NAQ.IG index. While spreads of credit default indices have generally decrease between December 2012 and December 2014, we observe a reversal in 2015. For instance, the spread of the CDX.NA.IG index is equal to 93.6 bps in September 2015 whereas it was only equal to 66.3% nine months ago. We observe a similar increase of 30 bps for the iTraxx Europe index. For the CDX.NA.HY index, it is more impressive with a variation of +150 bps in nine months.

TABLE 3.13: Historical spread of CDX/iTraxx indices (in bps)

Date	CDX			iTraxx		
	NA.IG	NA.HY	EM	Europe	Japan	Asia
Dec. 2012	94.1	484.4	208.6	117.0	159.1	108.8
Dec. 2013	62.3	305.6	272.4	70.1	67.5	129.0
Dec. 2014	66.3	357.2	341.0	62.8	67.0	106.0
Sep. 2015	93.6	505.3	381.2	90.6	82.2	160.5

3.1.3.4 Collateralized debt obligations

A collateralized debt obligation (CDO) is another form of multi-name credit default swaps. It corresponds to a pay-through ABS structure[35], whose securities are bonds linked to a series of tranches. If we consider the example given in Figure 3.20, they are 4 types of bonds, whose returns depends on the loss of the corresponding tranche (equity, mezzanine, senior and super senior). Each tranche is characterized by an attachment point A and a detachment point D. In our example, we have:

Tranche	Equity	Mezzanine	Senior	Super senior
A	0%	15%	25%	35%
D	15%	25%	35%	100%

The protection buyer of the tranche $[A, D]$ pays a coupon rate $\mathbf{c}^{[A,D]}$ on the outstanding nominal of the tranche to the protection seller. In return, he receives the protection leg, which is the loss of the tranche $[A, D]$. However, the losses satisfy a payment priority which is the following:

- the equity tranche is the most risky security, meaning that the first losses hit this tranche alone until the cumulative loss reaches the detachment point;
- from the time the portfolio loss is larger than the detachment point of the equity tranche, the equity tranche no longer exists and this is the

[35]See Figure 3.12 in page 154.

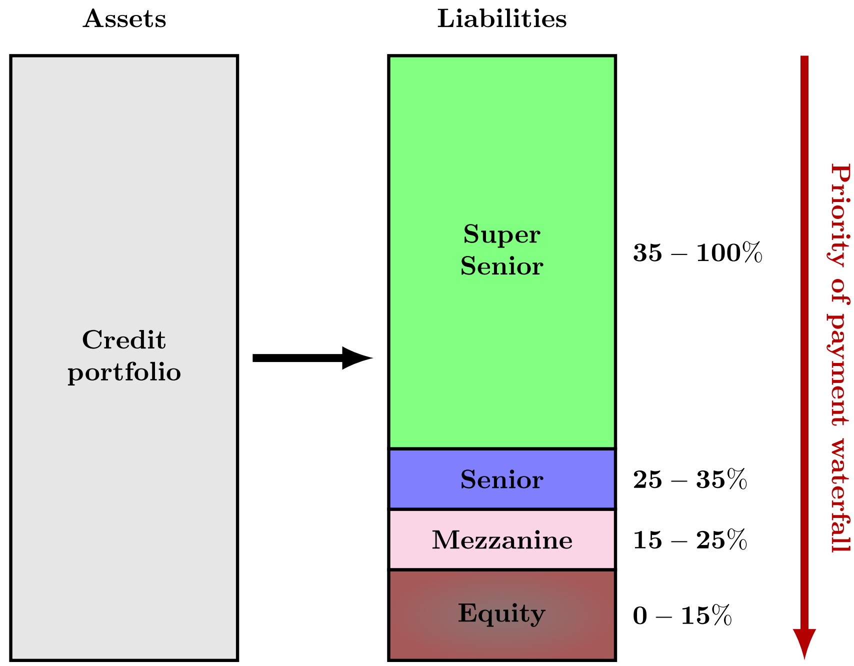

FIGURE 3.20: Structure of a collateralized debt obligation

protection seller of the mezzanine tranche, who will pay the next losses to the protection buyer of the mezzanine tranche;

- the protection buyer of a tranche pays the coupon from the inception of the CDO until the death of the tranche, when the cumulative loss is larger than the detachment point of the tranche; moreover, the premium payments are made on the reduced notional after each credit events of the tranche.

Each CDO tranche can then be viewed as a CDS with a time-varying notional principal to define the premium leg and a protection leg, which is paid if the portfolio loss is between the attachment and detachment points of the tranche. We can therefore interpret a CDO as a basket default swap, where the equity, mezzanine, senior and super senior tranches correspond respectively to a first-to-default, second-to-default, third-to-default and last-to-default swaps.

Let us now see the mathematical framework to price a CDO tranche. Assuming a portfolio of n credits, the cumulative loss is equal to:

$$L_t(u) = \sum_{i=1}^{n} N_i \times (1 - \boldsymbol{\mathcal{R}}_i) \times \mathbb{1}\{\boldsymbol{\tau}_i \leq u\}$$

whereas the loss of the tranche $[A, D]$ is given by[36]:

$$\begin{aligned} L_t^{[A,D]}(u) &= (L_t(u) - A) \times \mathbb{1}\{A \le L_t(u) \le D\} + \\ & \quad (D - A) \times \mathbb{1}\{L_t(u) > D\} \end{aligned}$$

where A and D are the attachment and detachment points expressed in \$. The outstanding nominal of the tranche is therefore:

$$N_t^{[A,D]}(u) = (D - A) - L_t^{[A,D]}(u)$$

This notional principal decreases then by the loss of the tranche. At the inception of the CDO, $N_t^{[A,D]}(t)$ is equal to the tranche width $(D - A)$. At the maturity date T, we have:

$$\begin{aligned} N_t^{[A,D]}(T) &= (D - A) - L_t^{[A,D]}(T) \\ &= \begin{cases} (D - A) \text{ if } L_t(T) \le A \\ (L_t(T) - A) \text{ if } A < L_t(T) \le D \\ 0 \text{ if } L_t(T) > D \end{cases} \end{aligned}$$

We deduce that the stochastic discounted values of the premium and default legs are:

$$SV_t(\mathcal{PL}) = \mathcal{c}^{[A,D]} \times \sum_{t_m \ge t} \Delta t_m \times N_t^{[A,D]}(t_m) \times e^{-\int_t^{t_m} r_s \, \mathrm{d}s}$$

and:

$$SV_t(\mathcal{DL}) = \sum_{t_m \ge t} \Delta L_t^{[A,B]}(t_m) \times e^{-\int_t^{t_m} r_s \, \mathrm{d}s}$$

We deduce that the spread of the CDO tranche is[37]:

$$\mathcal{s}^{[A,D]} = \frac{\mathbb{E}\left[\sum_{t_m \ge t} \Delta L_t^{[A,D]}(t_m) \times B_t(t_m)\right]}{\mathbb{E}\left[\sum_{t_m \ge t} \Delta t_m \times N_t^{[A,D]}(t_m) \times B_t(t_m)\right]} \tag{3.15}$$

We obviously have the following inequalities:

$$\mathcal{s}^{\text{Equity}} > \mathcal{s}^{\text{Mezzanine}} > \mathcal{s}^{\text{Senior}} > \mathcal{s}^{\text{Super senior}}$$

As in the case of k^{th}-to-default swaps, the distribution of these tranche spreads highly depend on the default correlation [38]. Depending on the model and

---

[36] Another expression is:

$$L_t^{[A,D]}(u) = \min\left(D - A, (L_t(u) - A)^+\right)$$

[37] This formula is obtained by assuming no upfront and accrued interests.

[38] See Section 15.5.2 in Chapter 15.

the parameters, we can therefore promote the protection buyer/seller of one specific tranche with respect to the other tranches.

When collateralized debt obligations emerged in the 1990s, they were used to transfer credit risk from the balance sheet of banks to investors (e.g. insurance companies). They were principally portfolios of loans (CLO) or asset-backed securities (ABS CDO). With these balanced-sheet CDO, banks could recover regulatory capital in order to issue new credits. In the 2000s, a new type of CDO was created by considering CDS portfolios as underlying assets. These synthetic CDO are also called arbitrage CDO, because they have used by investors to express their market views on credit.

The impressive success of CDO with investors before the 2008 financial crisis is due to the rating mechanism of tranches. Suppose that the underlying portfolio is composed of BB rated credits. It is obvious that the senior and super senior tranches will be rated higher than BB, because the probability that these tranches will be impacted is very low. The slicing approach of CDO enables then to create high-rated securities from medium or low-rated debts. Since the appetite of investors for AAA and AA rated bonds was very important, CDO were solutions to meet this demand. Moreover,this lead to rating method in order to provide an attractive spread. This explains that most of AAA-rated CDO tranches promised a return higher than AAA-rated sovereign and corporate bonds. In fact, the 2008 financial crisis has demonstrated that many CDO tranches were more risky than expected, because the riskiness of the assets were underestimated[39].

TABLE 3.14: List of Markit credit default tranches

Index name	Tranche					
CDX.NA.IG	$0-3$	$3-7$	$7-15$	$15-100$		
CDX.NA.HY	$0-10$	$10-15$	$15-25$	$25-35$	$35-100$	
LCDX	$0-5$	$5-8$	$8-12$	$12-15$	$15-100$	
iTraxx Europe	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$	$22-100$
iTraxx Europe XO	$0-10$	$10-15$	$15-25$	$25-35$	$35-100$	
iTraxx Asia	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$	
iTraxx Australia	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$	
iTraxx Japan	$0-3$	$3-6$	$6-9$	$9-12$	$12-22$	

Source: Markit (2014).

For some years now, CDO have been created using credit default indices as the underlying portfolio. For instance, Table 3.14 provides the list of available tranches on Markit indices[40]. We notice that attachment and detachment points differ from one index to another index. The first tranche always indi-

[39]More details of the impact of the securitization market on the 2008 financial crisis are developed in Chapter 12 dedicated to systemic risk.

[40]They are also called credit default tranches (CDT).

cates the equity tranche. For IG underlying assets, the notional corresponds to the first 3% losses of the portfolio, whereas the detachment point is higher for crossover or high yield assets. We also notice that some senior tranches are not traded (Asia, Australia and Japan). These products are mainly used in correlation trading and also served as benchmarks for all the other OTC credit debt obligations.

3.2 Capital requirements

This section deals with regulatory aspects of credit risk. From an historical point of view, this is the first risk, which are required regulatory capital before market risk. Nevertheless, the development of credit risk management is more recent and was accelerated with the Basel II Accord. Before presenting the different approaches for calculating capital requirements, we need to define more precisely what credit risk is.

It is the risk of loss on a debt instrument resulting from the failure of the borrower to make required payments. We generally distinguish two types of credit risk. The first one is the *default risk*, which arises when the borrower is unable to pay the principal or interests. An example is a student loan or a mortgage loan. The second type is the *downgrading risk*, which concerns debt securities. In this case, the debt holder may face a loss, because the price of the debt security is directly related to the credit risk of the borrower. For instance, the price of the bond may go down because the credit risk of the issuer increases and even if the borrower does not default. Of course, default risk and downgrading risk are highly correlated, because it is rare that a counterparty suddenly defaults without downgrading of its credit rating.

To measure credit risk, we first eed to define the default of the obligor. BCBS (2006) provides the following standard definition:

> "*A default is considered to have occurred with regard to a particular obligor when either or both of the two following events have taken place.*
>
> - *The bank considers that the obligor is unlikely to pay its credit obligations to the banking group in full, without recourse by the bank to actions such as realizing security (if held).*
> - *The obligor is past due more than 90 days on any material credit obligation to the banking group. Overdrafts will be considered as being past due once the customer has breached an advised limit or been advised of a limit smaller than current outstandings*" (BCBS, 2006, page 100).

This definition contains both objective elements (when a payment has been missed or delayed) and subjective elements (when a loss becomes highly probable). However, this last case generally corresponds to an extreme situation, where a specific provisioning is declared. The Basel definition of default covers then two types of credit: debts under litigation and doubtful debts.

Downgrading risk is more difficult to define. If the counterparty is rated by an agency, it can be measured by a single or multi-notch downgrade. However, it is not always the case in practice, because the credit quality decreases before the downgrade announcement. A second measure is to consider a market-based approach by using CDS spreads. However, we notice that the two methods concern counterparties, which are able to issue debt securities, in particular bonds. For instance, the concept of downgrading risk is difficult to apply for retail assets.

The distinction between default risk and downgrading risk has an impact about the credit risk measure. For loans and debt-like instruments that can not be traded in a market, the horizon time for managing credit risk is the maturity of the credit. Contrary to this hold-to-maturity approach, the horizon time for managing debt securities is shorter, typically one year. In this case, the big issue is not to manage the default, but the mark-to-market of the credit exposure.

3.2.1 The Basel I framework

According to Tarullo (2008), two explanatory factors was behind the Basel I Accord. The first motivation was to increase capital levels of internationally banks, which were very low at that time and has continuously decreased since many years. For instance, the ratio of equity capital to total assets[41] was 5.15% in 1970 and only 3.83% in 1981 for the 17 largest US banks. In 1988, this capital ratio was equal to 2.55% on average for the five largest bank in the world. The second motivation concerned the distortion risk of competition resulting from heterogeneous national capital requirements. One point that was made repeatedly, especially by US bankers, was the growth of Japanese banks. In Table 3.15, we report the ranking of the 10 world's largest banks in 1981 and 1988. While there is only one Japanese bank in the top 10 in 1981, nine Japanese banks are included in the ranking seven years later. In this context, the underlying idea of the Basel I Accord was then to increase capital requirements and harmonize national regulations for international banks.

The Basel I Accord provides a detailed definition of bank capital C and risk weighted assets RWA. We recall that tier one capital consists mainly of common stock and disclosed reserves, whereas tier two capital includes undisclosed reserves, general provisions, hybrid debt capital instruments and subordinated term debt. Risk weighted assets are simply calculated as the product of the asset notional (or the exposure at default (EAD)) by a risk

[41] All the statistics of this section comes from Chapters 2 and 3 of Tarullo (2008).

TABLE 3.15: World's largest banks in 1981 and 1988

	1981		1988	
	Bank	Assets	Bank	Assets
1	Bank of America (US)	115.6	Dai-Ichi Kangyo (JP)	352.5
2	Citicorp (US)	112.7	Sumitomo (JP)	334.7
3	BNP (FR)	106.7	Fuji (JP)	327.8
4	Crédit Agricole (FR)	97.8	Mitsubishi (JP)	317.8
5	Crédit Lyonnais (FR)	93.7	Sanwa (JP)	307.4
6	Barclays (UK)	93.0	Industrial Bank (JP)	261.5
7	Société Générale (FR)	87.0	Norinchukin (JP)	231.7
8	Dai-Ichi Kangyo (JP)	85.5	Crédit Agricole (FR)	214.4
9	Deutsche Bank (DE)	84.5	Tokai (JP)	213.5
10	National Westminster (UK)	82.6	Mitsubishi Trust (JP)	206.0

Source: Tarullo (2008).

weight (RW). Table 3.16 shows the different values of RW with respect to the category of the asset. For off-balance-sheet assets, BCBS (2008) define credit conversion factor (CCF) for converting the amount E of a credit line or off-balance-sheet asset to an exposure at default:

$$\text{EAD} = E \times \text{CCF}$$

The CCF values are 100% for direct credit substitutes (standby letters of credit), sale and repurchase agreements, forward asset purchases, 50% for standby facilities and credit lines with an original maturity of over one year, note issuance facilities and revolving underwriting facilities, 20% for short-term self-liquidating trade-related contingencies and 0% for standby facilities and credit lines with an original maturity of up to one year. The above framework is used to calculate the Cooke ratio, which is in fact a set of two capital ratios. The core capital ratio includes only tier one capital whereas the total capital ratio considers both tier one C_1 and tier two C_2 capital:

$$\begin{aligned}\text{Tier 1 ratio} &= \frac{C_1}{\text{RWA}} \geq 4\% \\ \text{Tier 2 ratio} &= \frac{C_1 + C_2}{\text{RWA}} \geq 8\%\end{aligned}$$

Example 27 *The assets of the bank are composed of* \$100 *mn of US treasury bonds,* \$20 *mn of Mexico government bonds dominated in US Dollar,* \$20 *mn of Argentine debt dominated in Argentine Peso,* \$500 *mn of residential mortgage,* \$500 *mn of corporate loans,* \$20 *of non-used standby facilities for OECD governments and* \$100 *of retail credit lines, which are decomposed as follows:* \$40 *mn are used and* 70% *of non-used credit lines have a maturity higher than one year.*

TABLE 3.16: Risk weights by category of on-balance-sheet asset

RW	Instruments
0%	Cash Claims on central governments and central banks denominated in national currency and funded in that currency Other claims on OECD central governments and central banks Claims† collateralized by cash of OECD government securities
20%	Claims† on multilateral development banks Claims† on banks incorporated in the OECD and claims guaranteed by OECD incorporated banks Claims† on securities firms incorporated in the OECD subject to comparable supervisory and regulatory arrangements Claims† on banks incorporated in countries outside the OECD with a residual maturity of up to one year Claims† on non-domestic OECD public-sector entities Cash items in process of collection
50%	Loans fully secured by mortgage on residential property
100%	Claims on the private sector Claims on banks incorporated outside the OECD with a residual maturity of over one year Claims on central governments outside the OECD and non denominated in national currency All other assets

†or guaranteed by these entities.

Source: BCBS (1988).

For each asset, we calculate RWA by choosing the right risk weight and credit conversion factor for off-balance-sheet items. We obtain the results below. The risk-weighted assets of the bank are then equal to \$831 mn. We deduce that the required capital is \$33.24 mn for tier one.

Balance Sheet	Asset	E	CCF	EAD	RW	RWA
On	US bonds			100	0%	0
	Mexico bonds			20	100%	20
	Argentine debt			20	0%	0
	Home mortgage			500	50%	250
	Corporate loans			500	100%	500
	Credit lines			40	100%	40
Off	Standby facilities	20	100%	20	0%	0
	Credit lines ($>$ 1Y)	42	50%	21	100%	21
	Credit lines ($\leq$ 1Y)	18	0%	0	100%	0
	Total					831

3.2.2 The Basel II standardized approach

The main criticism of the Cooke ratio is the lack of economic rationale with respect to risk weights. Indeed, most of claims have a 100% risk weight and do not reflect the real credit risk of the borrower. Other reasons have been given to justify a reformulation of capital requirements for credit risk with the goal to:

- obtain a better credit risk measure by taking into account the default probability of the counterparty;
- avoid regulatory arbitrage, in particular by using credit derivatives;
- have a more coherent framework that supports credit risk mitigation.

3.2.2.1 Standardized risk weights

In Basel II, the default probability is the key parameter to define risk weights. For the standardized approach (SA), they depend directly on external ratings whereas they are based on internal rating for the IRB approach. Table 3.17 shows the new matrix of risk weights, when we consider the Standard & Poor's rating system[42]. We notice that there are four main categories of claims[43]: sovereigns, banks, corporates and retail portfolios.

TABLE 3.17: Risk weights of the SA approach (Basel II)

Rating		AAA to AA−	A+ to A−	BBB+ to BBB−	BB+ to B−	CCC+ to C	NR
Sovereigns		0%	20%	50%	100%	150%	100%
Banks	1	20%	50%	100%	100%	150%	100%
	2	20%	50%	50%	100%	150%	50%
	2 ST	20%	20%	20%	50%	150%	20%
Corporates				BBB+ to BB−		B+ to C	
		20%	50%	100%		150%	100%
Retail					75%		
Residential mortgages					35%		
Commercial mortgages					100%		

The sovereigns category include central governments and central banks, whereas non-central public sector entities are treated with the banks category. We note that there are two options for the latter, whose choice is left to the

[42] NR stands for non-rated entities.

[43] The regulatory framework is more comprehensive by considering three other categories (public sector entities, multilateral development banks and securities firms), which are treated as banks. For all other assets, the standard risk weight is 100%.

discretion of the national supervisors[44]. Under the first option, the risk weight depends on the rating of the country, where the bank is located. Under the second option, this is the rating of the bank that determines the risk weight, which is more more favorable for short-term claims (three months or less). The risk weight of a corporate is calculated with respect to the rating of the entity, but uses a slightly different breakdown of ratings than the second option of the banks category. Finally, the Basel Committee uses lower levels for retail portfolios than those provided in the Basel I Accord. Indeed, residential mortgages and retail loans are now weighted at 35% and 75% instead of 50% and 100% previously. Other comparisons between Basel I and Basel II (with the second option for banks) are shown in Table 3.18.

TABLE 3.18: Comparison of risk weights between Basel I and Basel II

Entity	Rating	Maturity	Basel I	Basel II
Sovereign (OECD)	AAA		0%	0%
Sovereign (OECD)	A-		0%	20%
Sovereign	BBB		100%	50%
Bank (OECD)	BBB	2Y	20%	50%
Bank	BBB	2M	100%	20%
Corporate	AA+		100%	20%
Corporate	BBB		100%	100%

The SA approach is based on external ratings and then depends on rating agencies. The three most famous are Standard & Poor's, Moody's and Fitch. However, they cover only large companies. This is why banks will also consider rating agencies specialized in a specific sector or a given countries[45]. Of course, rating agencies must be first registered and certified by national regulators in order to be used by the banks. The validation process consists of two steps, which are the assessment of the six required criteria (objectivity, independence, transparency, disclosure, resources and credibility) and the mapping process between the ratings and the Basel matrix of risk weights.

Table 3.19 shows the rating systems of S&P, Moody's and Fitch, which are very similar. Examples of S&P's rating are given in Tables 3.20, 3.21 and 3.22. We note that the rating of many sovereign counterparties has been downgrating by at least one notch, except China which has now a rating better than before the 2008 financial crisis. For some countries, the rating in local currency is different than the rating in foreign currency, for instance Argnetina, Brazil, Russia and Ukraine[46]. We observe the same evolution for

[44] The second option is more frequent and is implemented in Europe, US and Japan for instance.

[45] For instance, banks may use Japan Credit Rating Agency Ltd for Japanese public and corporate entities, DBRS Ratings Limited for bond issuers, Cerved Rating Agency for Italian small and medium-sized enterprises, etc..

[46] An SD rating is assigned in case of selective default of the obligor.

TABLE 3.19: Credit rating system of S&P, Moody's and Fitch

	Prime Maximum Safety	High Grade High Quality			Upper Medium Grade		
S&P/Fitch	AAA	AA+	AA	AA−	A+	A	A−
Moody's	Aaa	Aa1	Aa2	Aa3	A1	A2	A3

	Lower Medium Grade			Non Investment Grade Speculative		
S&P/Fitch	BBB+	BBB	BBB−	BB+	BB	BB−
Moody's	Baa1	Baa2	Baa3	Ba1	Ba2	Ba3

	Highly Speculative			Substantial Risk	In Poor Standing		Extremely Speculative
S&P/Fitch	B+	B	B−	CCC+	CCC	CCC−	CC
Moody's	B1	B2	B3	Caa1	Caa2	Caa3	Ca

TABLE 3.20: Examples of country's S&P rating

Country	Local currency		Foreign currency	
	Jun. 2009	Oct. 2015	Jun. 2009	Oct. 2015
Argentina	B-	CCC+	B-	SD
Brazil	BBB+	BBB-	BBB-	BB+
China	A+	AA-	A+	AA-
France	AAA	AA	AAA	AA
Italy	A+	BBB-	A+	BBB-
Japan	AA	A+	AA	A+
Russia	BBB+	BBB-	BBB	BB+
Spain	AA+	BBB+	AA+	BBB+
Ukraine	B-	CCC+	CCC+	SD
US	AAA	AA+	AA+	AA+

Source: Standard & Poor's, `www.standardandpoors.com`.

TABLE 3.21: Examples of bank's S&P rating

Bank	Oct. 2001	Jun. 2009	Oct. 2015
Barclays Bank PLC	AA	AA-	A-
Credit Agricole S.A.	AA	AA-	A
Deutsche Bank AG	AA	A+	BBB+
International Industrial Bank	CCC+	BB-	
JPMorgan Chase & Co.	AA-	A+	A
UBS AG	AA+	A+	A

Source: Standard & Poor's, `www.standardandpoors.com`.

TABLE 3.22: Examples of corporate's S&P rating

Corporate	Jul. 2009	Oct. 2015
Danone	A-	A-
Exxon Mobil Corp.	AAA	AAA
Ford Motor Co.	CCC+	BBB-
General Motors Corp.	D	BBB-
L'Oreal S.A.	NR	NR
Microsoft Corp.	AAA	AAA
Nestle S.A.	AA	AA
The Coca-Cola Co.	A+	AA
Unilever PLC	A+	A+

Source: Standard & Poor's, `www.standardandpoors.com`.

banks and it is now rare to find a bank with a AAA rating. This is not the case of corporate counterparties, which present more stable ratings across time.

Remark 31 *Credit conversion factors for off-balance sheet items are similar to those defined in the original Basel Accord. For instance, any commitments that are unconditionally cancelable receives a* 0% *CCF. A CCF of* 20% *(resp.* 50%*) is applied to commitments with an original maturity up to one year (resp. higher than one year). For Revolving underwriting facilities, the CCF is equal to* 50% *whereas it is equal to* 100% *for other off-balance sheet items (e.g. direct credit substitutes, sale and repurchase agreements, forward asset purchases).*

3.2.2.2 Credit risk mitigation

Credit risk mitigation (CRM) refers to the various techniques used by banks for reducing the credit risk. These methods allow to decrease the credit exposure or to increase the recovery in case of default. The most common approaches are collateralized transactions, guarantees, credit derivatives and netting agreements.

Collateralized transactions In such operations, the credit exposure of the bank is partially hedged by collateral posted by the counterparty. BCBS (2006) defines then the following eligible instruments:

1. Cash and comparable instruments;
2. Gold;
3. Debt securities which are rated AAA to BB- when issued by sovereigns or AAA to BBB- when issued by other entities or at least A-3/P-3 for short-term debt instruments;

4. Debt securities which are not rated but fulfill certain criteria (senior debt issued by banks, listed on a recognisee exchange and sufficiently liquid);

5. Equities that are included in a main index;

6. UCITS and mutual funds, whose assets are eligible instruments and which offer a daily liquidity;

7. Equities which are listed on a recognized exchange and UCITS/mutual funds which include such equities.

The bank has the choice between two approaches to take into account collateralized transactions. In the simple approach[47], the risk weight of the collateral (with a floor of 20%) is applied to the market value of the collateral where the non-hedged exposure $(E - C)$ receives the risk weight of the counterparty:

$$\mathrm{RWA} = (\mathrm{EAD} - C) \times \mathrm{RW} + C \times \max\left(\mathrm{RW}_C, 20\%\right) \tag{3.16}$$

where EAD is the exposure at default, C is the market value of the collateral, RW is the risk weight appropriate to the exposure and RW_C is the risk weight of the collateral. The second method, called the comprehensive approach, is based on haircuts. The risk weighted asset after risk mitigation is $\mathrm{RWA} = \mathrm{RW} \times \mathrm{EAD}^\star$ where $\mathrm{EAD}^\star$ is the modified exposure at default defined as follows:

$$\mathrm{EAD}^\star = \max\left(0, (1 + H_E) \times \mathrm{EAD} - (1 - H_C - H_{FX}) \times C\right) \tag{3.17}$$

where H_E is the haircut applied to the exposure, H_C is the haircut applied to the collateral and H_{FX} is the haircut for currency risk. Table 3.23 gives the standard supervisory values of haircuts. If the bank uses an internal model to calculate haircuts, they must be based on a value-at-risk with a 99% confidence level and an holding period which depends on the collateral type and the frequency of remargining. The standard supervisory haircuts have been calibrated by assuming daily mark-to-market, daily remargining and a 10-business day holding period.

Exercise 28 *We consider a 10-year credit of* \$100 *mn to a corporate rated* A. *The credit is guaranteed by five collateral instruments: a cash deposit (*\$2 *mn), a gold deposit (*\$5 *mn), a sovereign bond rated* AA *with a 2-year residual maturity (*\$15 *mn) and repurchase transactions on Microsoft stocks (*\$20 *mn) and Wirecard*[48] *stocks (*\$20 *mn).*

[47] Collateral instruments (7) are not eligible for this approach.

[48] Wirecard is a German financial company specialized in payment processing and issuing services. The stock belongs to the MSCI Small Cap Europe Index.

TABLE 3.23: Standardized supervisory haircuts for collateralized transactions

Rating	Residual Maturity	Sovereigns	Others
AAA to AA−	0–1Y	0.5%	1%
	1–5Y	2%	4%
	5Y+	4%	8%
A+ to BBB−	0–1Y	1%	2%
	1–5Y	3%	6%
	5Y+	6%	12%
BB+ to BB−		15%	
Cash		0%	
Gold		15%	
Main index equities		15%	
Equities listed on a recognized exchange		25%	
FX risk		8%	

Before credit risk mitigation, the risk weight asset is equal to:

$$\text{RWA} = 100 \times 50\% = \$50 \text{ mn}$$

If we consider the simple approach, the repurchase transaction on Wirecard stocks is not eligible, because it does not fall within categories (1)-(6). The risk weight asset becomes[49]:

$$\begin{aligned}\text{RWA} &= (100 - 2 - 5 - 15 - 20) \times 50\% + (2 + 5 + 15 + 20) \times 20\% \\ &= \$37.40 \text{ mn}\end{aligned}$$

The repurchase transaction on Wirecard stocks is eligible in the comprehensive approach, because these equity stocks are traded in Börse Francfurt. The haircuts are 15% for gold, 2% for the sovereign bond and 15% for Microsoft stocks[50]. For Wirecard stocks, a first haircut of 25% is applied because these instruments belong to the category (7) and a second haircut of 8% is applied because there is a foreign exchange risk. The adjusted exposure at default is then equal to:

$$\begin{aligned}\text{EAD}^{\star} &= (1 + 8\%) \times 100 - 2 - (1 - 15\%) \times 5 - (1 - 2\%) \times 15 - \\ &\quad (1 - 15\%) \times 20 - (1 - 25\% - 8\%) \times 20 \\ &= \$73.65 \text{ mn}\end{aligned}$$

It follows that:

$$\text{RWA} = 73.65 \times 50\% = \$36.82 \text{ mn}$$

[49] The floor of 20% is applied to the cash, gold and sovereign bond collateral instruments. The risk weight for Microsoft stocks is 20% because the rating of Microsoft is AAA.

[50] Because Microsoft belongs to the S&P 500 index, which is a main equity index.

Guarantees and credit derivatives Banks can use these credit protection instruments if then are direct, explicit, irrevocable and unconditional. In this case, banks use the simple approach given by Equation (3.16). The case of credit default tranches is covered by rules described in the securitization framework.

Maturity mismatches A maturity mismatch occurs when the residual maturity of the hedge is less than that of the underlying asset. In this case, the bank uses the following adjustment:

$$C_A = C \times \frac{\min(T_G, T, 5) - 0.25}{\min(T, 5) - 0.25} \tag{3.18}$$

where T is the residual maturity of the exposure and T_C is the residual maturity of the collateral (or guarantee).

Exercise 29 *The bank A has granted a credit of* \$30 *mn to a corporate company B, which is rated* BB. *In order to hedge the default risk, the bank A buy* \$20 *mn of a 3-year CDS protection on B to the bank C, which is rated* A+.

If the residual maturity of the credit is lower than 3 years, we obtain:

$$\text{RWA} = (30 - 20) \times 100\% + 20 \times 50\% = \$20 \text{ mn}$$

If the residual maturity is higher than 3 years, we first have to calculate the adjusted value of the guarantee. Assuming that the residual maturity is 4 years, we have:

$$G_A = 20 \times \frac{\min(3, 4, 5) - 0.25}{\min(4, 5) - 0.25} = \$14.67 \text{ mn}$$

It follows that:

$$\text{RWA} = (30 - 14.67) \times 100\% + 14.67 \times 50\% = \$22.67 \text{ mn}$$

3.2.3 The Basel II internal ratings-based approach

The completion of the internal ratings-based (IRB) approach was a complex task, because it required many negotiations between regulators, banks and politics. Tarullo (2008) points out that the publication of the first consultative paper (CP1) in June 1999 was both "*anticlimactic and contentious*". The paper is curiously vague without a precise direction. The only tangible proposal is the use of external ratings. The second consultative paper is released in January 2001 and includes in particular the IRB approach, which has been essentially developed by US members of the Basel Committee with the support of large international banks. The press release dated 16 January 2001 indicated that the Basel Committee would finalize the New Accord by the end of 2001, for an implementation in 2004. However, it has taken much longer than originally anticipated and the final version of the New Accord was published in June 2004 and implemented from December 2006[51]. The main reason is the difficulty of calibrating the IRB approach in order to satisfy a large part of international banks. The IRB formulas of June 2004 are significantly different from the original ones and reflect compromises between the different participants without really being satisfactory.

3.2.3.1 The general framework

Contrary to the standardized approach, the IRB approach is based on internal rating systems. With such a method, the objectives of the Basel Committee are to propose a more sensitive credit risk measure and define a common basis between internal credit risk models. The IRB approach must been seen as an external credit risk model with internal parameters. Therefore, it is not an internal model, but a first step to harmonize the internal risk management practices by focusing on the main risk components, which are:

- the exposure at default (EAD);
- the probability of default (PD);
- the loss given default (LGD);
- the effective maturity (M).

The exposure at default is defined as the outstanding debt at the time of default. For instance, it is equal to the principal amount for a loan. The loss given default is the expected percentage of exposure at default that is lost if the debtor defaults. At first approximation, one can consider that $\mathrm{LGD} \simeq 1 - \boldsymbol{\mathcal{R}}$. While EAD is expressed in \$, LGD is measured in %. For example, if EAD is equal to \$10 mn and LGD is set to 70%, the expected loss due to default is equal to \$7 mn. The probability of default measures the default risk of the debtor. In Basel II, the horizon time of PD is set to one year. When the duration of the credit is not equal to one year, one has to specify its effective maturity M. This is the combination of the one-year default probability PD and the effective maturity M that measure the default risk of the debtor until the duration of the credit.

In this approach, the credit risk measure is the sum of individual risk contributions:

$$\mathcal{R}(w) = \sum_{i=1}^{n} \mathcal{RC}_i$$

where $\mathcal{RC}_i$ is a function of the four risk components:

$$\mathcal{RC}_i = f_{\mathrm{IRB}}\left(\mathrm{EAD}_i, \mathrm{LGD}_i, \mathrm{PD}_i, \mathrm{M}_i\right)$$

[51]See Chapter 4 entitled "*Negotiating Basel II*" of Tarullo (2008) for a comprehensive story of the Basel II Accord.

and f_{IRB} is the IRB fomula. In fact, there are two IRB methodologies. In the foundation IRB approach (FIRB), banks use their internal estimates of PD whereas the values of the other components (EAD, LGD and M) are set by regulators. Banks that adopt the advanced IRB approach (AIRB) may calculate all the four parameters (PD, EAD, LGD and M) using their own internal models and not only the probability of default. The mechanism of the IRB approach is then the following:

- a classification of exposures (sovereigns, banks, corporates, retail portfolios, etc.);
- for each credit i, the bank estimates the probability of default PD_i;
- it uses the standard regulatory values of the other risk components (EAD_i, LGD_i and M_i) or estimates them in the case of AIRB;
- the bank calculate then the risk weighted asset RWA_i of the credit by applying the right IRB formula f_{IRB} to the risk components;

Internal ratings are central to the IRB approach. Table 3.24 gives an example of internal rating system, where risk increases with the number grade (1, 2, 3, etc.). Another approach is to consider alphabetical letter grades[52]. A third approach is to use an internal rating scale similar to that of S&P[53].

3.2.3.2 The credit risk model of Basel II

Decomposing the value-at-risk into risk contributions BCBS (2004) used the Merton-Vasicek model (Merton, 1974; Vasicek 2002) to derive the IRB formulas. In this framework, the loss portfolio is equal to:

$$L = \sum_{i=1}^{n} w_i \times \text{LGD}_i \times \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\} \tag{3.19}$$

where w_i and T_i are the exposure at default and the residual maturity of the i^{th} credit. We assume that the loss given default LGD_i is a random variable and the default time $\boldsymbol{\tau}_i$ depends on a set of risk factors X, whose probability distribution is denoted $\mathbf{H}$. Let $p_i(X)$ be the conditional default probability. It follows that the (unconditional or long-term) default probability is:

$$\begin{aligned} p_i &= \mathbb{E}_X\left[\mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}\right] \\ &= \mathbb{E}_X\left[p_i(X)\right] \end{aligned}$$

We also introduce the notation $D_i = \mathbb{1}\left\{\boldsymbol{\tau}_i \leq T_i\right\}$, which is the default indicator function. Conditionally to the risk factors, D_i is a Bernoulli random

[52] For instance, the rating system of Crédit Agricole is: A+, A, B+, B, C+, C, C-, D+, D, D-, E+, E and E- (source: Credit Agricole, Annual Financial Report 2014, page 201).

[53] This is the case of JPMorgan Chase & Co. (source: JPMorgan Chase & Co., Annual Report 2014, page 104).

TABLE 3.24: Example of Internal Rating Systems

Rating		Degree of risk	Definition	Borrower category by self-assessment
1		No essential risk	Extremely high degree of certainty of repayment	Normal
2		Negligible risk	High degree of certainty of repayment	
3		Some risk	Sufficient certainty of repayment	
4	A B C	Better than average	There is certainty of repayment but substantial changes in the environment in the future may have some impact on this uncertainty	
5	A B C	Average	There are no problems foreseeable in the future, but a strong likelihood of impact from changes in the environment	
6	A B C	Tolerable	There are no problems foreseeable in the future, but the future cannot be considered entirely safe	
7		Lower than average	There are no problems at the current time but the financial position of the borrower is relatively weak	
8	A B	Needs preventive management	There are problems with lending terms or fulfilment, or the borrower's business conditions are poor or unstable, or there are other factors requiring careful management	Needs attention
9		Needs serious management	There is a high likelihood of bankruptcy in the future	In danger of bankruptcy
10	I		The borrower is in serious financial straits and "effectively bankrupt"	Effectively bankruptcy
	II		The borrower is bankrupt	Bankrupt

Source: Ieda, Marumo and Yoshiba (2000).

variable with probability $p_i(X)$. If we consider the standard assumption that the loss given default is independent from the default time and also assume that the default times are conditionally independent[54], we obtain:

$$\begin{aligned} \mathbb{E}[L \mid X] &= \sum_{i=1}^{n} w_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}[D_i \mid X] \\ &= \sum_{i=1}^{n} w_i \times \mathbb{E}[\mathrm{LGD}_i] \times p_i(X) \end{aligned} \tag{3.20}$$

[54]The default times are not independent, because they depend on the common risk factors X. However, conditionally to these factors, they become independent because idiosyncratic risk factors are not correlated.

and[55]:

$$\begin{aligned}
\sigma^2(L \mid X) &= \mathbb{E}\left[L^2 \mid X\right] - \mathbb{E}^2\left[L \mid X\right] \\
&= \sum_{i=1}^n w_i^2 \times \left(\mathbb{E}\left[\mathrm{LGD}_i^2\right] \times \mathbb{E}\left[D_i^2 \mid X\right] - \mathbb{E}^2\left[\mathrm{LGD}_i\right] \times p_i^2(X)\right)
\end{aligned}$$

We have $\mathbb{E}\left[D_i^2 \mid X\right] = p_i(X)$ and $\mathbb{E}\left[\mathrm{LGD}_i^2\right] = \sigma^2(\mathrm{LGD}_i) + \mathbb{E}^2\left[\mathrm{LGD}_i\right]$. We deduce that:

$$\sigma^2(L \mid X) = \sum_{i=1}^n w_i^2 \times A_i \tag{3.21}$$

with:

$$A_i = \mathbb{E}^2\left[\mathrm{LGD}_i\right] \times p_i(X) \times (1 - p_i(X)) + \sigma^2(\mathrm{LGD}_i) \times p_i(X)$$

BCBS (2004) assumes that the portfolio is infinitely fine-grained, which means that there is no concentration:

$$\lim_{n \to \infty} \max \frac{w_i}{\sum_{i=1}^n w_i} = 0 \tag{3.22}$$

In this case, Gordy (2003) shows that the conditional distribution of L degenerates to its conditional expectation $\mathbb{E}[L \mid X]$. The intuition of this result is given by Wilde (2001). He considers a fine-grained portfolio equivalent to the original portfolio by replacing the original credit i by m credits with the same default probability p_i, the same loss given default LGD_i but an exposure at default divided by m. Let L_m be the loss of the equivalent fine-grained portfolio. We have:

$$\begin{aligned}
\mathbb{E}[L_m \mid X] &= \sum_{i=1}^n \left(\sum_{j=1}^m \frac{w_i}{m}\right) \times \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}[D_i \mid X] \\
&= \sum_{i=1}^n w_i \times \mathbb{E}[\mathrm{LGD}_i] \times p_i(X) \\
&= \mathbb{E}[L \mid X]
\end{aligned}$$

and:

$$\begin{aligned}
\sigma^2(L_m \mid X) &= \sum_{i=1}^n \left(\sum_{j=1}^m \frac{w_i^2}{m^2}\right) \times A_i \\
&= \frac{1}{m} \sum_{i=1}^n w_i^2 \times A_i \\
&= \frac{1}{m} \sigma^2(L_m \mid X)
\end{aligned}$$

[55]Because the conditional covariance between D_i and D_j is equal to zero.

When m tends to ∞, we obtain the infinitely fine-grained portfolio. We note that $\mathbb{E}[L_\infty \mid X] = \mathbb{E}[L \mid X]$ and $\sigma^2(L_\infty \mid X) = 0$. Conditionally to the risk factors X, the portfolio loss L_∞ is equal to the unconditional mean $\mathbb{E}[L \mid X]$. The associated probability distribution $\mathbf{F}$ is then:

$$\begin{aligned}
\mathbf{F}(\ell) &= \Pr\{L_\infty \leq \ell\} \\
&= \Pr\{\mathbb{E}[L \mid X] \leq \ell\} \\
&= \Pr\left\{\sum_{i=1}^{n} w_i \times \mathbb{E}[\mathrm{LGD}_i] \times p_i(X) \leq \ell\right\}
\end{aligned}$$

Let $g(x)$ be the function $\sum_{i=1}^{n} w_i \times \mathbb{E}[\mathrm{LGD}_i] \times p_i(x)$. We have:

$$\mathbf{F}(\ell) = \int \cdots \int \mathbb{1}\{g(x) \leq \ell\} \,\mathrm{d}\mathbf{H}(x)$$

However, it is not possible to obtain a closed-form formula for the value-at-risk $\mathbf{F}^{-1}(\alpha)$ defined as follows:

$$\mathbf{F}^{-1}(\alpha) = \{\ell : \Pr\{g(X) \leq \ell\} = \alpha\}$$

If we consider a single risk factor and assume that $g(x)$ is an increasing function, we obtain:

$$\begin{aligned}
\Pr\{g(X) \leq \ell\} = \alpha &\Leftrightarrow \Pr\left\{X \leq g^{-1}(\ell)\right\} = \alpha \\
&\Leftrightarrow \mathbf{H}\left(g^{-1}(\ell)\right) = \alpha \\
&\Leftrightarrow \ell = g\left(\mathbf{H}^{-1}(\alpha)\right)
\end{aligned}$$

We finally deduce that the value-at-risk has the following expression:

$$\begin{aligned}
\mathbf{F}^{-1}(\alpha) &= g\left(\mathbf{H}^{-1}(\alpha)\right) \\
&= \sum_{i=1}^{n} w_i \times \mathbb{E}[\mathrm{LGD}_i] \times p_i\left(\mathbf{H}^{-1}(\alpha)\right) \qquad (3.23)
\end{aligned}$$

Equation (3.23) is appealing because the value-at-risk satisfies the Euler decomposition. Indeed, we have:

$$\begin{aligned}
\mathcal{RC}_i &= w_i \times \frac{\partial\, \mathbf{F}^{-1}(\alpha)}{\partial\, w_i} \\
&= w_i \times \mathbb{E}[\mathrm{LGD}_i] \times p_i\left(\mathbf{H}^{-1}(\alpha)\right) \qquad (3.24)
\end{aligned}$$

and:

$$\sum_{i=1}^{n} \mathcal{RC}_i = \mathbf{F}^{-1}(\alpha)$$

Remark 32 *If $g(x)$ is a decreasing function, we obtain* $\Pr\left\{X \geq g^{-1}(\ell)\right\} = \alpha$ *and:*

$$\mathbf{F}^{-1}(\alpha) = \sum_{i=1}^{n} w_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i\left(\mathbf{H}^{-1}(1-\alpha)\right)$$

The risk contribution becomes:

$$\mathcal{RC}_i = w_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i\left(\mathbf{H}^{-1}(1-\alpha)\right) \tag{3.25}$$

We recall that Equation (3.24) has been obtained under the following assumptions:

$\mathcal{H}_1$ The loss given default LGD_i is independent from the default time $\boldsymbol{\tau}_i$.

$\mathcal{H}_2$ The default times $(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ depends on a single risk factor X and are conditionally independent with respect to X.

$\mathcal{H}_3$ The portfolio is infinitely fine-grained, meaning that there is no exposure concentration.

Equation (3.24) is a very important result for two main reasons. First, it implies that, under the previous assumptions, the value-at-risk of an infinitely fine-grained portfolio can be decomposed as a sum of independent risk contributions. Indeed, $\mathcal{RC}_i$ depends solely on the characteristics of the i^{th} credit (exposure at default, loss given default and probability of default). This facilitates the calculation of the value-at-risk of large portfolios. Second, the risk contribution $\mathcal{RC}_i$ is related to the expected value of the loss given default. We don't need to model the probability distribution of LGD_i, only the mean $\mathbb{E}\left[\mathrm{LGD}_i\right]$ is taken into account.

Closed-form formula of the value-at-risk In order to obtain a closed-form formula, we need a model of default times. BCBS (2004) has selected the one-factor model of Merton (1974), which has been formalized by Vasicek (1991). Let Z_i be the normalized asset value of the entity i. In the Merton model, the default occurs when Z_i is below a given barrier B_i:

$$D_i = 1 \Leftrightarrow Z_i < B_i$$

We deduce that:

$$\begin{aligned} p_i &= \Pr\left\{D_i = 1\right\} \\ &= \Pr\left\{Z_i < B_i\right\} \\ &= \Phi\left(B_i\right) \end{aligned}$$

The value of the barrier B_i is then $\Phi^{-1}(p_i)$. We assume that the asset value Z_i depends on a common risk factor X and an idiosyncratic risk factor ε_i as follows:

$$Z_i = \sqrt{\rho} X + \sqrt{1-\rho}\,\varepsilon_i$$

X and ε_i are two independent standard normal random variables. We note that[56]:

$$\begin{aligned} \mathbb{E}\left[Z_i Z_j\right] &= \mathbb{E}\left[\left(\sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i\right)\left(\sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_j\right)\right] \\ &= \mathbb{E}\left[\rho X^2 + (1-\rho)\varepsilon_i \varepsilon_j + X\sqrt{\rho(1-\rho)}\left(\varepsilon_{i+}\varepsilon_j\right)\right] \\ &= \rho \end{aligned}$$

We interest ρ as the constant asset correlation. We now calculate the conditional default probability:

$$\begin{aligned} p_i(X) &= \Pr\left\{D_i = 1 \mid X\right\} \\ &= \Pr\left\{Z_i < B_i \mid X\right\} \\ &= \Pr\left\{\sqrt{\rho} X + \sqrt{1-\rho}\varepsilon_i < B_i\right\} \\ &= \Pr\left\{\varepsilon_i < \frac{B_i - \sqrt{\rho} X}{\sqrt{1-\rho}}\right\} \\ &= \Phi\left(\frac{B_i - \sqrt{\rho} X}{\sqrt{1-\rho}}\right) \end{aligned}$$

Using the framework of the previous paragraph, we obtain:

$$\begin{aligned} g(x) &= \sum_{i=1}^{n} w_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times p_i(x) \\ &= \sum_{i=1}^{n} w_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho} X}{\sqrt{1-\rho}}\right) \end{aligned}$$

We note that $g(x)$ is a decreasing function if $w_i \geq 0$. Using Equation (3.25) and using the relationship $\Phi^{-1}(1-\alpha) = -\Phi^{-1}(\alpha)$, it follows that:

$$\mathcal{RC}_i = w_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \tag{3.26}$$

Remark 33 *We verify that p_i is the unconditional default probability. Indeed, we have:*

$$\begin{aligned} \mathbb{E}_X\left[p_i(X)\right] &= \mathbb{E}_X\left[\Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho} X}{\sqrt{1-\rho}}\right)\right] \\ &= \int_{-\infty}^{\infty} \Phi\left(\frac{\Phi^{-1}(p_i) - \sqrt{\rho} x}{\sqrt{1-\rho}}\right)\phi(x)\,\mathrm{d}x \end{aligned}$$

We recognize the integral function analyzed in Appendix A.2.2.4 in page 457.

[56]We have $\mathbb{E}\left[\varepsilon_i \varepsilon_j\right] = 0$ because ε_i and ε_j are two specific risk factors.

We deduce that:

$$\begin{aligned}\mathbb{E}_X\left[p_i(X)\right] &= \Phi_2\left(\infty, \frac{\Phi^{-1}(p_i)}{\sqrt{1-\rho}} \times \left(\frac{1}{1-\rho}\right)^{-1/2}; \frac{\sqrt{\rho}}{\sqrt{1-\rho}}\left(\frac{1}{1-\rho}\right)^{-1/2}\right) \\ &= \Phi_2\left(\infty, \Phi^{-1}(p_i); \sqrt{\rho}\right) \\ &= \Phi\left(\Phi^{-1}(p_i)\right) \\ &= p_i\end{aligned}$$

Example 30 *We consider a homogeneous portfolio with* 100 *credits. For each credit, the exposure at default, the expected LGD and the probability of default are set to* \$1 *mn,* 50% *and* 5%*.*

Let us assume that the asset correlation ρ is equal to 10% We have reported the numerical values of $\mathbf{F}^{-1}(\alpha)$ for different values of α in Table 3.25. If we are interested in the cumulative distribution function, $\mathbf{F}(\ell)$ is equal to the numerical solution α of the equation $\mathbf{F}^{-1}(\alpha) = \ell$. Using a bisection algorithm, we find the probabilities given in Table 3.25. For instance, the probability to have a loss less than or equal to \$3 mn is equal to 70.44%. Finally, to calculate the probability density function of the portfolio loss, we use the following relationship[57]:

$$f(x) = \frac{1}{\partial_\alpha \mathbf{F}^{-1}(\mathbf{F}(x))}$$

with:

$$\begin{aligned}\partial_\alpha \mathbf{F}^{-1}(\alpha) &= \sum_{i=1}^{n} w_i \times \mathbb{E}\left[\mathrm{LGD}_i\right] \times \sqrt{\frac{\rho}{1-\rho}} \times \frac{1}{\phi\left(\Phi^{-1}(\alpha)\right)} \times \\ &\quad \phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right)\end{aligned}$$

In Figure 3.21, we compare the probability functions for two different asset correlations. We note that the level of ρ has a big impact on the quantile function and the shape of the density function.

TABLE 3.25: Numerical values of $f(\ell)$, $\mathbf{F}(\ell)$ and $\mathbf{F}^{-1}(\alpha)$ when ρ is equal to 10%

α	(in %)	10.00	25.00	50.00	75.00	90.00	95.00
$\mathbf{F}^{-1}(\alpha)$	(in \$ mn)	0.77	1.25	2.07	3.28	4.78	5.90
ℓ	(in \$ mn)	0.10	1.00	2.00	3.00	4.00	5.00
$\mathbf{F}(\ell)$	(in %)	0.03	16.86	47.98	70.44	83.80	91.26
$f(\ell)$	(in %)	1.04	31.19	27.74	17.39	9.90	5.43

[57]See Appendix A.2.2.2 in page 456.

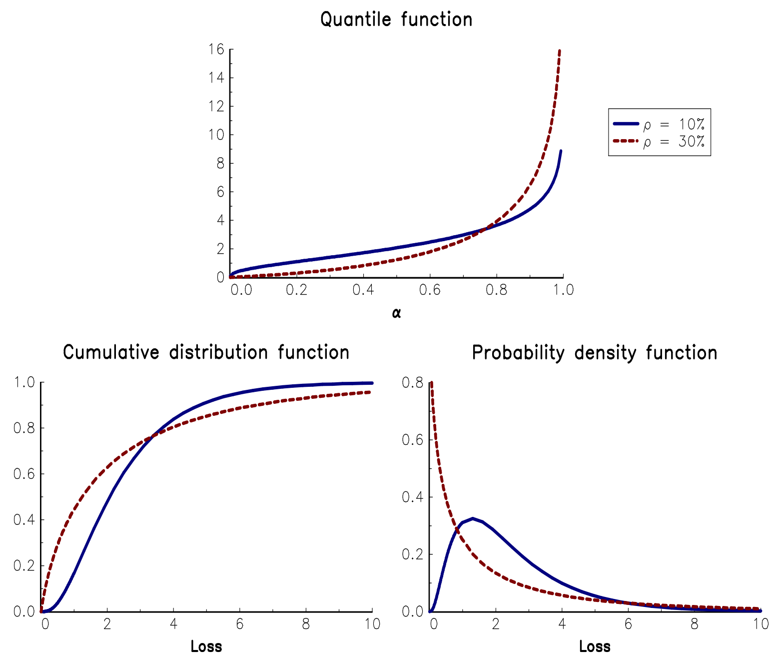

FIGURE 3.21: Probability functions of the credit portfolio loss

The risk contribution $\mathcal{RC}_i$ depends on three credit parameters (the exposure at default w_i, the expected loss given default $\mathbb{E}[\mathrm{LGD}_i]$ and the probability of default p_i) and two model parameters (the asset correlation ρ and the confidence level α of the value-at-risk). It is obvious that $\mathcal{RC}_i$ is an increasing function of the different parameters with the exception of the correlation. We obtain:

$$\operatorname{sgn} \frac{\partial \mathcal{RC}_i}{\partial \rho} = \operatorname{sgn} \frac{1}{2(1-\rho)^{3/2}} \left(\Phi^{-1}(p_i) + \frac{\Phi^{-1}(\alpha)}{\sqrt{\rho}} \right)$$

We deduce that the risk contribution is not a monotone function with respect to ρ. It increases if the term $\sqrt{\rho}\Phi^{-1}(p_i) + \Phi^{-1}(\alpha)$ is positive. This implies that the risk contribution may decrease if the probability of default is very low and the confidence level is larger than 50%. The two limiting cases are $\rho = 0$ and $\rho = 1$. In the first case, the risk contribution is equal to the expected loss:

$$\mathcal{RC}_i = \mathbb{E}[L_i] = w_i \times \mathbb{E}[\mathrm{LGD}_i] \times p_i$$

In the second case, the risk contribution depends on the value of the probability of default:

$$\lim_{\rho \to 1} \mathcal{RC}_i = \begin{cases} 0 & \text{if } p_i < 1-\alpha \\ 0.5 \times w_i \times \mathbb{E}[\mathrm{LGD}_i] & \text{if } p_i = 1-\alpha \\ w_i \times \mathbb{E}[\mathrm{LGD}_i] & \text{if } p_i > 1-\alpha \end{cases}$$

The behavior of the risk contribution is illustrated in Figure 3.22 with the following base parameter values: $w_i = 100$, $\mathbb{E}[\mathrm{LGD}_i] = 70\%$, $\rho = 20\%$ and $\alpha = 90\%$. We verify that the risk contribution is an increasing function of $\mathbb{E}[\mathrm{LGD}_i]$ (top/left panel) and α (top/right panel). When p_i and α are set to 10% and 90%, the risk contribution increases with ρ and reaches the value 35, which corresponds to half of nominal loss given default. When p_i and α are set to 5% and 90%, the risk contribution increases in a first time and then decreases (bottom/left panel). The maximum is reached for the value[58] $\rho^{\star} = 60.70\%$. When α is equal to 99%, this behavior vanishes (bottom/right panel).

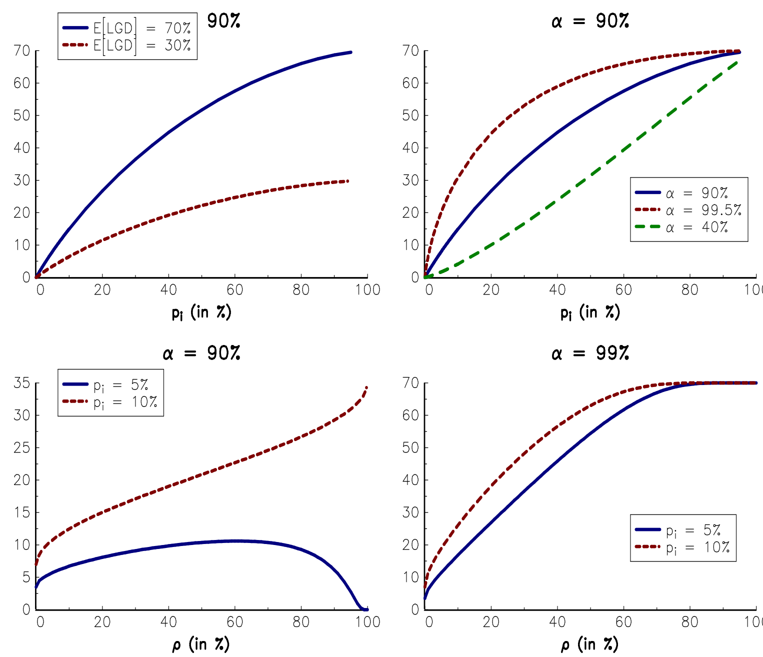

FIGURE 3.22: Relationship between the risk contribution $\mathcal{RC}_i$ and model parameters

In this model, the maturity T_i is taken into account through the probability of default. Indeed, we have $p_i = \Pr\{\boldsymbol{\tau} \leq T_i\}$. Let us denote PD_i the annual default probability of the obligor. If we assume that the default time

[58]We have:

$$\rho^{\star} = \max{}^2\left(0, -\frac{\Phi^{-1}(\alpha)}{\Phi^{-1}(p_i)}\right) = \left(\frac{1.282}{1.645}\right)^2 = 60.70\%$$

is Markovian, we have the following relationship:

$$\begin{aligned} p_i &= 1 - \Pr\{\boldsymbol{\tau} > T_i\} \\ &= 1 - (1 - \mathrm{PD}_i)^{T_i} \end{aligned}$$

We can then rewrite Equation (3.26) such that the risk contribution depends on the exposure at default, the expected loss given default, the annualized probability of default and the maturity, which are the 4 parameters of the IRB approach.

3.2.3.3 The IRB formulas

A long process to obtain the finalized formulas The IRB formula of the second consultative portfolio was calibrated with $\alpha = 99.5\%$, $\rho = 20\%$ and a standard maturity of three years. To measure the impact of this approach, the Basel Committee conducted a quantitative impact study (QIS) in April 2001. A QIS is an Excel workbook to be filled by the bank. It allows the Basel Committee to gauge the impact of the different proposals for capital requirements. The answers are then gathered and analyzed at the industry level. Results are published in November 2001. Overall, 138 banks from 25 countries participated in the QIS. Not all participating banks managed to calculate the capital requirements under the three methods (SA, FIRB and AIRB). However, 127 banks provided complete information on the SA approach and 55 banks on the FIRB approach. Only 22 banks were able to calculate the AIRB approach for all portfolios.

TABLE 3.26: Percentage change in capital requirements under CP2 proposals

		SA	FIRB	AIRB
G10	Group 1	6%	14%	−5%
	Group 2	1%		
EU	Group 1	6%	10%	−1%
	Group 2	−1%		
Others		5%		

Source: Basel Committee on Banking Supervision (2001b).

In Table 3.26, we report the difference in capital requirements between CP2 proposals and Basel I. Group 1 corresponds to diversified, internationally active banks with Tier 1 capital of at least €3 bn whereas Group 2 consists of smaller or more specialized banks. BCBS (2001b) concluded that "*on average, the QIS2 results indicate that the CP2 proposals for credit risk would deliver an increase in capital requirements for all groups under both the SA and FIRB approaches*". It was obvious that these figures were not satisfactory. The Basel Committee considered then several modifications in order to (1) maintain equivalence on average between current required capital and the revised SA approach and (2) provide incentives under the FIRB approach. A third motivation has emerged rapidly. According to many studies[59], Basel II may considerably increase the procyclicality of capital requirements. Indeed, capital requirements may increase in an economic meltdown, because LGD increases in bad times and credits received lower ratings. In this case, capital requirements may move in an opposite direction than the macro-economic cycle, leading banks to reduce their supply of credit. In this scenario, Basel II proposals may amplify credit crises and economic downturns. All these reasons explain the long period to finalize the Basel II Accord. After two new QIS (QIS 2.5 in July 2002 and QIS 3 in May 2003) and a troubled period at the end of 2003, the new Capital Accord is finally published in June 2004. However, there is a shared feeling that it is more a compromise than a terminated task. Thus, several issues are unresolved and new QIS will be conducted in 2004 and 2005 before the implementation in order to confirm the calibration.

The supervisory formula If we use the notations of the Basel Committee, the risk contribution has the following expression:

$$\mathcal{RC} = \mathrm{EAD} \times \mathrm{LGD} \times \Phi \left(\frac{\Phi^{-1}\left(1-(1-\mathrm{PD})^{\mathrm{M}}\right) + \sqrt{\rho}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right)$$

where EAD is the exposure at default, LGD is the (expected) loss given default, PD is the (one-year) probability of default and M is the effective maturity. Because $\mathcal{RC}$ is directly the capital requirement ($\mathcal{RC} = 8\% \times \mathrm{RWA}$), we deduce that the risk-weighted asset is equal to:

$$\mathrm{RWA} = 12.50 \times \mathrm{EAD} \times \boldsymbol{\mathcal{K}}^{\star} \tag{3.27}$$

where $\boldsymbol{\mathcal{K}}^{\star}$ is the normalized required capital for a unit exposure:

$$\boldsymbol{\mathcal{K}}^{\star} = \mathrm{LGD} \times \Phi \left(\frac{\Phi^{-1}\left(1-(1-\mathrm{PD})^{\mathrm{M}}\right) + \sqrt{\rho}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right) \tag{3.28}$$

In order to obtain the finalized formulas, the Basel Committee has introduced the following modifications:

- a maturity adjustment $\varphi(\mathrm{M})$ has been added in order to separate the impact of the one-year probability of default and the effect of the maturity; the function $\varphi(\mathrm{M})$ has then been calibrated such that Expression (3.28) becomes:

$$\boldsymbol{\mathcal{K}}^{\star} \approx \mathrm{LGD} \times \Phi \left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho}\,\Phi^{-1}(\alpha)}{\sqrt{1-\rho}} \right) \times \varphi(\mathrm{M}) \tag{3.29}$$

[59]See for instance Goodhart *et al.* (2004) or Kashyap and Stein (2004).

- it has used a confidence level of 99.9% instead of the 99.5% value;
- it has defined a parametric function $\rho(\mathrm{PD})$ for the default correlation in order that low ratings are not too penalizing for capital requirements;
- it has considered the unexpected loss as the credit risk measure:

$$\mathrm{UL}_\alpha = \mathrm{VaR}_\alpha - \mathbb{E}[L]$$

In summary, the risk-weighted asset in the IRB approach is calculated using Equation (3.27) and the following normalized required capital:

$$\mathcal{K}^\star = \left(\mathrm{LGD} \times \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})}\Phi^{-1}(99.9\%)}{\sqrt{1-\rho(\mathrm{PD})}}\right) - \mathrm{LGD} \times \mathrm{PD}\right) \times \varphi(\mathrm{M}) \tag{3.30}$$

Risk-weighted assets for corporate, sovereign, and bank exposures The three asset classes uses the same formula:

$$\begin{aligned}\mathcal{K}^\star \quad = \quad & \left(\mathrm{LGD} \times \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})}\Phi^{-1}(99.9\%)}{\sqrt{1-\rho(\mathrm{PD})}}\right) - \mathrm{LGD} \times \mathrm{PD}\right) \times \\ & \left(\frac{1 + (\mathrm{M} - 2.5) \times b(\mathrm{PD})}{1 - 1.5 \times b(\mathrm{PD})}\right)\end{aligned} \tag{3.31}$$

with $b(\mathrm{PD}) = (0.11852 - 0.05478 \times \ln(\mathrm{PD}))^2$ and:

$$\rho(\mathrm{PD}) = 12\% \times \frac{1 - e^{-50 \times \mathrm{PD}}}{1 - e^{-50}} + 24\% \times \frac{1 - \left(1 - e^{-50 \times \mathrm{PD}}\right)}{1 - e^{-50}} \tag{3.32}$$

We note that the maturity adjustment $\varphi(\mathrm{M})$ vanishes when the effective maturity is one year. For a defaulted exposure, we have:

$$\mathcal{K}^\star = \max(0, \mathrm{LGD} - \mathrm{EL})$$

where EL is the bank's best estimate of the expected loss[60].

For Small and medium-sized enterprises[61], a firm-size adjustment is introduced by defining a new parametric function for the default correlation:

$$\rho^{\mathrm{SME}}(\mathrm{PD}) = \rho(\mathrm{PD}) - 4\% \times \left(1 - \frac{(\max(S, 5) - 5)}{45}\right)$$

where S is the reported sales expressed in € mn. This adjustment has the effect to reduce the default correlation and then the risk-weighted asset. Similarly,

[60] We can assimilate it to specific provisions.

[61] They are defined as corporate entities where the reported sales for the consolidated group of which the firm is a part is less than € 50 mn.

the Basel Committee proposes specific arrangements for specialized lending and high-volatility commercial real estate (HVCRE).

In the foundation IRB approach, the banks estimates the probability of default, but uses standard values for the other parameters. In the advanced IRB approach, the bank always estimates the parameters PD and M, and may uses its own estimates for the parameters EAD and LGD subject to certain minimum requirements. The risk components are defined as follows:

1. The exposure of default is the amount of the claim, without taking into account specific provisions or partial write-offs. For off-balance sheet positions, the bank uses the same credit conversion factors for the FIRB approach as for the SA approach. In the AIRB approach, the bank may use its own internal measures of CCF.

2. In the FIRB approach, the loss given default is set to 45% for senior claims and 75% for subordinated claims. In the AIRB approach, the bank may use its own estimates of LGD. However, they must be conservative and take into account adverse economic conditions. Moreover, they must include all the recovery costs (litigation cost, administrative cost, etc.).

3. PD is the one-year probability of default calculated with the internal rating system. For corporate and bank exposures, a floor of 0.03% is applied.

4. The maturity is set to 2.5 years in the FIRB approach. In the advanced approach, M is the weighted average time of the cash flows, with a one-year floor and a five-year cap.

Exercise 31 *We consider a senior debt of* \$3 *mn on a corporate firm. The residual maturity of the debt is equal to 2 years. We estimate the one-year probability of default at* 5%*.*

To determine the capital charge, we first calculate the default correlation:

$$\begin{aligned} \rho(\mathrm{PD}) &= 12\% \times \frac{1 - e^{-50 \times 0.05}}{1 - e^{-50}} + 24\% \times \frac{1 - \left(1 - e^{-50 \times 0.05}\right)}{1 - e^{-50}} \\ &= 12.985\% \end{aligned}$$

We have:

$$\begin{aligned} b(\mathrm{PD}) &= \left(0.11852 - 0.05478 \times \ln(0.05)\right)^2 \\ &= 0.0799 \end{aligned}$$

It follows that the maturity adjustment is equal to:

$$\begin{aligned} \varphi(M) &= \frac{1 + (2 - 2.5) \times 0.0799}{1 - 1.5 \times 0.0799} \\ &= 1.0908 \end{aligned}$$

The normalized capital charge with a one-year maturity is:

$$\begin{aligned}\mathcal{K}^{\star} &= 45\% \times \Phi\left(\frac{\Phi^{-1}(5\%) + \sqrt{12.985\%}\Phi^{-1}(99.9\%)}{\sqrt{1-12.985\%}}\right) - 45\% \times 5\% \\ &= 0.1055\end{aligned}$$

When the maturity is two years, we obtain:

$$\begin{aligned}\mathcal{K}^{\star} &= 0.1055 \times 1.0908 \\ &= 0.1151\end{aligned}$$

We deduce the value taken by the risk weight:

$$\begin{aligned}\mathrm{RW} &= 12.5 \times 0.1151 \\ &= 143.87\%\end{aligned}$$

It follows that the risk-weighted asset is equal to \$4.316 mn whereas the capital charge is \$345 287. Using the same process, we have calculated the values of risk weight for different values of PD, LGD and $\mathbb{M}$ in Table 3.270. The last two columns are for a SME claim by considering that sales are equal to €5 mn.

TABLE 3.27: IRB Risk weights (in %) for corporate exposures

PD	M = 1		M = 2.5		M = 2.5 (SME)	
(in %)	45.0	75.0	45.0	75.0	45.0	75.0
0.10	18.7	31.1	29.6	49.4	23.3	38.8
0.50	52.2	86.9	69.6	116.0	54.9	91.5
1.00	73.3	122.1	92.3	153.9	72.4	120.7
2.00	95.8	159.6	114.8	191.4	88.5	147.6
5.00	131.9	219.8	149.8	249.8	112.3	187.1
10.00	175.7	292.9	193.1	321.8	146.5	244.2
20.00	223.0	371.6	238.2	397.0	188.4	314.0

Risk-weighted assets for retail exposures Claims can be included in the regulatory retail portfolio if they meet certain criteria: in particular, the exposure must be to an individual person or persons or to a small business; it satisfies the granularity criterion, meaning that no aggregate exposure to one counterpart can exceed 0.2% of the overall regulatory retail portfolio; the aggregated exposure to one counterparty cannot exceed €1 mn. In these cases, the bank use the following IRB formula:

$$\mathcal{K}^{\star} = \mathrm{LGD} \times \Phi\left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})}\Phi^{-1}(99.9\%)}{\sqrt{1-\rho(\mathrm{PD})}}\right) - \mathrm{LGD} \times \mathrm{PD}$$

We note that the IRB formula uses a one-year fixed maturity. The value of the default correlation depends on the categories. For residential mortgage exposures, we have $\rho(\text{PD}) = 15\%$ whereas the default correlation $\rho(\text{PD})$ is equal to 4% for qualifying revolving retail exposures. For other retail exposures, it is defined as follows:

$$\rho(\text{PD}) = 3\% \times \frac{1 - e^{-35 \times \text{PD}}}{1 - e^{-35}} + 16\% \times \frac{1 - \left(1 - e^{-35 \times \text{PD}}\right)}{1 - e^{-35}}$$

in Table 3.28, we report the corresponding risk weights for the three categories and for two different values of LGD.

TABLE 3.28: IRB Risk weights (in %) for retail exposures

PD	Mortgage		Revolving		Other retail	
(in %)	45.0	25.0	45.0	85.0	45.0	85.0
0.10	10.7	5.9	2.7	5.1	11.2	21.1
0.50	35.1	19.5	10.0	19.0	32.4	61.1
1.00	56.4	31.3	17.2	32.5	45.8	86.5
2.00	87.9	48.9	28.9	54.6	58.0	109.5
5.00	148.2	82.3	54.7	103.4	66.4	125.5
10.00	204.4	113.6	83.9	158.5	75.5	142.7
20.00	253.1	140.6	118.0	222.9	100.3	189.4

The two other pillars The first pillar of Basel II, which concerns minimum capital requirements, is completed by two other pillars. The second pillar is the supervisory review process (SRP) and is composed of two main processes: the supervisory review and evaluation process (SREP) and the internal capital adequacy assessment process (ICAAP). SREP defines the regulatory response to the first pillar, in particular the validation processes. ICAAP addresses risks that are not captured in Pillar 1 like concentration risk or non-granular portfolios in the case of credit risk. For instance, stress tests are part of Pillar 2. The goal of the second pillar is then to encourage banks to continuously improve their internal models and processes for assessing the adequacy of their capital and to ensure that supervisors have the adequate tools to control them. The third pillar, which is also called market discipline, requires banks to publish comprehensive information about their risk management process. This is particularly true since the publication in January 2015 of the revised pillar 3 disclosure requirements. Indeed, BCBS (2015a) imposes the use of templates for quantitative disclosure with a fixed format in order to facilitate the comparison between banks.

3.2.4 The securitization framework

3.2.5 Basel IV proposals

3.3 Credit risk modeling

We now address the problem of parameter specification. This mainly concerns the exposure at default, the loss given default and the probability of default because the effective maturity is well defined. This section also analyzes default correlations and non granular portfolios when the bank develop its own credit model for calculating economic capital and satisfying Pillar 2 requirements.

3.3.1 Exposure at default

3.3.2 Loss given default

3.3.2.1 Economic modeling

3.3.2.2 Stochastic modeling

3.3.3 Probability of default

3.3.3.1 Survival function

The survival function is the main tool to characterize the probability of default. It is also known as reduced form modeling.

Definition and main properties Let τ be a default (or survival) time. The survival function[62] is defined as follows:

$$\begin{aligned} \mathbf{S}(t) &= \Pr\{\boldsymbol{\tau} > t\} \\ &= 1 - \mathbf{F}(t) \end{aligned}$$

where $\mathbf{F}$ is the cumulative distribution function. We deduce that the density function is related to the survival function in the following manner:

$$f(t) = -\frac{\partial\,\mathbf{S}(t)}{\partial\,t} \tag{3.33}$$

In survival models, the key concept is the hazard function $\lambda(t)$, which is the instantaneous default rate given that the default has not occurred before t:

$$\lambda(t) = \lim_{\mathrm{d}t \to 0^+} \frac{\Pr\{t \leq \tau \leq t + \mathrm{d}t \mid \tau \geq t\}}{\mathrm{d}t}$$

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[62] Previously, we have noted the survival function as $\mathbf{S}_{t_0}(t)$. Here, we assume that the current time t_0 is 0.

We deduce that:

$$\begin{aligned}\lambda(t) &= \lim_{\mathrm{d}t \to 0^+} \frac{\Pr\{t \le \tau \le t + \mathrm{d}t\}}{\mathrm{d}t} \times \frac{1}{\Pr\{\tau \ge t\}} \\ &= \frac{f(t)}{\mathbf{S}(t)}\end{aligned}$$

Using Equation (3.33), another expression of the hazard function is:

$$\begin{aligned}\lambda(t) &= -\frac{\partial_t \mathbf{S}(t)}{\mathbf{S}(t)} \\ &= -\frac{\partial \ln \mathbf{S}(t)}{\partial t}\end{aligned}$$

The survival function can then be rewritten with respect to the hazard function and we have:

$$\mathbf{S}(t) = e^{-\int_0^t \lambda(s)\,\mathrm{d}s} \tag{3.34}$$

In Table 3.29, we have reported the most common hazard and survival functions. They can be extended by adding explanatory variables in order to obtain proportional hazard models (Cox, 1972). In this case, the expression of the hazard function is $\lambda(t) = \lambda_0(t) \exp\left(\beta^\top x\right)$ where $\lambda_0(t)$ is the baseline hazard rate function and x is the vector of explanatory variables, which are not dependent on time.

TABLE 3.29: Common survival functions

Model	$\mathbf{S}(t)$	$\lambda(t)$
Exponential	$\exp(-\lambda t)$	λ
Weibull	$\exp(-\lambda t^\gamma)$	$\lambda\gamma t^{\gamma-1}$
Log-normal	$1 - \Phi(\gamma \ln(\lambda t))$	$\gamma t^{-1}\phi(\gamma \ln(\lambda t)) / (1 - \Phi(\gamma \ln(\lambda t)))$
Log-logistic	$1/\left(1 + \lambda t^{\frac{1}{\gamma}}\right)$	$\lambda\gamma^{-1} t^{\frac{1}{\gamma}} / \left(t + \lambda t^{1+\frac{1}{\gamma}}\right)$
Gompertz	$\exp(\lambda(1 - e^{\gamma t}))$	$\lambda\gamma \exp(\gamma t)$

The exponential model holds a special place in default time models. It can be justified by the following problem in physics:

> "*Assume that a system consists of n identical components which are connected in series. This means that the system fails as soon as one of the components fails. One can assume that the components function independently. Assume further that the random time interval until the failure of the system is one n^{th} of the time interval of component failure*" (Galambos, 1982).

We have $\Pr\{\min(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n) \le t\} = \Pr\{\tau_i \le n \times t\}$. The problem is then equivalent to solve the functional equation $\mathbf{S}(t) = \mathbf{S}^n(t/n)$ with $\mathbf{S}(t) = \Pr\{\tau_1 > t\}$. We can show that the unique solution for $n \ge 1$ is the exponential distribution. Following Galambos and Kotz (1978), its other main properties are:

1. the mean residual life $\mathbb{E}[\boldsymbol{\tau} \mid \boldsymbol{\tau} \geq t]$ is constant;
2. it satisfies the famous lack of memory property:

$$\Pr\{\boldsymbol{\tau} \geq t + u \mid \boldsymbol{\tau} \geq t\} = \Pr\{\boldsymbol{\tau} \geq u\}$$

   or equivalently $\mathbf{S}(t+u) = \mathbf{S}(t)\,\mathbf{S}(u)$;
3. the probability distribution of $n \times \boldsymbol{\tau}_{1:n}$ is the same than this of $\boldsymbol{\tau}_i$.

Piecewise exponential model In credit risk models, the standard distribution to define default times is a generalization of the exponential model by considering piecewise constant hazard rates:

$$\begin{aligned} \lambda(t) &= \sum_{m=1}^{M} \lambda_m \times \mathbb{1}\left\{t_{m-1}^{\star} < t \leq t_m^{\star}\right\} \\ &= \lambda_m \qquad \text{if } t \in \left]t_{m-1}^{\star}, t_m^{\star}\right] \end{aligned}$$

where $t_m^{\star}$ are the knots of the function[63]. For $t \in \left]t_{m-1}^{\star}, t_m^{\star}\right]$, the expression of the survival function becomes:

$$\begin{aligned} \mathbf{S}(t) &= \exp\left(-\sum_{k=1}^{m-1} \lambda_k \left(t_k^{\star} - t_{k-1}^{\star}\right) - \lambda_m \left(t - t_{m-1}^{\star}\right)\right) \\ &= \mathbf{S}\left(t_{m-1}^{\star}\right) e^{-\lambda_m \left(t - t_{m-1}^{\star}\right)} \end{aligned}$$

It follows that the density function is equal to [64]:

$$f(t) = \lambda_m \exp\left(-\sum_{k=1}^{m-1} \lambda_k \left(t_k^{\star} - t_{k-1}^{\star}\right) - \lambda_m \left(t - t_{m-1}^{\star}\right)\right)$$

In Figure 3.23, we have reported the hazard, survival and density functions for three set of parameters $\left\{\left(t_m^{\star}, \lambda_m\right), m = 1, \ldots, M\right\}$:

$$\begin{array}{ll} \left\{(1, 1\%), (2, 1.5\%), (3, 2\%), (4, 2.5\%), (\infty, 3\%)\right\} & \text{for } \lambda_1(t) \\ \left\{(1, 10\%), (2, 7\%), (5, 5\%), (7, 4.5\%), (\infty, 6\%)\right\} & \text{for } \lambda_2(t) \end{array}$$

and $\lambda_3(t) = 4\%$. We note the special shape of the density function, which is not smooth at the knots.

---

[63] We have $t_0^{\star} = 0$ and $t_{M+1}^{\star} = \infty$.

[64] We verify that:

$$\frac{f(t)}{\mathbf{S}(t)} = \lambda_m \qquad \text{if } t \in \left]t_{m-1}^{\star}, t_m^{\star}\right]$$

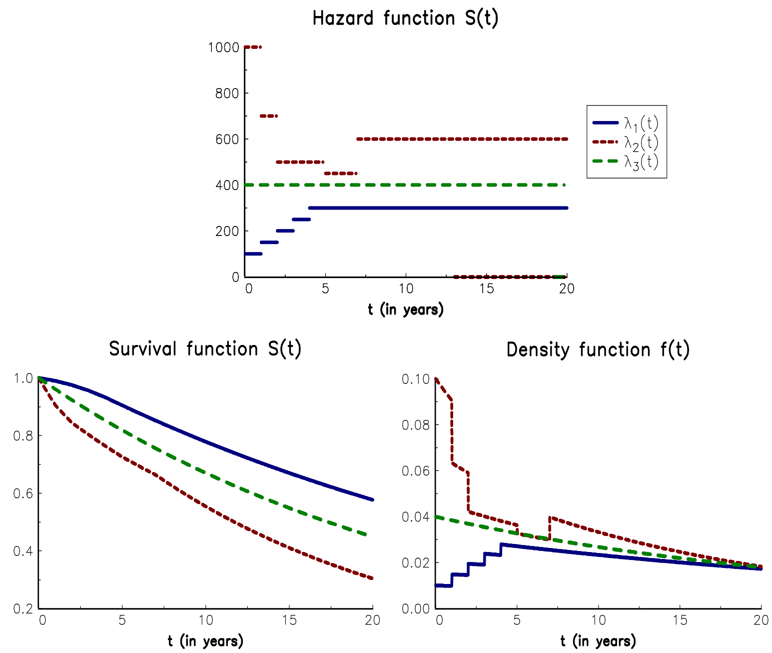

FIGURE 3.23: Examples of piecewise exponential model

Estimation To estimate the parameters of the survival function, we can use the cohort approach. Under this method, we estimate the empirical survival function by counting the number of entities for a given population that do not default over the period Δt:

$$\hat{\mathbf{S}}(\Delta t) = 1 - \frac{\sum_{i=1}^{n} \mathbb{1}\{t < \boldsymbol{\tau}_i \leq t + \Delta t\}}{n}$$

where n is the number of entities that compose the population. We can then fit the survival function by using for instance the least squares method.

Example 32 *We consider a population of 1000 companies. The number of defaults $n_D(\Delta t)$ over the period Δt is given in the table below:*

Δt *(in months)*	3	6	9	12	15	18	21	22
$n_D(\Delta t)$	2	5	9	12	16	20	25	29

We obtain $\hat{\mathbf{S}}(0.25) = 0.998$, $\hat{\mathbf{S}}(0.50) = 0.995$, $\hat{\mathbf{S}}(0.75) = 0.991$, $\hat{\mathbf{S}}(1.00) = 0.988$, $\hat{\mathbf{S}}(1.25) = 0.984$, $\hat{\mathbf{S}}(1.50) = 0.980$, $\hat{\mathbf{S}}(1.75) = 0.975$ and $\hat{\mathbf{S}}(2.00) = 0.971$. For the exponential model, the least squares estimator $\hat{\lambda}$ is equal to 1.375%. In the case of the Gompertz survival function, we obtain $\hat{\lambda} = 2.718\%$ and $\hat{\gamma} = 0.370$. If we consider the piecewise exponential model, whose knots correspond to the different periods Δt, we have $\hat{\lambda}_1 = 0.796\%$, $\hat{\lambda}_2 = 1.206\%$,

$\hat{\lambda}_3 = 1.611\%$, $\hat{\lambda}_4 = 1.216\%$, $\hat{\lambda}_5 = 1.617\%$, $\hat{\lambda}_6 = 1.640\%$, $\hat{\lambda}_7 = 2.044\%$ and $\hat{\lambda}_8 = 1.642\%$. To compare these three calibrations, we report the corresponding hazard functions in Figure 3.24. We deduce that the one-year default probability[65] is respectively equal to 1.366%, 1.211% and 1.200%.

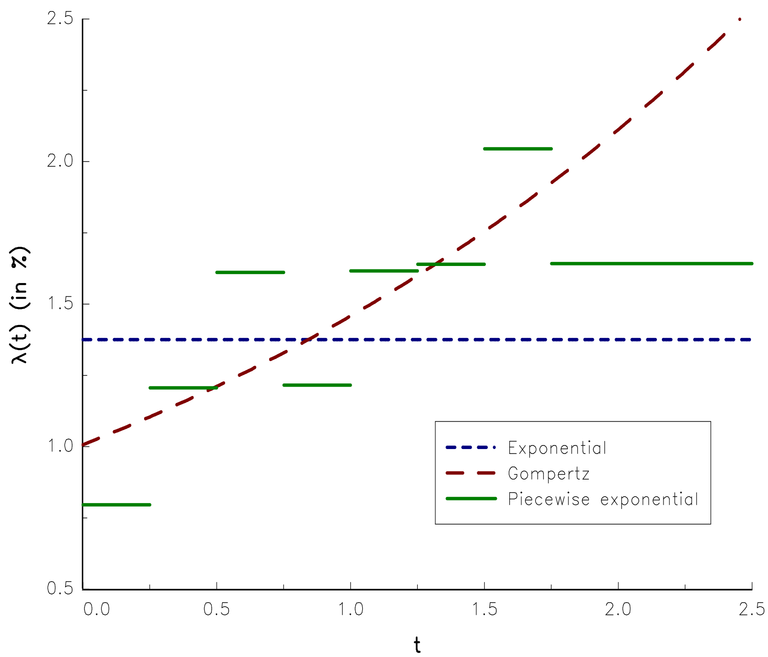

FIGURE 3.24: Estimated hazard function

In the piecewise exponential model, we can specify an arbitrary number of knots. In the previous example, we use the same number of knots than the number of observations to calibrate. In such case, we can calibrate the parameters using the following iterative process:

1. We first estimate the parameter λ_1 for the earliest maturity Δt_1.
2. Assuming that $\left(\hat{\lambda}_1, \ldots, \hat{\lambda}_{i-1}\right)$ have been estimated, we calculate $\hat{\lambda}_i$ for the next maturity Δt_i.
3. We iterate Step 2 until the last maturity Δt_m.

This algorithm works well if the knots $t_m^\star$ exactly match the maturities. It is known as the bootstrap method and is very popular to estimate the survival function from market prices. Let $\{\mathcal{s}(T_1), \ldots, \mathcal{s}(T_M)\}$ be a set of CDS spreads for a given name. Assuming that $T_1 < T_2 < \ldots < T_M$, we consider the

[65] We have $\mathrm{PD} = 1 - \mathbf{S}(1)$.

piecewise exponential model with $t_m^\star = T_m$. We first estimate $\hat{\lambda}_1$ such that the theoretical spread is equal to $\mathcal{s}(T_1)$. We then calibrate the hazard function in order to retrieve the spread $\mathcal{s}(T_2)$ for the second maturity. This means to consider that $\lambda(t)$ is known and equal to $\hat{\lambda}_1$ until time T_1 whereas $\lambda(t)$ is unknown from T_1 to T_2:

$$\lambda(t) = \begin{cases} \hat{\lambda}_1 & \text{if } t \in \left]0, T_1\right] \\ \lambda_2 & \text{if } t \in \left]T_1, T_2\right] \end{cases}$$

Estimating $\hat{\lambda}_2$ is therefore straightforward because it is equivalent to solve one equation with one variable. We proceed in a similar way for the other maturities.

Example 33 *We assume that the term structure of interest rates is generated by the Nelson-Siegel model with $\theta_1 = 5\%$, $\theta_2 = -5\%$, $\theta_3 = 6\%$ and $\theta_4 = 10$. We consider three credit curves, whose CDS spreads expressed in bps are given in the following table:*

Maturity (in years)	#1	#2	#3
1	50	50	350
3	60	60	370
5	70	90	390
7	80	115	385
10	90	125	370

The recovery rate is set to 40%.

TABLE 3.30: Calibrated piecewise exponential model from CDS prices

Maturity (in years)	#1	#2	#3
1	83.3	83.3	582.9
3	110.1	110.1	637.5
5	140.3	235.0	702.0
7	182.1	289.6	589.4
10	194.1	241.9	498.5

Using the bootstrap method, we obtain results in Table 3.30. We notice that the piecewise exponential model coincide for the credit curves #1 and #2 for $t < 3$ years. This is normal because the CDS spreads of the two credit curves are equal when the maturity is less or equal than 3 years. The third credit curve illustrates that the bootstrap method is highly sensitive to small differences. Indeed, the calibrated intensity parameter varies from 499 to 702 bps while the CDS spreads varies from 350 to 390 bps. Finally, the survival function associated to these 3 bootstrap calibrations are shown in Figure 3.25.

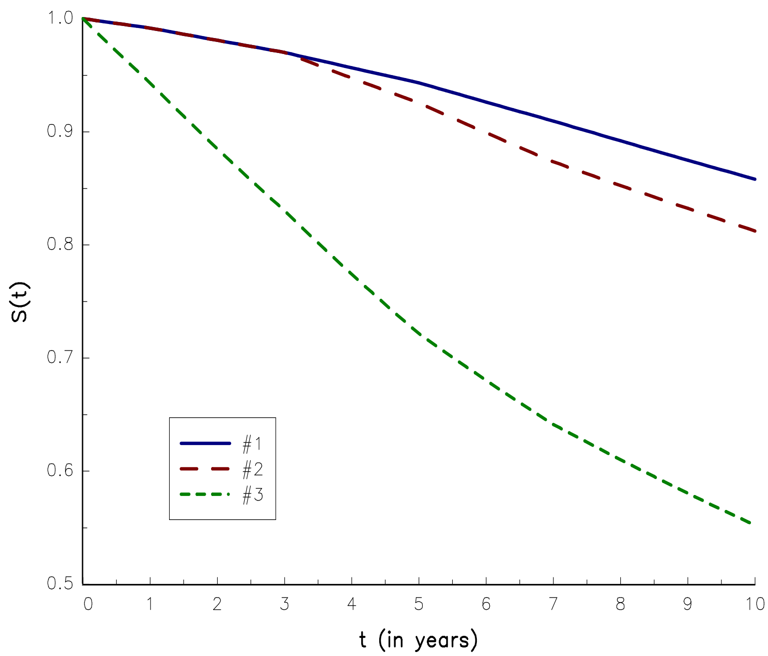

FIGURE 3.25: Calibrated survival function from CDS prices

Remark 34 *Other methods for estimating the probability of default are presented in Chapter 19 dedicated to credit scoring models.*

3.3.3.2 Transition probability matrix

When dealing with risk classes, it is convenient to model the matrix of transition probabilities. For instance, this approach is used for modeling credit rating migration.

Discrete time modeling We consider a time-homogeneous Markov chain $\mathfrak{R}$, whose the transition matrix is $P = (p_{i,j})$. We note $\mathcal{S} = \{1, 2, \ldots, K\}$ the state space of the chain and $p_{i,j}$ is the probability that the entity migrates from rating i to rating j. The matrix P satisfies the following properties:

- $\forall i, j \in \mathcal{S}$, $p_{i,j} \geq 0$;
- $\forall i \in \mathcal{S}$, $\sum_{j=1}^{K} p_{i,j} = 1$.

In credit risk, we generally assume that K is the absorbing state (or the default state), implying that any entity which has reached this state remains in this state. In this case, we have $p_{K,K} = 1$. Let $\mathfrak{R}(t)$ be the value of the state at time t. We define $p(s, i; t, j)$ as the probability that the entity reaches the state j at time t given that it has reached the state i at time s . We have:

$$\begin{aligned} p(s,i;t,j) &= \Pr\{\mathfrak{R}(t)=j \mid \mathfrak{R}(s)=i\} \\ &= p_{i,j}^{(t-s)} \end{aligned}$$

This probability only depends on the duration between s and t because of the Markov property. Therefore, we can restrict the analysis by calculating the n-step transition probability:

$$p_{i,j}^{(n)} = \Pr\{\mathfrak{R}(t+n)=j \mid \mathfrak{R}(t)=i\}$$

and the associated n-step transition matrix $P^{(n)} = \left(p_{i,j}^{(n)}\right)$. For $n = 2$, we obtain:

$$\begin{aligned} p_{i,j}^{(2)} &= \Pr\{\mathfrak{R}(t+2)=j \mid \mathfrak{R}(t)=i\} \\ &= \sum_{k=1}^{K} \Pr\{\mathfrak{R}(t+2)=j, \mathfrak{R}(t+1)=k \mid \mathfrak{R}(t)=i\} \\ &= \sum_{k=1}^{K} \Pr\{\mathfrak{R}(t+2)=j \mid \mathfrak{R}(t+1)=k\} \times \Pr\{\mathfrak{R}(t+1)=k \mid \mathfrak{R}(t)=i\} \\ &= \sum_{k=1}^{K} p_{i,k} \times p_{k,j} \end{aligned}$$

In a similar way, we obtain:

$$p_{i,j}^{(n+m)} = \sum_{k=1}^{K} p_{i,k}^{(n)} \times p_{k,j}^{(m)} \qquad \forall n, m > 0 \tag{3.35}$$

This equation is called the Chapman-Kolmogorov equation. In matrix form, we have:

$$P^{(n+m)} = P^{(n)} \times P^{(m)}$$

with the convention $P^{(0)} = I$. In particular, we have:

$$\begin{aligned} P^{(n)} &= P^{(n-1)} \times P^{(1)} \\ &= P^{(n-2)} \times P^{(1)} \times P^{(1)} \\ &= \prod_{t=1}^{n} P^{(1)} \\ &= P^{n} \end{aligned}$$

We deduce that:

$$p(t,i;t+n,j) = p_{i,j}^{(n)} = \mathbf{e}_i^\top P^n \mathbf{e}_j \tag{3.36}$$

When we apply this framework to credit risk, $\mathfrak{R}(t)$ denotes the rating (or the risk class) of the firm at time t, $p_{i,j}$ is the one-period transition probability from rating i to rating j and $p_{i,K}$ is the one-period default probability of rating i. In Table 3.31, we report the S&P one-year transition matrix for corporate bonds estimated by Kavvathas (2001). We read the figures as follows[66]: a firm rated AAA has a one-year probability of 92.83% to remain AAA; its probability to become AA is 6.50%; a firm rated CCC defaults one year later with a probability equal to 23.50%; etc.

TABLE 3.31: Example of credit migration matrix (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	92.82	6.50	0.56	0.06	0.06	0.00	0.00	0.00
AA	0.63	91.87	6.64	0.65	0.06	0.11	0.04	0.00
A	0.08	2.26	91.66	5.11	0.61	0.23	0.01	0.04
BBB	0.05	0.27	5.84	87.74	4.74	0.98	0.16	0.22
BB	0.04	0.11	0.64	7.85	81.14	8.27	0.89	1.06
B	0.00	0.11	0.30	0.42	6.75	83.07	3.86	5.49
CCC	0.19	0.00	0.38	0.75	2.44	12.03	60.71	23.50
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Source: Kavvathas (2001).

In Table 3.32 and 3.33, we have reported the two-year and five-year transition matrices. We detail below the calculation of $p^{(2)}_{\mathsf{AAA},\mathsf{AAA}}$:

$$\begin{aligned}
p^{(2)}_{\mathsf{AAA},\mathsf{AAA}} &= p_{\mathsf{AAA},\mathsf{AAA}} \times p_{\mathsf{AAA},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{AA}} \times p_{\mathsf{AA},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{A}} \times p_{\mathsf{A},\mathsf{AAA}} + \\
&\quad p_{\mathsf{AAA},\mathsf{BBB}} \times p_{\mathsf{BBB},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{BB}} \times p_{\mathsf{BB},\mathsf{AAA}} + p_{\mathsf{AAA},\mathsf{B}} \times p_{\mathsf{B},\mathsf{AAA}} + \\
&\quad p_{\mathsf{AAA},\mathsf{CCC}} \times p_{\mathsf{CCC},\mathsf{AAA}} \\
&= 0.9283^2 + 0.0650 \times 0.0063 + 0.0056 \times 0.0008 + \\
&\quad 0.0006 \times 0.0005 + 0.0006 \times 0.0004 \\
&= 86.1970\%
\end{aligned}$$

We note $\pi_i^{(n)}$ the probability of the state i at time n:

$$\pi_i^{(n)} = \Pr\left\{\mathfrak{R}(n) = i\right\}$$

and $\pi^{(n)} = \left(\pi_1^{(n)}, \ldots, \pi_K^{(n)}\right)$ the probability distribution. By construction, we have:

$$\pi^{(n+1)} = P^\top \pi^{(n)}$$

The Markov chain $\mathfrak{R}$ admits a stationary distribution $\pi^\star$ if[67]:

$$\pi^\star = P^\top \pi^\star$$

[66]The rows represent the initial rating whereas the columns indicate the final rating.
[67]Not all Markov chain behave in this way, meaning that $\pi^\star$ does not necessarily exist.

TABLE 3.32: Two-year transition matrix P^2 (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	86.20	12.02	1.47	0.18	0.11	0.01	0.00	0.00
AA	1.17	84.59	12.23	1.51	0.18	0.22	0.07	0.02
A	0.16	4.17	84.47	9.23	1.31	0.51	0.04	0.11
BBB	0.10	0.63	10.53	77.66	8.11	2.10	0.32	0.56
BB	0.08	0.24	1.60	13.33	66.79	13.77	1.59	2.60
B	0.01	0.21	0.61	1.29	11.20	70.03	5.61	11.03
CCC	0.29	0.04	0.68	1.37	4.31	17.51	37.34	38.45
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

TABLE 3.33: Five-year transition matrix P^5 (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	69.23	23.85	5.49	0.96	0.31	0.12	0.02	0.03
AA	2.35	66.96	24.14	4.76	0.86	0.62	0.13	0.19
A	0.43	8.26	68.17	17.34	3.53	1.55	0.18	0.55
BBB	0.24	1.96	19.69	56.62	13.19	5.32	0.75	2.22
BB	0.17	0.73	5.17	21.23	40.72	20.53	2.71	8.74
B	0.07	0.47	1.73	4.67	16.53	44.95	5.91	25.68
CCC	0.38	0.24	1.37	2.92	7.13	18.51	9.92	59.53
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

In this case, $\pi_i^\star$ is the limiting probability of state i:

$$\lim_{n=\infty} p_{k,i}^{(n)} = \pi_i^\star$$

We can interpret $\pi_i^\star$ as the average duration spent by the chain $\mathfrak{R}$ in the state i. Let $\mathcal{T}_i$ be the return period[68] of state i:

$$\mathcal{T}_i = \inf \{n : \mathfrak{R}(n) = i \mid \mathfrak{R}(0) = i\}$$

The average return period is then equal to:

$$\mathbb{E}[\mathcal{T}_i] = \frac{1}{\pi_i^\star}$$

For credit migration matrices, there is no stationary distribution because the long-term rating $\mathfrak{R}(\infty)$ is the absorbing state as noted by Jafry and Schuermann:

> "*Given sufficient time, all firms will eventually sink to the default state. This behavior is clearly a mathematical artifact, stemming*

[68]This concept plays an important role when designing stress scenarios (see Chapter 18).

from the idealized linear, time invariant assumptions inherent in the simple Markov model. In reality the economy (and hence the migration matrix) will change on time-scales far shorter than required to reach the idealized default steady-state proscribed by an assumed constant migration matrix" (Jafry and Schuermann, 2004, page 2609).

We note that the survival function $\mathbf{S}_i(t)$ of a firm whose initial rating is the state i is given by:

$$\begin{aligned}\mathbf{S}_i(t) &= 1 - \Pr\{\Re(t) = K \mid \Re(0) = i\} \\ &= 1 - \mathbf{e}_i^\top P^t \mathbf{e}_K \qquad (3.37)\end{aligned}$$

In the piecewise exponential model, we recall that the survival function has the following expression:

$$\mathbf{S}(t) = \mathbf{S}\left(t_{m-1}^\star\right) e^{-\lambda_m\left(t - t_{m-1}^\star\right)}$$

for $t \in \left]t_{m-1}^\star, t_m^\star\right]$. We deduce that $\mathbf{S}(t_m^\star) = \mathbf{S}\left(t_{m-1}^\star\right) e^{-\lambda_m\left(t_m^\star - t_{m-1}^\star\right)}$, implying that:

$$\ln \mathbf{S}(t_m^\star) = \ln \mathbf{S}\left(t_{m-1}^\star\right) - \lambda_m\left(t_m^\star - t_{m-1}^\star\right)$$

and:

$$\lambda_m = \frac{\ln \mathbf{S}\left(t_{m-1}^\star\right) - \ln \mathbf{S}(t_m^\star)}{t_m^\star - t_{m-1}^\star}$$

It is then straightforward to estimate the piecewise hazard function:

- the knots of the piecewise function are the years $m \in \mathbb{N}^*$;
- for each initial rating i, the hazard function $\lambda_i(t)$ is defined as:

  $$\lambda_i(t) = \lambda_{i,m} \qquad \text{if } t \in \left]m-1, m\right]$$

  with:

  $$\begin{aligned}\lambda_{i,m} &= \frac{\ln \mathbf{S}_i(m-1) - \ln \mathbf{S}_i(m)}{m - (m-1)} \\ &= \ln\left(\frac{1 - \mathbf{e}_i^\top P^{m-1}\mathbf{e}_K}{1 - \mathbf{e}_i^\top P^m \mathbf{e}_K}\right)\end{aligned}$$

  and $P^0 = I$.

If we consider the credit migration matrix given in Table 3.31 and estimate the piecewise exponential model, we obtain the hazard function[69] $\lambda_i(t)$ shown in Figure 3.26. For good initial ratings, hazard rates are low for short maturities

[69]Contrary to the graph suggests, $\lambda_i(t)$ is a piecewise constant function (see details of the curve in the fifth panel for very short maturities).

and increase with time. For bad initial ratings, we obtain the opposite effect, because the firm can only improve its rating if it did not default. We observe that the hazard function of all ratings converges to the same level, which is equal to 102.63 bps. This indicates the long-term hazard rate of the Markov chain, meaning that 1.02% of firms default every year.

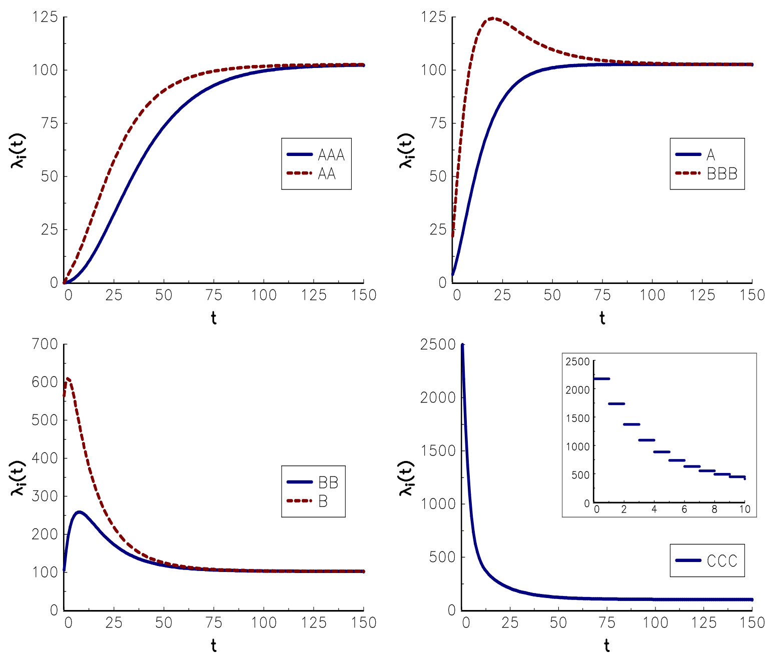

FIGURE 3.26: Estimated hazard function $\lambda_i(t)$ from the credit migration matrix

Continuous time modeling We now consider the case $t \in \mathbb{R}_+$. We note $P(s;t)$ the transition matrix defined as follows:

$$\begin{aligned} P_{i,j}(s;t) &= p(s,i;t,j) \\ &= \Pr\{\mathfrak{R}(t) = j \mid \mathfrak{R}(s) = i\} \end{aligned}$$

Assuming that the Markov chain is time-homogenous, we have $P(t) = P(0;t)$. Jarrow *et al.* (1997) introduce the generator matrix $\Lambda = (\lambda_{i,j})$ where $\lambda_{i,j} \geq 0$ for all $i \neq j$ and:

$$\lambda_{i,i} = -\sum_{j \neq i}^{K} \lambda_{i,j}$$

In this case, the transition matrix satisfies the following relationship:

$$P(t) = \exp(t\Lambda) \tag{3.38}$$

where $\exp(A)$ is the matrix exponential of A. Let us give a probabilistic interpretation of Λ. If we assume that the probability of jumping from rating i to rating j in a short time period Δt is proportional to Δt, we have:

$$p(t, i; t + \Delta t, j) = \lambda_{i,j} \Delta t$$

The matrix form of this equation is $P(t; t + \Delta t) = \Lambda \, \Delta t$. We deduce that:

$$\begin{aligned} P(t + \Delta t) &= P(t) \, P(t; t + \Delta t) \\ &= P(t) \, \Lambda \, \Delta t \end{aligned}$$

We deduce that:

$$\mathrm{d}P(t) = P(t) \, \Lambda \, \mathrm{d}t$$

Because we have $\exp(\mathbf{0}) = I$, we obtain the solution $P(t) = \exp(t\Lambda)$. We then interpret $\lambda_{i,j}$ as the instantaneous transition rate of jumping from rating i to rating j.

Remark 35 *In Appendix A.1.1.2, we present the matrix exponential function and its mathematical properties. In particular, we have* $e^{A+B} = e^A e^B$ *and* $e^{A(s+t)} = e^{As} e^{At}$ *where* A *and* B *are two square matrices such that* $AB = BA$ *and* s *and* t *are two real numbers.*

Example 34 *We consider a rating system with three states: A (good rating), B (bad rating) and D (default). The Markov generator is equal to:*

$$\Lambda = \begin{pmatrix} -0.30 & 0.20 & 0.10 \\ 0.15 & -0.40 & 0.25 \\ 0.00 & 0.00 & 0.00 \end{pmatrix}$$

The one-year transition matrix is equal to:

$$P(1) = e^{\Lambda} = \begin{pmatrix} 75.16\% & 14.17\% & 10.67\% \\ 10.63\% & 68.07\% & 21.30\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

For the two-year maturity, we get:

$$P(2) = e^{2\Lambda} = \begin{pmatrix} 58.00\% & 20.30\% & 21.71\% \\ 15.22\% & 47.85\% & 36.93\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

We verify that $P(2) = P(1)^2$. This derives from the property of the matrix exponential:

$$P(t) = e^{t\Lambda} = \left(e^{\Lambda}\right)^t = P(1)^t$$

The continuous-time framework allows to calculate transition matrices for non-integer maturities, which do not correspond to full years. For instance, the one-month transition matrix of the previous example is equal to:

$$P(2) = e^{\frac{1}{12}\Lambda} = \begin{pmatrix} 97.54\% & 1.62\% & 0.84\% \\ 1.21\% & 96.73\% & 2.05\% \\ 0.00\% & 0.00\% & 100.00\% \end{pmatrix}$$

One of the issue with the continuous time framework is to estimate the Markov generator Λ. One solution consists in using the empirical transition matrix $\hat{P}(t)$, which have been calculated for a given time horizon t. In this case, the estimate $\hat{\Lambda}$ must satisfy the relationship $\hat{P}(t) = \exp\left(t\hat{\Lambda}\right)$. We deduce that:

$$\hat{\Lambda} = \frac{1}{t}\ln\left(\hat{P}(t)\right)$$

where $\ln(A)$ is the matrix logarithm of A. However, the matrix $\hat{\Lambda}$ can not verify the Markov conditions $\hat{\lambda}_{i,j} \geq 0$ for all $i \neq j$ and $\sum_{j=1}^{K} \lambda_{i,j} = 0$. For instance, if we consider the previous S&P transition matrix, we obtain the generator $\hat{\Lambda}$ given in Table 3.34. We notice that six off-diagonal elements of the matrix are negative[70]. This implies that we can obtain transition probabilities which are negative for short maturities.

[70] We have also calculated the estimator described in Israel *et al.* (2001):

$$\breve{\Lambda} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\left(\hat{P} - I\right)^n}{n}$$

We do not obtain the same matrix as for the estimator $\hat{\Lambda}$, but there are also six negative off-diagonal elements (see Table 3.35).

TABLE 3.34: Markov generator $\hat{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−747.49	703.67	35.21	3.04	6.56	−0.79	−0.22	0.02
AA	67.94	−859.31	722.46	51.60	2.57	10.95	4.92	−1.13
A	7.69	245.59	−898.16	567.70	53.96	20.65	−0.22	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	−0.84	11.83	30.11	8.71	818.31	−1936.82	539.18	529.52
CCC	25.11	−2.89	44.11	84.87	272.05	1678.69	−5043.00	2941.06
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 3.35: Markov generator $\breve{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−745.85	699.11	38.57	2.80	6.27	−0.70	−0.16	−0.05
AA	67.54	−855.70	716.56	54.37	2.81	10.81	4.62	−1.01
A	7.77	243.62	−891.46	560.45	56.33	20.70	0.07	2.53
BBB	5.06	22.68	641.55	−1335.03	542.46	91.05	16.09	16.15
BB	4.18	10.12	48.00	903.40	−2111.65	965.71	98.28	81.96
B	−0.56	11.61	29.31	19.39	789.99	−1887.69	491.46	546.49
CCC	23.33	−1.94	42.22	81.25	272.44	1530.66	−4725.22	2777.25
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 3.36: Markov generator $\bar{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−748.50	703.67	35.21	3.04	6.56	0.00	0.00	0.02
AA	67.94	−860.44	722.46	51.60	2.57	10.95	4.92	0.00
A	7.69	245.59	−898.38	567.70	53.96	20.65	0.00	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	0.00	11.83	30.11	8.71	818.31	−1937.66	539.18	529.52
CCC	25.11	0.00	44.11	84.87	272.05	1678.69	−5045.89	2941.06
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

TABLE 3.37: Markov generator $\tilde{\Lambda}$ (in bps)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	−747.99	703.19	35.19	3.04	6.55	0.00	0.00	0.02
AA	67.90	−859.88	721.98	51.57	2.57	10.94	4.92	0.00
A	7.69	245.56	−898.27	567.63	53.95	20.65	0.00	2.80
BBB	5.07	21.53	650.21	−1352.28	557.64	85.56	16.08	16.19
BB	4.22	10.22	41.74	930.55	−2159.67	999.62	97.35	75.96
B	0.00	11.83	30.10	8.71	818.14	−1937.24	539.06	529.40
CCC	25.10	0.00	44.10	84.84	271.97	1678.21	−5044.45	2940.22
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

In this case, Israel *et al.* (2001) propose two estimators to obtain a valid generator:

1. the first approach consists in adding the negative values back into the diagonal values:

$$\left\{\begin{array}{ll} \bar{\lambda}_{i,j} = \max\left(\hat{\lambda}_{i,j}, 0\right) & i \neq j \\ \bar{\lambda}_{i,i} = \hat{\lambda}_{i,i} + \sum_{j \neq i} \min\left(\hat{\lambda}_{i,j}, 0\right) & \end{array}\right.$$

2. in the second method, we carry forward the negative values on the matrix entries which have the correct sign:

$$\left\{\begin{array}{l} G_i = \left|\hat{\lambda}_{i,i}\right| + \sum_{j \neq i} \max\left(\hat{\lambda}_{i,j}, 0\right) \\ B_i = \sum_{j \neq i} \max\left(-\hat{\lambda}_{i,j}, 0\right) \\ \tilde{\lambda}_{i,j} = \left\{\begin{array}{ll} 0 & \text{if } i \neq j \text{ and } \hat{\lambda}_{i,j} < 0 \\ \hat{\lambda}_{i,j} - B_i \left|\hat{\lambda}_{i,j}\right| / G_i & \text{if } G_i > 0 \\ \hat{\lambda}_{i,j} & \text{if } G_i = 0 \end{array}\right. \end{array}\right.$$

Using the estimator $\hat{\Lambda}$ and the two previous algorithm, we obtain the valid generators given in Tables 3.36 and 3.37. We find that $\left\|\hat{P} - \exp\left(\bar{\Lambda}\right)\right\|_2 = 11.02 \times 10^{-4}$ and $\left\|\hat{P} - \exp\left(\tilde{\Lambda}\right)\right\|_2 = 10.95 \times 10^{-4}$, meaning that the Markov generator $\tilde{\Lambda}$ is the estimator that minimizes the distance to $\hat{P}$. We can then calculate the transition matrix for all maturities, and not only for calendar years. For instance, we report the 207-day transition matrix $P\left(\frac{207}{305}\right) = \exp\left(\frac{207}{365}\tilde{\Lambda}\right)$ in Table 3.38.

TABLE 3.38: 207-day transition matrix (in %)

	AAA	AA	A	BBB	BB	B	CCC	D
AAA	95.85	3.81	0.27	0.03	0.04	0.00	0.00	0.00
AA	0.37	95.28	3.90	0.34	0.03	0.06	0.02	0.00
A	0.04	1.33	95.12	3.03	0.33	0.12	0.00	0.02
BBB	0.03	0.14	3.47	92.75	2.88	0.53	0.09	0.11
BB	0.02	0.06	0.31	4.79	88.67	5.09	0.53	0.53
B	0.00	0.06	0.17	0.16	4.16	89.84	2.52	3.08
CCC	0.12	0.01	0.23	0.45	1.45	7.86	75.24	14.64
D	0.00	0.00	0.00	0.00	0.00	0.00	0.00	100.00

Remark 36 *The continuous time framework is more flexible when modeling credit risk. For instance, the expression of the survival function becomes:*

$$\begin{aligned} \mathbf{S}_i(t) &= \Pr\{\mathfrak{R}(t) = K \mid \mathfrak{R}(0) = i\} \\ &= 1 - \mathbf{e}_i^\top \exp(t\Lambda)\,\mathbf{e}_K \end{aligned}$$

We can therefore calculate the density function in a more easy way:

$$\begin{aligned} f_i(t) &= -\partial_t \mathbf{S}_i(t) \\ &= \mathbf{e}_i^\top \Lambda \exp(t\Lambda)\, \mathbf{e}_K \end{aligned}$$

For illustration purposes, we represent the density function of S&P ratings estimated with the valid generator $\tilde{\Lambda}$ in Figure 3.27.

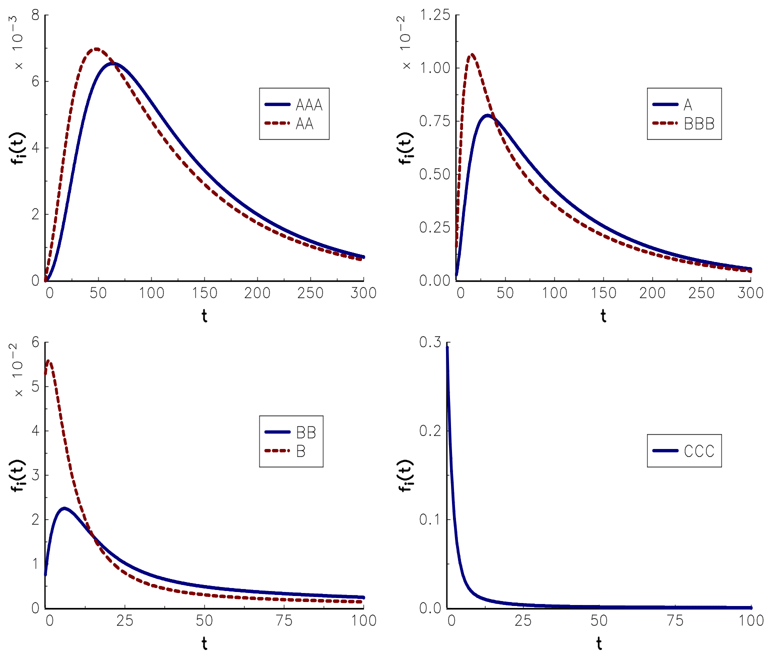

FIGURE 3.27: Density function $f_i(t)$ of S&P ratings

3.3.3.3 Structural models

3.3.4 Default correlation

3.3.4.1 Factor models

3.3.4.2 Copula models

3.3.4.3 Estimation methods

3.3.5 Granularity and concentration

3.3.5.1 Difference between fine-grained and concentrated portfolios

3.3.5.2 Granularity adjustment

3.4 Exercises

3.4.1 Single and multi-name credit default swaps

1. We assume that the default time $\boldsymbol{\tau}$ follows an exponential distribution with parameter λ. Write the cumulative distribution function $\mathbf{F}$, the survival function $\mathbf{S}$ and the density function f of the random variable $\boldsymbol{\tau}$. How do we simulate this default time?

2. We consider a CDS 3M with two-year maturity and \$1 mn notional principal. The recovery rate $\mathcal{R}$ is equal to 40% whereas the spread $\mathcal{s}$ is equal to 150 bps at the inception date. We assume that the protection leg is paid at the default time.

   (a) Give the cash flow chart. What is the P&L of the protection seller A if the reference entity does not default? What is the PnL of the protection buyer B if the reference entity defaults in one year and two months?

   (b) What is the relationship between $\mathcal{s}$, $\mathcal{R}$ and λ? What is the implied one-year default probability at the inception date?

   (c) Seven months later, the CDS spread has increased and is equal to 450 bps. Estimate the new default probability. The protection buyer B decides to realize his P&L. For that, he reassigns the CDS contract to the counterparty C. Explain the offsetting mechanism if the risky PV01 is equal to 1.189.

3. We consider the following CDS spread curves for three reference entities:

Maturity	#1	#2	#3
6M	130 bps	1 280 bps	30 bps
1Y	135 bps	970 bps	35 bps
3Y	140 bps	750 bps	50 bps
5Y	150 bps	600 bps	80 bps

   (a) Define the notion of credit curve. Comment the previous spread curves.

   (b) Using the Merton Model, we estimate that the one-year default probability is equal to 2.5% for #1, 5% for #2 and 2% for #3 at a five-year horizon time. Which arbitrage position could we consider about the reference entity #2?

4. We consider a basket of n single-name CDS.

   (a) What is a first-to-default (FtD), a second-to-default (StD) and a last-to-default (LtD)?

(b) Define the notion of default correlation. What is its impact on three previous spreads?

(c) We assume that $n = 3$. Show the following relationship:

$$\mathcal{S}_1^{\mathrm{CDS}} + \mathcal{S}_2^{\mathrm{CDS}} + \mathcal{S}_3^{\mathrm{CDS}} = \mathcal{S}^{\mathrm{FtD}} + \mathcal{S}^{\mathrm{StD}} + \mathcal{S}^{\mathrm{LtD}}$$

where $\mathcal{S}_1^{\mathrm{CDS}}$ is the CDS spread of the i^{th} reference entity.

(d) Many professionals and academics believe that the subprime crisis is due to the use of the Normal copula. Using the results of the previous question, what could you conclude?

3.4.2 Risk contribution in the Basel II model

1. We note L the portfolio loss of n credit and w_i the exposure at default of the i^{th} credit. We have:

$$L(w) = w^\top \varepsilon = \sum_{i=1}^{n} w_i \times \varepsilon_i \tag{3.39}$$

where ε_i is the unit loss of the i^{th} credit. Let $\mathbf{F}$ be the cumulative distribution function of $L(w)$.

(a) We assume that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \sim \mathcal{N}(\mathbf{0}, \Sigma)$. Compute the value-at-risk $\mathrm{VaR}_\alpha(w)$ of the portfolio when the confidence level is equal to α.

(b) Deduce the marginal value-at-risk of the i^{th} credit. Define then the risk contribution $\mathcal{RC}_i$ of the i^{th} credit.

(c) Check that the marginal value-at-risk is equal to:

$$\frac{\partial\, \mathrm{VaR}_\alpha(w)}{\partial\, w_i} = \mathbb{E}\left[\varepsilon_i \mid L(w) = \mathbf{F}^{-1}(\alpha)\right]$$

Comment on this result.

2. We consider the Basel II model of credit risk and the value-at-risk risk measure. The expression of the portfolio loss is given by:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathrm{LGD}_i \times \mathbb{1}\{\boldsymbol{\tau}_i < M_i\} \tag{3.40}$$

(a) Define the different parameters EAD_i, LGD_i, $\boldsymbol{\tau}_i$ and M_i. Show that Model (3.40) can be written as Model (3.39) by identifying w_i and ε_i.

(b) What are the necessary assumptions $(\mathcal{H})$ to obtain this result:

$$\mathbb{E}\left[\varepsilon_i \mid L = \mathbf{F}^{-1}(\alpha)\right] = \mathbb{E}\left[\mathrm{LGD}_i\right] \times \mathbb{E}\left[D_i \mid L = \mathbf{F}^{-1}(\alpha)\right]$$

with $D_i = \mathbb{1}\{\boldsymbol{\tau}_i < M_i\}$.

(c) Deduce the risk contribution $\mathcal{RC}_i$ of the i^{th} credit and the value-at-risk of the credit portfolio.

(d) We assume that the credit i defaults before the maturity M_i if a latent variable Z_i goes below a barrier B_i:

$$\tau_i \leq M_i \Leftrightarrow Z_i \leq B_i$$

We consider that $Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i$ where Z_i, X and ε_i are three independent Gaussian variables $\mathcal{N}(0,1)$. X is the factor (or the systemic risk) and ε_i is the idiosyncratic risk.

i. Interpret the parameter ρ.

ii. Calculate the unconditional default probability:

$$p_i = \Pr\{\boldsymbol{\tau}_i \leq M_i\}$$

iii. Calculate the conditional default probability:

$$p_i(x) = \Pr\{\boldsymbol{\tau}_i \leq M_i \mid X = x\}$$

(e) Show that, under the previous assumptions ($\mathcal{H}$), the risk contribution $\mathcal{RC}_i$ of the i^{th} credit is:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\rho}\Phi^{-1}(\alpha)}{\sqrt{1-\rho}}\right) \tag{3.41}$$

when the risk measure is the value-at-risk.

3. We now assume that the risk measure is the expected shortfall:

$$\mathrm{ES}_\alpha(w) = \mathbb{E}[L \mid L \geq \mathrm{VaR}_\alpha(w)]$$

(a) In the case of the Basel II framework, show that we have:

$$\mathrm{ES}_\alpha(w) = \sum_{i=1}^{n} \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \mathbb{E}\left[p_i(X) \mid X \leq \Phi^{-1}(1-\alpha)\right]$$

(b) By using the following result:

$$\int_{-\infty}^{c} \Phi(a+bx)\phi(x)\,\mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

where $\Phi_2(x,y;\rho)$ is the cdf of the bivariate Gaussian distribution with correlation ρ on the space $[-\infty, x] \times [-\infty, y]$, deduce that the risk contribution $\mathcal{RC}_i$ of the i^{th} credit in the Basel II model is:

$$\mathcal{RC}_i = \mathrm{EAD}_i \times \mathbb{E}[\mathrm{LGD}_i] \times \frac{\mathbf{C}\left(1-\alpha, p_i; \sqrt{\rho}\right)}{1-\alpha} \tag{3.42}$$

when the risk measure is the expected shortfall. Here $\mathbf{C}(u_1, u_2; \theta)$ is the Normal copula with parameter θ.

(c) What do the results (3.41) and (3.42) become if the correlation ρ is equal to zero? Same question if $\rho = 1$.

4. The risk contributions (3.41) and (3.42) were obtained considering the assumptions $(\mathcal{H})$ and the default model defined in Question 2(d). What are the implications in terms of Pillar 2?

3.4.3 Calibration of the piecewise exponential model

1. We denote by $\mathbf{F}$ and $\mathbf{S}$ the distribution and survival functions of the default time $\boldsymbol{\tau}$. Define the function $\mathbf{S}(t)$ and deduce the expression of the associated density function $f(t)$.

2. Define the hazard rate $\lambda(t)$. Deduce that the exponential model corresponds to the particular case $\lambda(t) = \lambda$.

3. We assume that the interest rate r is constant. In a continuous-time model, we recall that the CDS spread is given by the following expression:
$$\mathcal{s}(T) = \frac{(1 - \boldsymbol{\mathcal{R}}) \times \int_0^T e^{-rt} f(t)\,\mathrm{d}t}{\int_0^T e^{-rt} \mathbf{S}(t)\,\mathrm{d}t} \tag{3.43}$$
where $\boldsymbol{\mathcal{R}}$ is the recovery rate and T is the maturity of the CDS. Find the triangle relationship when $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$.

4. Let us assume that:
$$\lambda(t) = \begin{cases} \lambda_1 & \text{if } t \leq 3 \\ \lambda_2 & \text{if } 3 < t \leq 5 \\ \lambda_3 & \text{if } t > 5 \end{cases}$$

(a) Give the expression of the survival distribution $\mathbf{S}(t)$ and calculate the density function $f(t)$. Verify that the hazard rate $\lambda(t)$ is a piecewise constant function.

(b) Find the expression of the CDS spread using Equation (3.43).

(c) We consider three credit default swaps, whose maturity is respectively 3, 5 and 7 years. Show that the calibration of the piecewise exponential model implies to derive a set of 3 equations and then solve for the unknown variables λ_1, λ_2 and λ_3. What is the name of this calibration method?

(d) Find an approximated solution when r is equal to zero and λ_m is small. Comment on this result.

(e) We consider the following numerical application: $r = 5\%$, $\mathcal{s}(3) = 100$ bps, $\mathcal{s}(5) = 150$ bps, $\mathcal{s}(7) = 160$ bps and $\boldsymbol{\mathcal{R}} = 40\%$. Estimate the implied hazard rate function.

(f) Using the previous numerical results, simulate the default time with the uniform random numbers 0.96, 0.23, 0.90 and 0.80.

3.4.4 Modeling loss given default

1. What is the difference between the recovery rate and the loss given default?

2. We consider a bank that grants 250 000 credits per year. The average amount of a credit is equal to \$50 000. We estimate that the average default probability is equal to 1% and the average recovery rate is equal to 65%. The total annual cost of the litigation department is equal to \$12.5 mn. Give an estimation of the loss given default?

3. The probability density function of the Beta probability distribution $\mathcal{B}(a, b)$ is:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\mathbf{B}(a, b)}$$

   where $\mathbf{B}(a, b) = \int_0^1 u^{a-1}(1-u)^{b-1}\, \mathrm{d}u$.

   (a) Why is the Beta probability distribution a good candidate to model the loss given default? Which parameter pair (a, b) correspond to the uniform probability distribution?

   (b) Let us consider a sample $(x_1, \ldots, x_n)$ of n losses in case of default. Write the log-likelihood function. Deduce the first order conditions of the maximum likelihood estimator.

   (c) We recall that the first two moments of the Beta probability distribution are:

$$\begin{aligned} \mathbb{E}[X] &= \frac{a}{a+b} \\ \sigma^2(X) &= \frac{ab}{(a+b)^2(a+b+1)} \end{aligned}$$

   Find the method of moments estimator.

4. We consider a risk class $\mathcal{C}$ corresponding to a customer/product segmentation specific to retail banking. A statistical analysis of 1 000 loss data available for this risk class gives the following results:

LGD_k	0%	25%	50%	75%	100%
n_k	100	100	600	100	100

   where n_k is the number of data corresponding to LGD_k.

   (a) We consider a portfolio of 100 homogeneous credits, which belong to the risk class $\mathcal{C}$. The notional is \$10 000 whereas the annual default probability is equal to 1%. Calculate the expected loss of this credit portfolio with a one-year horizon time if we use the previous empirical distribution to model the LGD parameter.

(b) We assume that the LGD parameter follows a Beta distribution $\mathcal{B}(a, b)$. Calibrate the parameters a and b with the method of moments.

(c) We assume that the Basel II model is valid. We consider the portfolio described in Question 4(a) and calculate the unexpected loss. What is the impact if we use a uniform probability distribution instead of the calibrated Beta probability distribution? Why does this result hold even if we consider different factors to model the default time?

3.4.5 Modeling default times with a Markov chain

We consider a rating system with 4 risk classes (A, B, C and D), where rating D represents the default. The transition probability matrix with a two-year horizon time is equal to[71]:

$$P(2) = \begin{pmatrix} 94 & 3 & 2 & 1 \\ 10 & 80 & 5 & 5 \\ 10 & 10 & 60 & 20 \\ 0 & 0 & 0 & 100 \end{pmatrix}$$

We also have:

$$P(4) = \begin{pmatrix} 88.860 & 5.420 & 3.230 & 2.490 \\ 17.900 & 64.800 & 7.200 & 10.100 \\ 16.400 & 14.300 & 36.700 & 32.600 \\ 0.000 & 0.000 & 0.000 & 100.000 \end{pmatrix}$$

and:

$$P(6) = \begin{pmatrix} 84.393 & 7.325 & 3.986 & 4.296 \\ 24.026 & 53.097 & 7.918 & 14.959 \\ 20.516 & 15.602 & 23.063 & 40.819 \\ 0.000 & 0.000 & 0.000 & 100.000 \end{pmatrix}$$

Let us denote by $\mathbf{S}_A(t)$, $\mathbf{S}_B(t)$ and $\mathbf{S}_C(t)$ the survival functions of each risk class A, B and C.

1. How are calculated the matrices $P(4)$ and $P(6)$?

2. Assuming a piecewise exponential distribution, calibrate the hazard rate function for each risk classes for $0 < t \leq 2$, $2 < t \leq 4$ and $4 < t \leq 6$.

3. Give the definition of a Markovian generator. How can we estimate the generator Λ associated to the transition probability matrices? Verify

[71] The transition probabilities are expressed in %.

numerically that the direct estimator is equal to:

$$\hat{\Lambda} = \begin{pmatrix} -3.254 & 1.652 & 1.264 & 0.337 \\ 5.578 & -11.488 & 3.533 & 2.377 \\ 6.215 & 7.108 & -25.916 & 12.593 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{pmatrix} \times 10^{-2}$$

4. In Figure 3.28, we show the hazard function $\lambda(t)$ deduced from Questions 2 and 3. Explain how do we calculate $\lambda(t)$ in both cases. Why do we obtain an increasing curve for rating A, a decreasing curve for rating C and an inverted U-shaped curve for rating B?

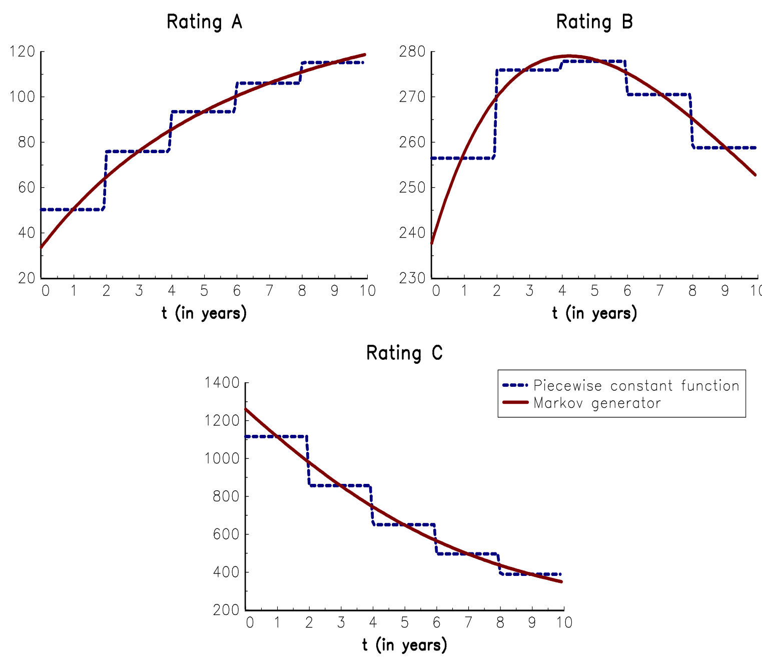

FIGURE 3.28: Hazard function $\lambda(t)$ (in bps) estimated respectively with the piecewise exponential model and the Markov generator

3.4.6 Calculating the credit value-at-risk with non granular portfolios

3.4.7 Understanding CDO cash flows

3.4.8 Modeling default correlations

Chapter 4

Counterparty Credit Risk and Collateral Risk

4.1 Counterparty credit risk

We generally make the distinction between credit risk (CR) and counterparty credit risk (CCR). The counterparty credit risk on market transactions is the risk that the counterparty could default before the final settlement of the transaction's cash flows. For instance, if the bank buy a CDS protection on a firm and the seller of the CDS protection could default before the maturity's contract, the bank could not be hedged to the default of the firm. Another example of CCR is the delivery/settlement risk. Indeed, few financial transactions are settled on the same-day basis and the difference between the payment date and the delivery date is generally between one and five business days. There is then a counterparty credit risk if one counterparty defaults when the payment date is not synchronized with the delivery date. This settlement risk is low when it is expressed as a percent of the notional because the maturity mismatch is short, but it concerns large amounts. In a similar way, when an OTC contract has a positive mark-to-market, the bank suffers a loss if the counterparty defaults. To reduce this risk, the bank can put in place bilateral netting agreements. We note that this risk disappears (or decreases) when the bank uses an exchange, because the counterparty credit risk is transferred to the central counterparty clearing house, which guarantees the expected cash flows.

4.1.1 Definition

BCBS (2004) measures the counterparty credit risk by the replacement cost of the OTC derivative. Let us consider two banks A and B that have entered into an OTC contract. We assume that the bank B defaults before the maturity of the contract. According to Pykhtin and Zhu (2006), Bank A can then face two situations:

- The current value of the contract is negative. In this case, Bank A close out the position and pays the market value of the contract to Bank B. To replace the contract, Bank A can enter with another counterparty C

into a similar contract. For that, Bank A receives the market value of the contract and the loss of the bank is equal to zero.

- The current value of the contract is positive. In this case, Bank A close out the position, but receives nothing from Bank B. To replace the contract, Bank A can enter with another counterparty C into a similar contract. For that, Bank A pays the market value of the contract to C. In this case, the loss of the bank is exactly equal to the market value.

We note that the counterparty exposure is then the maximum of the market value and zero. However, the counterparty credit risk differs from the credit risk by two main aspects (Canabarro and Duffie, 2003):

1. The counterparty risk is bilateral, meaning that both counterparties may face losses. In the previous example, Bank B is also exposed to the risk that Bank A defaults.
2. The exposure at default is uncertain, because we don't know what will be the replacement cost of the contract when the counterparty defaults.

Using the notations introduced in the previous chapter, we deduce that the credit loss of an OTC portfolio is:

$$L = \sum_{i=1}^{n} \mathrm{EAD}_i(\tau_i) \times \mathrm{LGD}_i \times \mathbf{1}\{\boldsymbol{\tau}_i \le T_i\}$$

This is the formula of a credit portfolio loss, except that the exposure at default is random and depends on different factors: the default time of the counterparty, the evolution of market risk factors and the correlation between the market value of the OTC contract and the default of the counterparty.

Let $\mathrm{MtM}(t)$ be the mark-to-market value of the OTC contract at time t. The exposure at default is defined as:

$$\mathrm{EAD} = \max(\mathrm{MtM}(\tau), 0)$$

If we consider a portfolio of OTC derivatives with the same counterparty entity, the exposure at default is the sum of positive market values:

$$\mathrm{EAD} = \sum_{i=1}^{n} \max(\mathrm{MtM}_i(\tau), 0)$$

This is why the bank may be interested in putting in place a global netting agreement:

$$\begin{aligned} \mathrm{EAD} &= \max\left(\sum_{i=1}^{n} \mathrm{MtM}_i(\tau), 0\right) \\ &\le \sum_{i=1}^{n} \max(\mathrm{MtM}_i(\tau), 0) \end{aligned}$$

In practice, it is extremely complicated and rare that two counterparties succeed in signing such agreement. Most of the time, there are several netting agreements on different trading perimeters (equities, bonds, interest-rate options, etc.). In this case, the exposure at default is:

$$\mathrm{EAD} = \sum_k \max\left(\sum_{i\in\mathcal{N}_k} \mathrm{MtM}_i\left(\tau\right), 0\right) + \sum_{i\notin\cup\mathcal{N}_k} \max\left(\mathrm{MtM}_i\left(\tau\right), 0\right)$$

where $\mathcal{N}_k$ corresponds to the k^{th} netting arrangement and defines a netting set.

Example 35 *Banks A and B have traded five OTC products, whose mark-to-market values*[1] *are given in the table below:*

t	1	2	3	4	5	6	7	8
$\mathcal{C}_1$	5	5	3	0	−4	0	5	8
$\mathcal{C}_2$	−5	10	5	−3	−2	−8	−7	−10
$\mathcal{C}_3$	0	2	−3	−4	−6	−3	0	5
$\mathcal{C}_4$	2	−5	−5	−5	2	3	5	7
$\mathcal{C}_5$	−1	−3	−4	−5	−7	−6	−7	−6

If we suppose that there is no netting agreement, the counterparty exposure of Bank A corresponds to the second row in Table 4.1. We notice that the exposure changes over time. If there is a netting agreement, we obtain lower exposures. We now consider a more complicated situation. We assume that Banks A and B have two netting agreements: one on equity OTC contracts ($\mathcal{C}_1$ and $\mathcal{C}_2$) and one on fixed-income OTC contracts ($\mathcal{C}_3$ and $\mathcal{C}_4$). In this case, we obtain results given in the last row in Table 4.1. For instance, the exposure at default for $t = 8$ is calculated as follows:

$$\begin{aligned}\mathrm{EAD} &= \max\left(8-10,0\right)+\max\left(5+7,0\right)+\max\left(-6,0\right)\\ &= 12\end{aligned}$$

TABLE 4.1: Counterparty exposure of Bank A

t	1	2	3	4	5	6	7	8
No netting	7	17	8	0	2	3	10	20
Global netting	1	9	0	0	0	0	0	4
Partial netting	2	15	8	0	0	0	5	12

If we consider Bank B, the counterparty exposure are given in Table 4.2. This illustrates the bilateral nature of the counterparty credit risk. Indeed, except if there is a global netting arrangement, both banks have a positive counterparty exposure.

[1] They are calculated from the viewpoint of Bank A.

TABLE 4.2: Counterparty exposure of Bank B

t	1	2	3	4	5	6	7	8
No netting	6	8	12	17	19	17	14	16
Global netting	0	0	4	17	17	14	4	0
Partial netting	1	6	12	17	17	14	9	8

Remark 37 *In the previous example, we have assumed that the mark-to-market value of the OTC contract for one bank is exactly the opposite of the mark-to-market value for the other bank. In practice, banks use mark-to-model prices, implying that they can differ from one bank to another one.*

4.1.2 Modeling the exposure at default

4.1.2.1 An illustrative example

Example 36 *We consider a bank that buys* 1 000 *ATM call options, whose maturity is one-year. The current value of the underlying asset is equal to* \$100*. We assume that the interest rate and the cost-of-carry parameter are equal to* 5%*. Moreover, the implied volatility of the option is considered as a constant and is equal to* 20%*.*

By considering the previous parameters, the value $\mathcal{C}_0$ of the call option is equal to \$10.45. At time t, the mark-to-market of this derivative exposure is defined by:

$$\mathrm{MtM}(t) = n_C \times (\mathcal{C}(t) - \mathcal{C}_0)$$

where n_C and $\mathcal{C}(t)$ are the number and the value of call options. Let $e(t)$ be the exposure at default. We have:

$$e(t) = \max(\mathrm{MtM}(t), 0)$$

At the initial date of the trade, the mark-to-market value and the counterparty exposure are zero. When $t > 0$, the mark-to-market value is not equal to zero, implying that the counterparty exposure $e(t)$ may be positive. In Table 4.3, we have reported the values taken by $\mathcal{C}(t)$, $\mathrm{MtM}(t)$ and $e(t)$ for two scenarios of the underlying price $S(t)$. If we consider the first scenario, the counterparty exposure is equal to zero during the fist three months, because the market-to-market value was negative. The counterparty exposure is then positive for five months. For instance, it is equal to \$2 519 at the end of the fourth month[2]. In the case of the second scenario, the counterparty exposure is always equal to zero except for two months. Therefore, we notice that the counterparty

[2]We have:

$$\mathrm{MtM}(t) = 1\,000 \times (12.969 - 10.450) = \$2\,519$$

exposure is time-varying and depends of the trajectory of the underlying price. This implies that the counterparty exposure cannot be calculated once and for all at the initial date of the trade. Indeed, the counterparty exposure changes with time. Moreover, we don't known what will be the future price of the underlying asset. That's why we are going to simulate it.

TABLE 4.3: Mark-to-market and counterparty exposure of the call option

t	Scenario #1				Scenario #2			
	$S(t)$	$\mathcal{C}(t)$	$\text{MtM}(t)$	$e(t)$	$S(t)$	$\mathcal{C}(t)$	$\text{MtM}(t)$	$e(t)$
1M	97.58	8.44	−2 013	0	91.63	5.36	−5 092	0
2M	98.19	8.25	−2 199	0	89.17	3.89	−6 564	0
3M	95.59	6.26	−4 188	0	97.60	7.35	−3 099	0
4M	106.97	12.97	2 519	2 519	97.59	6.77	−3 683	0
5M	104.95	10.83	382	382	96.29	5.48	−4 970	0
6M	110.73	14.68	4 232	4 232	97.14	5.29	−5 157	0
7M	113.20	16.15	5 700	5 700	107.71	11.55	1 098	1 098
8M	102.04	6.69	−3 761	0	105.71	9.27	−1 182	0
9M	115.76	17.25	6 802	6 802	107.87	10.18	−272	0
10M	103.58	5.96	−4 487	0	108.40	9.82	−630	0
11M	104.28	5.41	−5 043	0	104.68	5.73	−4 720	0
1Y	104.80	4.80	−5 646	0	115.46	15.46	5 013	5 013

We note $\text{MtM}(t_1; t_2)$ the mark-to-market value between dates t_1 and t_2. By construction, we have:

$$\text{MtM}(0; t) = \text{MtM}(0; t_0) + \text{MtM}(t_0; t)$$

where 0 is the initial date of the trade, t_0 is the current date and t is the future date. This implies that the mark-to-market value at time t has two components:

1. the current mark-to-market value $\text{MtM}(0; t_0)$ that depends on the past trajectory of the underlying price;

2. and the the future mark-to-market value $\text{MtM}(t_0; t)$ that depends on the future trajectory of the underlying price.

In order to evaluate the second component, we need to define the probability distribution of $S(t)$. In our example, we can assume that the underlying price follows a geometric Brownian motion:

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t)$$

We face here an issue because we have to define the parameters μ and σ. There are two approaches:

1. The first method uses the historical probability measure $\mathbb{P}$, meaning that the parameters μ and σ are estimated using historical data.

2. The second method considers the risk-neutral probability measure $\mathbb{Q}$, which is used to price the OTC derivative.

While the first approach is the more relevant to calculate the counterparty exposure, the second approach is more frequent because it is easier for a bank to implement it. Indeed, $\mathbb{Q}$ is already available because of the hedging portfolio, which is not the case of $\mathbb{P}$. In our example, this is equivalent to set μ and σ to the interest rate r and the implied volatility Σ.

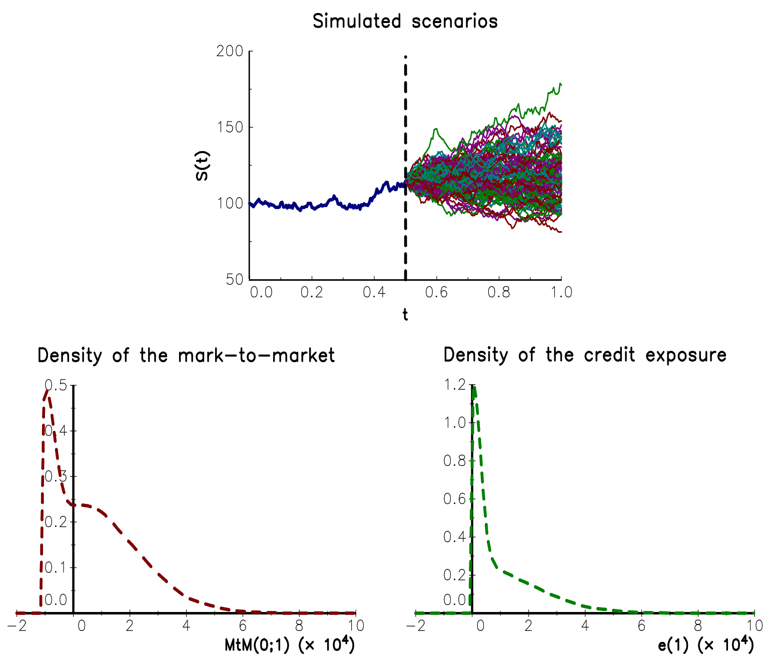

FIGURE 4.1: Density function of the counterparty exposure after six months

In Figure 4.1, we report an illustration of scenario generation when the current date t_0 is 6 months. This means that the trajectory of the asset price $S(t)$ is given when $t \leq t_0$ whereas it is simulated when $t > t_0$. At time $t_0 = 0.5$, the asset price is equal to \$114.77. We deduce that the option price $\mathcal{C}(t_0)$ is equal to \$18.17. The mark-to-market value is then positive and equal to \$7 716. Using 10 000 simulated scenarios, we estimate the probability density function of the mark-to-market value $\text{MtM}(0; 1)$ at the maturity date (bottom/left panel in Figure 4.1) and deduce the probability density function of the counterparty exposure $e(1)$ (bottom/right panel in Figure 4.1). We notice that the probability that the mark-to-market value is negative at the maturity date is significant. Indeed, it is equal to 36% because it remains 6 months and the asset price may sufficiently decrease.

Remark 38 *The mark-to-market value presents a very high skew, because it is bounded. Indeed, the worst-case scenario is reached when $S(1)$ is lower than the strike. In this case, we obtain:*

$$\begin{aligned} \mathrm{MtM}(0;1) &= 1\,000 \times (0 - 10.45) \\ &= -\$10\,450 \end{aligned}$$

We suppose now that the current date is nine months. During the last three months, the asset price has changed and it is now equal to \$129.49. The current counterparty exposure has then increased and is equal to[3] \$20 294. In Figure 4.2, we observe that the shape of the probability density functions has changed. Indeed, the skew has been highly reduced. The reason is that it only remains three months before the maturity date. The price is sufficiently high that the probability to obtain a positive mark-to-market at the settlement date is quasi equal to 100%. This is why the two probability density functions are very similar.

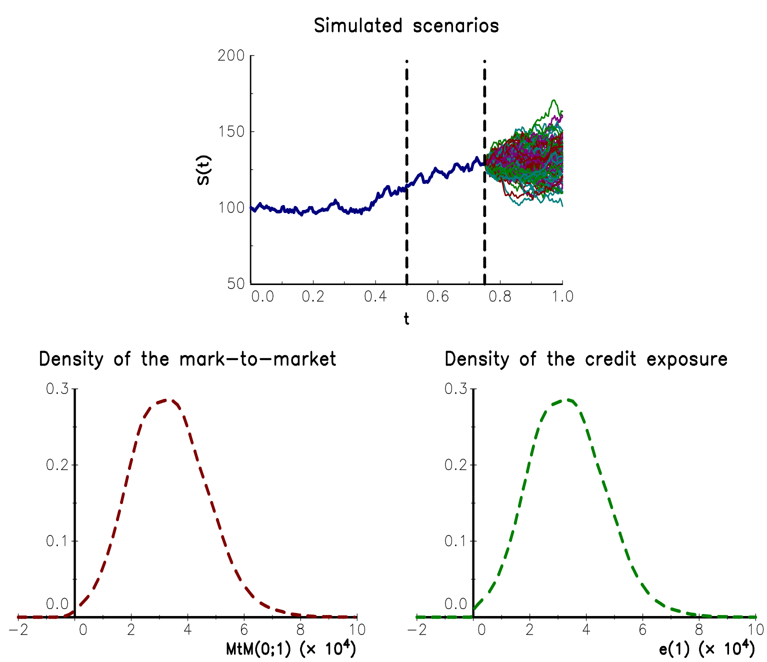

FIGURE 4.2: Density function of the counterparty exposure after nine months

[3]Using the previous parameters, the Black-Scholes price of the call option is now \$30.74.

We can use the previous approach of scenario generation in order to represent the evolution of counterparty exposure. In Figure 4.3, we consider two observed trajectories of the asset price. For each trajectory, we report the current exposure, the expected exposure and the 95% quantile of the counterparty exposure at the maturity date. All these counterparty measures converge at the maturity date, but differ before because of the uncertainty between the current date and the maturity date.

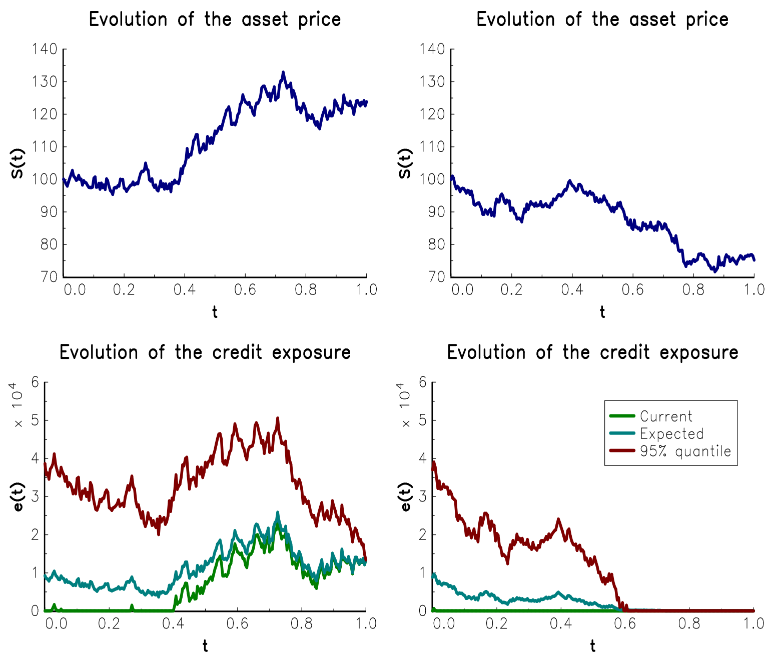

FIGURE 4.3: Evolution of the counterparty exposure

4.1.2.2 Measuring the counterparty exposure

We define the counterparty exposure at time t as the random credit exposure[4]:

$$e(t) = \max(\mathrm{MtM}(0;t), 0) \tag{4.1}$$

This counterparty exposure is also known as the potential future exposure (PFE). When the current date t_0 is not equal to the initial date 0, this coun-

[4]The definitions introduced in this paragraph come from Canabarro and Duffie (2003) and the Basel II framework.

terparty exposure can be decomposed in two parts:

$$\begin{aligned} e(t) &= \max\left(\mathrm{MtM}(0;t_0) + \mathrm{MtM}(t_0;t), 0\right) \\ &= \max\left(\mathrm{MtM}(0;t_0), 0\right) + \\ &\quad \left(\max\left(\mathrm{MtM}(0;t_0) + \mathrm{MtM}(t_0;t), 0\right) - \max\left(\mathrm{MtM}(0;t_0), 0\right)\right) \end{aligned}$$

The first component is the current exposure, which is always positive:

$$\mathrm{CE}(t_0) = \max\left(\mathrm{MtM}(0;t_0), 0\right)$$

The second component is the credit variation between t_0 and t. It is a positive value if the current mark-to-market value is negative. However, the credit variation may also be negative if the future mark-to-market value is negative. Let us denote by $\mathbf{F}_{[0,t]}$ the cumulative distribution function of the potential future exposure $e(t)$. The peak exposure (PE) is the quantile of the counterparty exposure at the confidence level α:

$$\begin{aligned} \mathrm{PE}_\alpha(t) &= \mathbf{F}_{[0,t]}^{-1}(\alpha) \\ &= \left\{\inf x : \Pr\left\{e(t) \leq x\right\} \geq \alpha\right\} \end{aligned} \tag{4.2}$$

The maximum value of the peak exposure is referred as the maximum peak exposure[5] (MPE):

$$\mathrm{MPE}_\alpha(0;t) = \sup_s \mathrm{PE}_\alpha(0;s) \tag{4.3}$$

We now introduce the traditional counterparty credit risk measures:

- The expected exposure (EE) is the average of the distribution of the counterparty exposure at the future date t:

$$\begin{aligned} \mathrm{EE}(t) &= \mathbb{E}\left[e(t)\right] \\ &= \int_0^\infty x \, \mathrm{d}\mathbf{F}_{[0,t]}(x) \end{aligned} \tag{4.4}$$

- The expected positive exposure (EPE) is the weighted average over time $[0,t]$ of expected exposures:

$$\begin{aligned} \mathrm{EPE}(0;t) &= \mathbb{E}\left[\frac{1}{t}\int_0^t e(s)\,\mathrm{d}s\right] \\ &= \frac{1}{t}\int_0^t \mathrm{EE}(s)\,\mathrm{d}s \end{aligned} \tag{4.5}$$

- The effective expected exposure (EEE) is the maximum expected exposure that occurs at the future date t or any prior date:

$$\begin{aligned} \mathrm{EEE}(t) &= \sup_{s \leq t} \mathrm{EE}(s) \\ &= \max\left(\mathrm{EEE}\left(t^-\right), \mathrm{EE}(t)\right) \end{aligned} \tag{4.6}$$

[5]It is also known as the maximum potential future exposure (MPFE).

- Finally, the effective expected positive exposure (EEPE) is the weighted average over time $[0,t]$ of effective expected exposures:

$$\mathrm{EEPE}\,(0;t) = \frac{1}{t}\int_0^t \mathrm{EEE}\,(s)\;\mathrm{d}s \tag{4.7}$$

We can make several observations concerning the previous measures. Some of them are defined with respect to the future date t. This is the case of $\mathrm{PE}_\alpha\,(t)$, $\mathrm{EE}\,(t)$ and $\mathrm{EEE}\,(t)$. The others depend on a time period $[0;t]$, typically a one-year time horizon. Previously, we have considered the counterparty measure $e\,(t)$, which defines the potential future exposure between the initial date 0 and the future date t. We can also use other credit measures like the potential future exposure between the current date t_0 and the future date t:

$$e\,(t) = \max\left(\mathrm{MtM}\,(t_0;t)\,,0\right)$$

The counterparty exposure $e\,(t)$ can be defined with respect to one contract or to a basket of contracts. In this last case, we have to take into account netting arrangements.

4.1.2.3 Practical implementation for calculating counterparty exposure

We consider again Example 36 and assume that the current date t_0 is the initial date $t = 0$. Using 50 000 simulations, we have calculated the different credit measures with respect to the time t and reported them in Figure 4.4. For that, we have used the risk-neutral distribution probability $\mathbb{Q}$ in order to simulate the trajectory of the asset price $S\,(t)$. Let $\{t_0, t_1, \ldots, t_n\}$ be the set of discrete times. We note n_S the number of simulations and $S_j\,(t_i)$ the value of the asset price at time t_i for the j^{th} simulation. For each simulated trajectory, we then calculate the option price $\boldsymbol{\mathcal{C}}_j\,(t_i)$ and the mark-to-market value:

$$\mathrm{MtM}_j\,(t_i) = n_C \times \left(\boldsymbol{\mathcal{C}}_j\,(t_i) - \boldsymbol{\mathcal{C}}_0\right)$$

Therefore, we deduce the potential future exposure:

$$e_j\,(t_i) = \max\left(\mathrm{MtM}_j\,(t_i)\,,0\right)$$

The peak exposure at time t_i is estimated using order statistics:

$$\mathrm{PE}_\alpha\,(t_i) = e_{\alpha n_S:n_S}\,(t_i) \tag{4.8}$$

We use the empirical mean to calculate the expected exposure:

$$\mathrm{EE}\,(t_i) = \frac{1}{n_S}\sum_{j=1}^{n_S} e_j\,(t_i) \tag{4.9}$$

For the expected positive exposure, we approximate the integral by the following sum:

$$\mathrm{EPE}\,(0;t_i) = \frac{1}{t_i}\sum_{k=1}^{i}\mathrm{EE}\,(t_k)\,\Delta t_k \tag{4.10}$$

If we consider a fixed-interval scheme with $\Delta t_k = \Delta t$, we obtain:

$$\begin{aligned}\mathrm{EPE}\,(0;t_i) &= \frac{\Delta t}{t_i}\sum_{k=1}^{i}\mathrm{EE}\,(t_k) \\ &= \frac{1}{i}\sum_{k=1}^{i}\mathrm{EE}\,(t_k)\end{aligned} \tag{4.11}$$

By definition, the effective expected exposure is given by the following recursive formula:

$$\mathrm{EEE}\,(t_i) = \max\left(\mathrm{EEE}\,(t_{i-1})\,,\mathrm{EE}\,(t_i)\right) \tag{4.12}$$

where $\mathrm{EEE}\,(0)$ is initialized with the value $\mathrm{EE}\,(0)$. Finally, the effective expected positive exposure is given by:

$$\mathrm{EEPE}\,(0;t_i) = \frac{1}{t_i}\sum_{k=1}^{i}\mathrm{EEE}\,(t_k)\,\Delta t_k \tag{4.13}$$

In the case of a fixed-interval scheme, this formula becomes:

$$\mathrm{EEPE}\,(0;t_i) = \frac{1}{i}\sum_{k=1}^{i}\mathrm{EEE}\,(t_k) \tag{4.14}$$

If we consider Figure 4.4, we observe that the counterparty exposure is increasing with respect to the horizon time[6]. This property is due to the fact that the credit risk evolves according to a square-root-of-time rule $\sqrt{t}$. In the case of an interest rate swap, the counterparty exposure takes the form of a bell-shaped curve. In fact, there are two opposite effects that determine the counterparty exposure (Pykhtin and Zhu, 2007):

- the diffusion effect of risk factors increases the counterparty exposure over time, because the uncertainty is greater in the future and may produce very large potential future exposures compared to the current exposure;
- the amortization effect decreases the counterparty exposure over time, because it reduces the remaining cash flows that are exposed to default.

In Figure 4.5, we have reported counterparty exposure in the case of an interest

[6]This implies that $\mathrm{MPE}_\alpha\,(0;t) = \mathrm{PE}_\alpha\,(t)$ and $\mathrm{EEE}\,(t) = \mathrm{EE}\,(t)$.

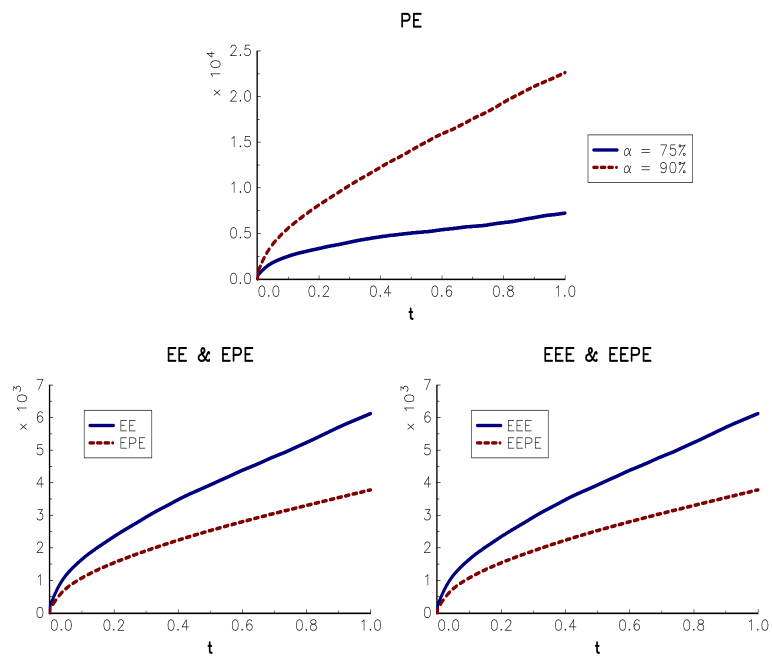

FIGURE 4.4: Counterparty exposure profile of option

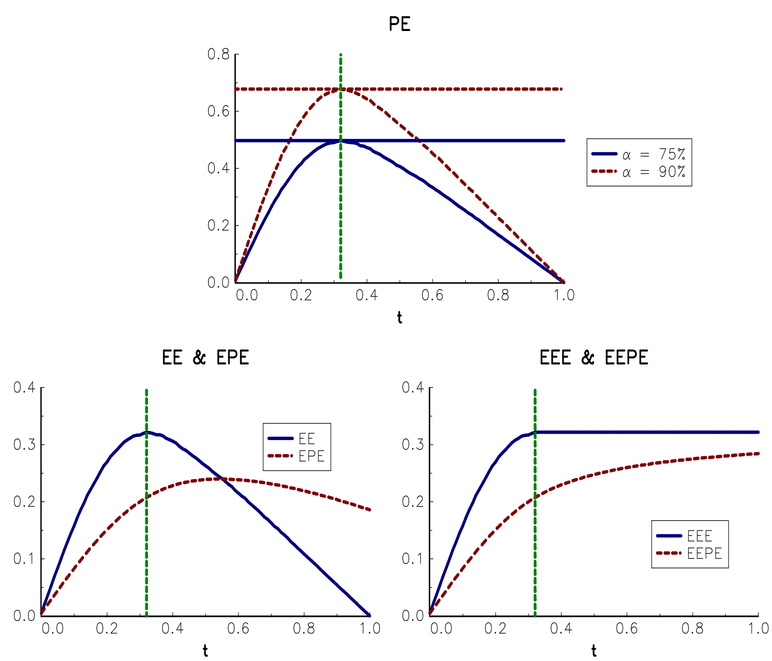

FIGURE 4.5: Counterparty exposure profile of interest rate swap

swap with a continuous amortization[7]. The peak exposure initially increases because of the diffusion effect and generally reaches its maximum at one-third of the remaining maturity. It then decreases because of the amortization effect. This is why it is equal to zero at the maturity date when the swap is fully amortized.

4.1.3 Regulatory capital

The Basel II Accord includes three approaches to calculate the capital requirement for the counterparty credit risk: current exposure method (CEM), standardized method (SM) and internal model method (IMM). In March 2014, the Basel Committee decided to replace non-internal model approaches (CEM and SM) by a more sensitive approach called standardized approach (or SA-CCR). The CEM and SM approaches continue to be valid until January 2017. After this date, only SA-CCR and IMM approaches can be theoretically used[8].

Each approach defines how the exposure at default EAD is calculated. The bank uses this estimate with the appropriated credit approach (SA or IRB) in order to measure the capital requirement. In the SA approach, the capital charge is equal to:

$$\mathcal{K} = 8\% \times \mathrm{EAD} \times \mathrm{RW}$$

where RW is the risk weight of the counterparty. In the IRB approach, we have:

$$\mathcal{K} = \mathrm{EAD} \times \mathrm{LGD} \times \left(\Phi \left(\frac{\Phi^{-1}(\mathrm{PD}) + \sqrt{\rho(\mathrm{PD})}\Phi^{-1}(0.999)}{\sqrt{1-\rho(\mathrm{PD})}} \right) - \mathrm{PD} \right) \times \varphi(\mathrm{M})$$

where LGD and PD are the loss given default and the probability of default, which apply to the counterparty. The correlation $\rho(\mathrm{PD})$ is calculated using the standard formula (3.32) given in page 200.

Remark 39 *Since the Basel III agreement, a* 1.25 *multiplier is applied to the correlation* $\rho(\mathrm{PD})$*:*

$$\begin{aligned}
\rho(\mathrm{PD}) &= 1.25 \times \left(12\% \times \frac{1-e^{-50\times \mathrm{PD}}}{1-e^{-50}} + 24\% \times \frac{1-\left(1-e^{-50\times \mathrm{PD}}\right)}{1-e^{-50}} \right) \\
&= 15\% \times \frac{1-e^{-50\times \mathrm{PD}}}{1-e^{-50}} + 30\% \times \frac{1-\left(1-e^{-50\times \mathrm{PD}}\right)}{1-e^{-50}}
\end{aligned}$$

for all exposures to systemically important financial intermediaries[9], implying that the correlation range increases from $12\% - 24\%$ *to* $15\% - 30\%$ *(BCBS, 2010).*

[7]The more realistic case of discrete amortization is addressed in Exercise 4.4.5 in page 259.

[8]For now, the SA-CCR approach has not been adopted by US and European legislations.

[9]They are regulated financial institutions (banks and insurance companies), with assets of at least \$100 bn and unregulated financial institutions, regardless of size.

4.1.3.1 Internal model method

In the internal model method, the exposure at default is calculated as the product of a scalar α and the one-year effective expected positive exposure[10]:

$$\text{EAD} = \alpha \times \text{EEPE}\,(0;1)$$

The Basel Committee has set the value α at 1.4. The maturity M used in the IRB formula is equal to one year if the remaining maturity is less or equal than one year. Otherwise, it is calculated as follows[11]:

$$\text{M} = \min\left(1 + \frac{\sum_{k=1} \mathbf{1}\{t_k > 1\}\,\text{EE}\,(t_k)\,\Delta t_k B_0\,(t_k)}{\sum_{k=1} \mathbf{1}\{t_k \leq 1\}\,\text{EEE}\,(t_k)\,\Delta t_k B_0\,(t_k)}, 5\right)$$

Under some conditions, the bank may uses its own estimates for α. Let LEE be the loan equivalent exposure such that:

$$\mathcal{K}\,(\text{LEE} \times \text{LGD} \times 1\{\tau \leq T\}) = \mathcal{K}\,(\text{EAD}\,(\tau) \times \text{LGD} \times 1\{\tau \leq T\})$$

The loan equivalent exposure is then the deterministic exposure at default, which gives the same capital than the random exposure at default $\text{EAD}\,(\tau)$. Using a one-factor credit risk model, Canabarro *et al.* (2003) showed that:

$$\alpha = \frac{\text{LEE}}{\text{EPE}}$$

This is the formula that banks must use in order to estimate α, subject to a floor of 1.2.

Example 37 *We assume that the one-year effective expected positive exposure with respect to a given counterparty is equal to* \$50.2 *mn.*

In Table 4.4, we have reported the required capital $\mathcal{K}$ for different values of PD under the foundation IRB approach. The maturity M is set equal to one year and we consider the 45% supervisory factor for the loss given default. The exposure at default is calculated with $\alpha = 1.4$. We show the impact of the Basel III multiplier applied to the correlation. In this example, if the default probability of the counterparty is equal to 1%, this induces an additional required capital of 27.77%.

[10]If the remaining maturity τ of the product is less than one year, the exposure at default becomes:

$$\text{EAD} = \alpha \times \text{EEPE}\,(0;\tau)$$

[11]The maturity has then a cap of five years.

TABLE 4.4: Capital charge of counterparty credit risk under the FIRB approach

Basel II	$\rho\,(\mathrm{PD})$ (in %)	19.28	16.41	14.68	13.62	12.99
	$\mathcal{K}$ (in \$ mn)	4.12	5.38	6.18	6.82	7.42
Basel III	$\rho\,(\mathrm{PD})$ (in %)	24.10	20.52	18.35	17.03	16.23
	$\mathcal{K}$ (in \$ mn)	5.26	6.69	7.55	8.25	8.89
	$\Delta\mathcal{K}$ (in %)	27.77	24.29	22.26	20.89	19.88

4.1.3.2 Non-internal models methods (Basel II)

Under the current exposure method, we have:

$$\mathrm{EAD} = \mathrm{CE}\,(0) + A$$

where CE (0) is the current exposure and A is the add-on value. For a single OTC transaction, A is the product of the notional and the add-on factor, which is given in Table 4.5. For a portfolio of OTC transactions with netting agreements, the exposure at default is the sum of the current net exposure plus a net add-one value A_N, which is defined as follows:

$$A_N = (0.5 + 0.6 \times \mathrm{NGR}) \times A_G$$

where $A_G = \sum_i A_i$ is the gross add-on, A_i is the add-on of the i^{th} transaction and NGR is the ratio between the current net exposure and the current gross exposure.

TABLE 4.5: Regulatory add-on factors for the current exposure method

Residual Maturity	Fixed Income	FX and Gold	Equity	Precious Metals	Other Commodities
0−1Y	0.0%	1.0%	8.0%	7.0%	10.0%
1Y−5Y	0.5%	5.0%	8.0%	7.0%	12.0%
5Y+	1.5%	7.5%	10.0%	8.0%	15.0%

Example 38 *We consider a portfolio of four OTC derivatives, which are traded with the same counterparty:*

Contract	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$
Asset class	*Fixed income*	*Fixed income*	*Equity*	*Equity*
Notional (in \$ mn)	100	40	20	10
Maturity	*2Y*	*6Y*	*6M*	*18M*
Mark-to-market (in \$ mn)	3.0	−2.0	2.0	−1.0

We assume that there is two netting arrangements: one concerning fixed income derivatives and another one for equity derivatives.

In the case where there is not netting agreement, we obtain the following results:

Contract	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	Sum
CE (0) (in \$ mn)	3.0	0.0	2.0	0.0	5.0
Add-on (in %)	0.5	1.5	8.0	8.0	
A (in \$ mn)	0.5	0.6	1.6	0.8	3.5

The exposure at default is then equal to \$8.5 mn. If we take into account the two netting agreements, the current net exposure becomes:

$$\mathrm{CE}(0) = \max(3-2,0) + \max(2-1,0) = \$2 \text{ mn}$$

We deduce that NGR is equal to $^2/_5$ or 40%. It follows that:

$$A_N = (0.5 + 0.6 \times 0.4) \times 3.5 = \$2.59 \text{ mn}$$

Finally, the exposure at default is equal to \$4.59 mn.

The standardized method was designed for banks that do not have the approval to apply the internal model method, but would like to have a more sensitive approach that the current exposure method. In this framework, the exposure at default is equal to:

$$\mathrm{EAD} = \beta \times \max\left(\sum_i \mathrm{CMV}_i, \sum_j \mathrm{CCF}_j \times \left|\sum_{i \in j} \mathrm{RPT}_i\right|\right)$$

where CMV_i is the current market value of transaction i, CCF_j is the supervisory credit conversion factor with respect to the hedging set j and RPT_i is the risk position from transaction i. The supervisory scaling factor β is set equal to 1.4. In this approach, the risk positions have to be grouped into hedging sets, which are defined by similar instruments (e.g. same commodity, same issuer, same currency, etc.). The risk position $\sum_{i \in j} \mathrm{RPT}_i$ is the sum of notional values of linear instruments and delta-equivalent notional values of non-linear instruments, which belong to the hedging set j. The credit conversion factors are given in Table 4.6.

TABLE 4.6: Supervisory credit conversion factors for the SM-CCR approach

Instruments	FX	Gold	Equity	Precious Metals	Electric Power	Other Commodities
CCF	2.5%	5.0%	7.0%	8.5%	4.0%	10.0%

Instruments	Debt: High specific risk	Debt: Low specific risk	Others
CCF	0.6%	0.3%	10%

4.1.3.3 SA-CCR method (Basel IV)

The SA-CCR has been adopted by the Basel Committee in March 2014 in order to replace non-internal models approaches in January 2017. The main motivation the Basel Committee was to propose a more-sensitive approach, which can easily be implemented:

> "*Although being more risk-sensitive than the CEM, the SM was also criticized for several weaknesses. Like the CEM, it did not differentiate between margined and unmargined transactions or sufficiently capture the level of volatilities observed over stress periods in the last five years. In addition, the definition of hedging set led to operational complexity resulting in an inability to implement the SM, or implementing it in inconsistent ways*" (BCBS, 2014b, page 1).

The exposure at default under the SA-CCR is defined as follows:

$$\mathrm{EAD} = \alpha \times (\mathrm{RC} + \mathrm{PFE})$$

where RC is the replacement cost (or the current exposure), PFE is the potential future exposure and α is equal to 1.4. We can view this formula as an approximation of the IMM calculation, meaning that $\mathrm{RC} + \mathrm{PFE}$ represents a stylized EEPE value. The PFE add-on is given by:

$$\mathrm{PFE} = \gamma \times \sum_{\mathcal{C}=1}^{5} A^{(\mathcal{C})}$$

where γ is the multiplier and $A^{(\mathcal{C})}$ is the add-on of the asset class $\mathcal{C}$ (interest rate, foreign exchange, credit, equity and commodity). We have:

$$\gamma = \min\left(1, 0.05 + 0.95 \times \exp\left(\frac{\mathrm{MtM}}{1.80 \times \sum_{\mathcal{C}=1}^{5} A^{(\mathcal{C})}}\right)\right)$$

where MtM is the mark-to-market value of the derivative transactions. We notice that γ is equal to 1 when the mark-to-market is positive and $\gamma \in [5\%, 1]$ when the the mark-to-market is negative. Figure 4.6 shows the relationship between the ratio $\mathrm{MtM} \Big/ \sum_{\mathcal{C}=1}^{5} A^{(\mathcal{C})}$ and the multiplier γ. The role of γ is then to reduce the potential future exposure in the case of negative mark-to-market.

The general steps for calculating the add-on are the following. First, we have to determine the primary risk factors of each transaction in order to classify the transaction into one or more asset classes. Second, we calculate an adjusted notional amount d_i at the transaction level[12] and a maturity factor

[12] The trade-level adjusted notional d_i is defined as the product of current price of one unit and the number of units for equity and commodity derivatives, the notional of the

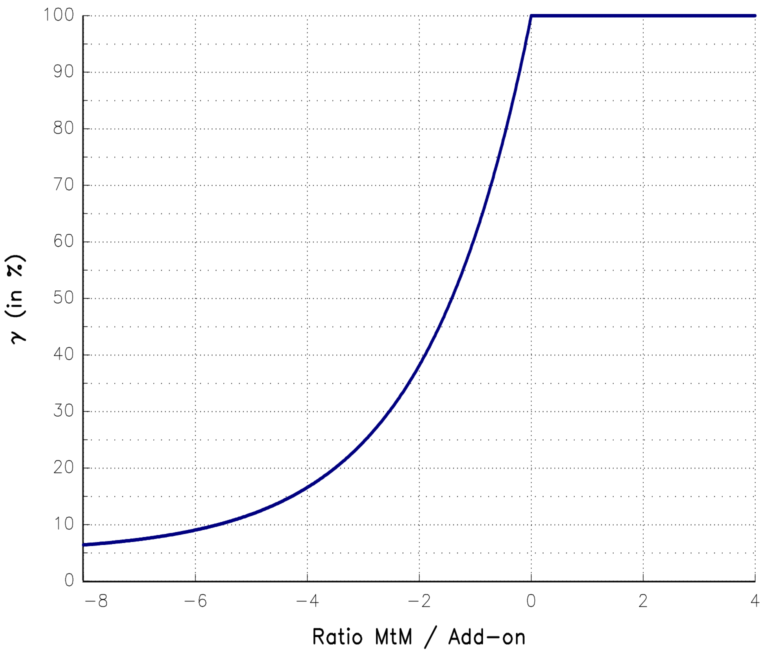

FIGURE 4.6: Impact of negative mark-to-market on the PFE multiplier

$\mathcal{MF}_i$, which reflects the time horizon appropriate for this type of transactions. For unmargined transactions, we have:

$$\mathcal{MF}_i = \sqrt{\min\left(\mathrm{M}_i, 1\right)}$$

where M_i is the remaining maturity of the transaction. For margined transactions, we have:

$$\mathcal{MF}_i = \frac{3}{2}\sqrt{\mathrm{M}_i^{\star}}$$

where $\mathrm{M}_i^{\star}$ is the appropriate margin period. Then, we apply a supervisory delta adjustment Δ_i to each transaction[13] and a supervisory factor $\mathcal{SF}_j$ to each hedging set j in order to take volatility into account. The add-on of one

foreign currency leg converted to domestic currency for foreign exchange derivatives and the product of the trade notional amount and the supervisory duration $\mathcal{SD}_i$ for interest rate and credit derivatives. The supervisory duration $\mathcal{SD}_i$ is defined as follows:

$$\mathcal{SD}_i = 20 \times \left(e^{-0.05 \times S_i} - e^{-0.05 \times E_i}\right)$$

where S_i and E_i are the start and end dates of the time period referenced by the derivative instrument.

[13]For instance Δ_i is equal to -1 for a short position, $+1$ for a long position, Black-Scholes' delta for option position, etc.

transaction i has then the following expression:

$$A_i = \mathcal{SF}_j \times (\Delta_i \times d_i \times \mathcal{MF}_i)$$

Finally, we apply an aggregation method to calculate the add-on $A^{(\mathcal{C})}$ of the asset class $\mathcal{C}$ by considering correlations between hedging sets. Here are the formulas that determine the add-on values[14]:

- The add-on for interest rate derivatives is equal to:

$$A^{(\mathrm{ir})} = \sum_j \mathcal{SF}_j \times \sqrt{\sum_{k=1}^{3} \sum_{k'=1}^{3} \rho_{k,k'} D_{j,k} D_{j,k'}}$$

where notations j and k refer to currency j and maturity bucket[15] k and the effective notional $D_{j,k}$ is calculated according to:

$$D_{j,k} = \sum_{i \in (j,k)} \Delta_i \times d_i \times \mathcal{MF}_i$$

- For foreign exchange derivatives, we obtain:

$$A^{(\mathrm{fx})} = \sum_j \mathcal{SF}_j \times \left| \sum_{i \in j} \Delta_i \times d_i \times \mathcal{MF}_i \right|$$

where the hedging set j refers to currency pair j.

- The add-on values for credit and equity derivatives use the same formula:

$$A^{(\mathrm{credit/equity})} = \sqrt{\left(\sum_k \rho_k A_k \right)^2 + \sum_k (1 - \rho_k)^2 A_k^2}$$

where k represents entity k and:

$$A_k = \mathcal{SF}_k \times \sum_{i \in k} \Delta_i \times d_i \times \mathcal{MF}_i$$

- In the case of commodity derivatives, we have:

$$A^{(\mathrm{commodity})} = \sum_j \sqrt{\left(\rho_j \sum_k A_{j,k} \right)^2 + (1 - \rho_j)^2 \sum_k A_{j,k}^2}$$

where j indicates the hedging set, k corresponds to the commodity type and:

$$A_{j,k} = \mathcal{SF}_{j,k} \times \sum_{i \in (j,k)} \Delta_i \times d_i \times \mathcal{MF}_i$$

[14]Derivation of these formulas is left as an exercise (see Exercise 4.4.6 in page 259).

[15]The three maturity buckets k are (1) less than one year, (2) between one and five years and (3) more than five years.

TABLE 4.7: Supervisory parameters for the SA-CCR approach

Asset class		$\mathcal{SF}_j$	ρ_k			Σ_i
Interest rate	0−1Y	0.50%	100%			50%
	1Y−5Y	0.50%	70%	100%		50%
	5Y+	0.50%	30%	70%	100%	50%
Foreign exchange		4.00%				15%
Credit	AAA	0.38%		50%		100%
	AA	0.38%		50%		100%
	A	0.42%		50%		100%
	BBB	0.54%		50%		100%
	BB	1.06%		50%		100%
	B	1.60%		50%		100%
	CCC	6.00%		50%		100%
	IG index	0.38%		80%		80%
	SG index	1.06%		80%		80%
Equity	Single name	32.00%		50%		120%
	Index	20.00%		80%		75%
Commodity	Electricity	40.00%		40%		150%
	Oil & gas	18.00%		40%		70%
	Metals	18.00%		40%		70%
	Agricultural	18.00%		40%		70%
	Other	18.00%		40%		70%

For interest rate, hedging sets correspond to all derivatives in the same currency (e.g. USD, EUR, JPY). For currency, they consists of all currency pairs (e.g. USD/EUR, USD/JPY, EUR/JPY). For credit and equity, there is a single hedging set, which contains all the entities (both single names and indexes). Finally, they are four hedging sets for commodity derivatives: energy (electricity, oil & gas), metals, agricultural and other. In Table 4.7, we give the supervisory parameters for the factor $\mathcal{SF}_j$, the correlation[16] ρ_k and the implied volatility Σ_i in order to calculate Black-Scholes' delta exposures.

Example 39 *The following netting set consists of four interest rate derivatives:*

Trade	*Instrument*	*Currency*	*Maturity*	*Swap*	*Notional*	MtM
1	*IRS*	*USD*	*9M*	*Payer*	4	0.10
2	*IRS*	*USD*	*4Y*	*Receiver*	20	−0.20
3	*IRS*	*USD*	*10Y*	*Payer*	20	0.70
4	*Swaption 10Y*	*USD*	*1Y*	*Receiver*	5	0.50

[16]We notice that we consider cross-correlations between the three time buckets for interest rate derivatives.

For the swaption, we assume that the forward rate swap is 6% *and the strike value is* 5%.

This netting set consists of only one hedging set, because the underlyings of all these derivative instruments are USD interest rates. We report the different calculations in the following table:

i	k	S_i	E_i	$\mathcal{SD}_i$	Δ_i	d_i	$\mathcal{MF}_i$	D_i
1	1	0.00	0.75	0.74	1.00	2.94	0.87	2.55
2	2	0.00	4.00	3.63	−1.00	72.51	1.00	−72.51
3	3	0.00	10.00	7.87	1.00	157.39	1.00	157.39
4	3	1.00	11.00	7.49	−0.27	37.43	1.00	−10.08

where k indicates the time bucket, S_i is the start date, E_i is the end date, $\mathcal{SD}_i$ is the supervisory duration, Δ_i is the delta, d_i is the adjusted notional, $\mathcal{MF}_i$ is the maturity factor and D_i is the effective notional. For instance, we obtain the following results for the swaption transaction:

$$\begin{aligned}
\mathcal{SD}_i &= 20 \times \left(e^{-0.05\times 1} - e^{-0.05\times 10}\right) = 7.49 \\
\Delta_i &= -\Phi\left(-\frac{\ln(6\%/5\%)}{0.5\times\sqrt{1}} + \frac{1}{2}\times 0.5\times\sqrt{1}\right) = -0.27 \\
d_i &= 7.49 \times 5 = 37.43 \\
\mathcal{MF}_i &= \sqrt{1} = 1 \\
D_i &= -0.27 \times 37.43 \times 1 = -10.08
\end{aligned}$$

We deduce that the effective notional of time buckets is respectively $D_1 = 2.55$, $D_2 = -72.51$ and $D_3 + D_4 = 147.30$. It follows that:

$$\begin{aligned}
\sum_{k=1}^{3}\sum_{k'=1}^{3}\rho_{k,k'}D_{j,k}D_{j,k'} &= 2.55^2 - 2\times 70\% \times 2.55 \times 72.51 + \\
&\quad\; 72.51^2 - 2\times 70\% \times 72.51 \times 147.30 + \\
&\quad\; 147.30^2 + 2 \times 30\% \times 2.55 \times 147.30 \\
&= 11\,976.1
\end{aligned}$$

While the supervisory factor is 0.50%, the add-on value $A^{(\mathrm{ir})}$ is then equal to 0.55. The replacement cost is:

$$\mathrm{RC} = \max(0.1 - 0.2 + 0.7 + 0.5, 0) = 1.1$$

Because the mark-to-market of the netting set is positive, the PFE multiplier is equal to 1. We finally deduce that:

$$\begin{aligned}
\mathrm{EAD} &= 1.4 \times (1.1 + 1 \times 0.55) \\
&= 2.31
\end{aligned}$$

Remark 40 *Annex 4 of BCBS (2014b) contains four examples of SA-CCR calculations. Exercise 4.4.7 in page 259 presents also several applications including different hedging sets, netting sets and asset classes.*

4.1.4 Impact of wrong-way risk

4.2 Credit valuation adjustment

CVA is the adjustment to the risk-free (or fair) value of derivative instruments to account for counterparty credit risk. Thus, CVA is commonly viewed as the market price of CCR. The concept of CVA was popularized after the 2008 financial crisis, even if investments bank started to use CVA in the early 1990s (Litzenberger, 1992; Duffie and Huang, 1996). Indeed, during the global financial crisis, banks suffered significant counterparty credit risk losses on their OTC derivatives portfolios. However, according to BCBS (2011), roughly two-thirds of these losses came from CVA markdowns on derivatives and only one-third were due to counterparty defaults. In a similar way, the Financial Service Authority concluded that CVA losses were five times larger than CCR losses for UK banks during the period 2007-2009. In this context, BCBS (2011) included CVA capital charge in the Basel III framework, whereas credit-related adjustments were introduced in the accounting standard IFRS 13 also called *Fair Value Measurement*[17]. Nevertheless, the complexity of CVA raises several issues (EBA, 2015a). This is why questions around the CVA are not stabilized and new standards are emerging, but they only provide partial answers.

4.2.1 Definition

4.2.1.1 Difference between CCR and CVA

In order to understand the credit valuation adjustment, it is important to make the distinction between CCR and CVA. CCR is the credit risk of OTC derivatives associated to the default of the counterparty, whereas CVA is the market risk of OTC derivatives associated to the credit migration of the two counterparties. This means that CCR occurs at the default time. On the contrary, CVA impacts the market value of OTC derivatives before the default time.

Let us consider an example with two banks A and B and an OTC contract $\mathcal{C}$. The P&L $\Pi_{A|B}$ of Bank A is equal to:

$$\Pi_{A|B} = \mathrm{MtM} - \mathrm{CVA}_B$$

where MtM is the risk-free mark-to-market value of $\mathcal{C}$ and CVA_B is the CVA with respect to Bank B. We assume that Bank A has traded the same contract with Bank C. It follows that:

$$\Pi_{A|C} = \mathrm{MtM} - \mathrm{CVA}_C$$

[17] IFRS 13 was originally issued in May 2011 and became effective after January 2013.

In a world where there is no counterparty credit risk, we have:

$$\Pi_{A|B} = \Pi_{A|C} = \text{MtM}$$

If we take into account the counterparty credit risk, the two P&Ls of the same contract are different because Bank A does not face the same risk:

$$\Pi_{A|B} \neq \Pi_{A|C}$$

In particular, if Bank A want to close the two exposures, it is obvious that the contact $\mathcal{C}$ with the counterparty B has more value than the contact $\mathcal{C}$ with the counterparty C if the credit risk of B is lower than the credit risk of C. In this context, the notion of mark-to-market is complex, because it depends on the credit risk of the counterparties.

Remark 41 *If the bank does not take into account CVA to price its OTC derivatives, it does not face CVA risk. This situation is now marginal, because of the accounting standards IFRS 13.*

4.2.1.2 CVA, DVA and bilateral CVA

In the previous section, we have defined the CVA as the market risk related to the credit risk of the counterparty. According to EBA (2015), it should reflect today's best estimate of the potential loss on the OTC derivative due to the default of the counterparty. In a similar way, we can define the debit value adjustment (DVA) as the credit-related adjustment capturing the entity's own credit risk. In this case, DVA should reflect the potential gain on the OTC derivative due to the entity's own default. If we consider our previous example, the expression of the P&L becomes:

$$\Pi_{A|B} = \text{MtM} + \underbrace{\text{DVA}_A - \text{CVA}_B}_{\text{Bilateral CVA}}$$

The combination of the two credit-related adjustments is called the bivariate CVA. We then obtain the following cases:

1. If the credit risk of Bank A is lower than the credit risk of Bank B ($\text{DVA}_A < \text{CVA}_B$), the bilateral CVA of Bank A is negative and reduces the value of the OTC portfolio from the perspective of Bank A.

2. If the credit risk of Bank A is higher than the credit risk of Bank B ($\text{DVA}_A > \text{CVA}_B$), the bilateral CVA of Bank A is positive and increases the value of the OTC portfolio from the perspective of Bank A.

3. If the credit risk of Bank A is equivalent to the credit risk of Bank B, the bilateral CVA is equal to zero.

We notice that the DVA of Bank A is the CVA of Bank A from the perspective of Bank B:

$$\mathrm{CVA}_A = \mathrm{DVA}_A$$

We also have $\mathrm{DVA}_B = \mathrm{CVA}_B$, which implies that the P&L of Bank B is equal to:

$$\begin{aligned} \Pi_{B|A} &= -\,\mathrm{MtM} + \mathrm{DVA}_B - \mathrm{CVA}_A \\ &= -\,\mathrm{MtM} + \mathrm{CVA}_B - \mathrm{DVA}_A \\ &= -\Pi_{A|B} \end{aligned}$$

We deduce that the P&Ls of Banks A and B are coherent in the bilateral CVA framework as in the risk-free MtM framework. This is not true if we only consider the (unilateral or one-sided) CVA or the DVA adjustments.

In order to define more precisely CVA and DVA, we introduce the following notations:

- The positive exposure $e^+(t)$ is the maximum between 0 and the risk-free mark-to-market:

  $$e^+(t) = \max(\mathrm{MtM}(t), 0)$$

  This quantity was previously noted $e(t)$ and corresponds to the potential future exposure in the CCR framework.

- The negative exposure $e^-(t)$ is the difference between the risk-free mark-to-market and the positive exposure:

  $$e^-(t) = \mathrm{MtM}(t) - e^+(t)$$

  We have also:

  $$\begin{aligned} e^-(t) &= -\min(\mathrm{MtM}(t), 0) \\ &= \max(-\,\mathrm{MtM}(t), 0) \end{aligned}$$

  The negative exposure is then the equivalent of the positive exposure from the perspective of the counterparty.

The credit value adjustment is the risk-neutral discounted expected value of the potential loss:

$$\mathrm{CVA} = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}\{\boldsymbol{\tau}_B \leq T\} \times e^{-\int_0^{\boldsymbol{\tau}_B} r_t\,\mathrm{d}t} \times L\right]$$

where T is the maturity of the OTC derivative, $\boldsymbol{\tau}_B$ is the default time of Bank B and:

$$L = (1 - \boldsymbol{\mathcal{R}}_B) \times e^+(\boldsymbol{\tau}_B)$$

Using usual assumptions[18], we obtain:

$$\mathrm{CVA} = (1 - \boldsymbol{\mathcal{R}}_B) \times \int_0^T B_0(t)\,\mathrm{EpE}(t)\;\mathrm{d}\mathbf{F}_B(t)$$

where $\mathrm{EpE}(t)$ is the risk-neutral discounted expected positive exposure:

$$\mathrm{EpE}(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{+}(t)\right]$$

and $\mathbf{F}_B$ is the cumulative distribution function of $\boldsymbol{\tau}_B$. Knowing that the survival function $\mathbf{S}_B(t)$ is equal to $1 - \mathbf{F}_B(t)$, we deduce that:

$$\mathrm{CVA} = (1 - \boldsymbol{\mathcal{R}}_B) \times \int_0^T -B_0(t)\,\mathrm{EpE}(t)\;\mathrm{d}\mathbf{S}_B(t) \tag{4.15}$$

In a similar way, the debit value adjustment is defined as the risk-neutral discounted expected value of the potential gain:

$$\mathrm{DVA} = \mathbb{E}^{\mathbb{Q}}\left[\mathbf{1}\{\boldsymbol{\tau}_A \le T\} \times e^{-\int_0^{\boldsymbol{\tau}_A} r_t\,\mathrm{d}t} \times G\right]$$

where $\boldsymbol{\tau}_A$ is the default time of Bank A and:

$$G = (1 - \boldsymbol{\mathcal{R}}_A) \times e^{-}(\boldsymbol{\tau}_A)$$

Using the same assumptions than previously, it follows that:

$$\mathrm{DVA} = (1 - \boldsymbol{\mathcal{R}}_A) \times \int_0^T -B_0(t)\,\mathrm{EnE}(t)\;\mathrm{d}\mathbf{S}_A(t) \tag{4.16}$$

where $\mathrm{EnE}(t)$ is the risk-neutral discounted expected negative exposure:

$$\mathrm{EnE}(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-}(t)\right]$$

We deduce that the bilateral CVA is:

$$\begin{aligned}\mathrm{BCVA} &= \mathrm{DVA} - \mathrm{CVA} \\ &= (1 - \boldsymbol{\mathcal{R}}_A) \times \int_0^T -B_0(t)\,\mathrm{EnE}(t)\;\mathrm{d}\mathbf{S}_A(t) - \\ &\quad\ (1 - \boldsymbol{\mathcal{R}}_B) \times \int_0^T -B_0(t)\,\mathrm{EpE}(t)\;\mathrm{d}\mathbf{S}_B(t) \end{aligned} \tag{4.17}$$

When we calculate the bilateral CVA as the difference between the DVA and the CVA, we consider that the DVA does not depend on $\boldsymbol{\tau}_B$ and the CVA does not depend on $\boldsymbol{\tau}_A$. In the more general case, we have:

$$\mathrm{BCVA} = \mathbb{E}^{\mathbb{Q}}\left[\begin{array}{c}\mathbf{1}\{\boldsymbol{\tau}_A \le \min(T, \boldsymbol{\tau}_B)\} \times e^{-\int_0^{\boldsymbol{\tau}_A} r_t\,\mathrm{d}t} \times G - \\ \mathbf{1}\{\boldsymbol{\tau}_B \le \min(T, \boldsymbol{\tau}_A)\} \times e^{-\int_0^{\boldsymbol{\tau}_B} r_t\,\mathrm{d}t} \times L\end{array}\right] \tag{4.18}$$

In this case, the calculation of the bilateral CVA requires to consider the joint survival function of $(\boldsymbol{\tau}_A, \boldsymbol{\tau}_B)$.

[18]The default time and the discount factor are independent and the recovery rate is constant.

Remark 42 *In practice, we calculate CVA and DVA by approximating the integral by a sum:*

$$\begin{aligned}\mathrm{CVA} &= (1-\boldsymbol{\mathcal{R}}_B)\times\sum_{t_i\leq T} B_0(t_i)\,\mathrm{EpE}(t_i)\left(\mathbf{S}_B(t_{i-1})-\mathbf{S}_B(t_i)\right)\\ \mathrm{DVA} &= (1-\boldsymbol{\mathcal{R}}_A)\times\sum_{t_i\leq T} B_0(t_i)\,\mathrm{EnE}(t_i)\left(\mathbf{S}_A(t_{i-1})-\mathbf{S}_A(t_i)\right)\end{aligned}$$

where $\{t_i\}$ is a partition of $[0,T]$. For the bilateral CVA, the expression (4.18) can be evaluated using Monte Carlo methods.

4.2.2 Regulatory capital

4.2.2.1 Advanced method

The advanced method (or AM-CVA) can be considered by banks that use IMM and VAR models. In this approach, we approximate the integral by the middle Riemann sum:

$$\mathrm{CVA}=\mathrm{LGD}_B\sum_{t_i\leq T}\left(\frac{\mathrm{EpE}(t_{i-1})\,B_0(t_{i-1})+B_0(t_i)\,\mathrm{EpE}(t_i)}{2}\right)\times\mathrm{PD}_B(t_{i-1},t_i)$$

where $\mathrm{LGD}=1-\boldsymbol{\mathcal{R}}_B$ is risk-neutral loss given default of the counterparty B and $\mathrm{PD}_B(t_{i-1},t_i)$ is the risk neutral probability of default between t_{i-1} and t_i. Using the credit triangle relationship, we know that the spread $\mathcal{s}$ of the CDS is related to the intensity λ:

$$\mathcal{s}_B(t)=(1-\boldsymbol{\mathcal{R}}_B)\times\lambda_B(t)$$

We deduce that:

$$\begin{aligned}\mathbf{S}_B(t) &= \exp(-\lambda_B(t)\times t)\\ &= \exp\left(-\frac{\mathcal{s}(t)}{\mathrm{LGD}_B}\times t\right)\end{aligned}$$

It follows that the risk neutral probability of default $\mathrm{PD}_B(t_{i-1},t_i)$ is equal to[19]:

$$\begin{aligned}\mathrm{PD}_B(t_{i-1},t_i) &= \max\left(\mathbf{S}_B(t_{i-1})-\mathbf{S}_B(t_i),0\right)\\ &= \max\left(\exp\left(-\frac{\mathcal{s}(t_{i-1})}{\mathrm{LGD}_B}\times t_{i-1}\right)-\exp\left(-\frac{\mathcal{s}(t_i)}{\mathrm{LGD}_B}\times t_i\right),0\right)\end{aligned}$$

In the advanced approach, the capital charge is equal to:

$$\boldsymbol{\mathcal{K}}=3\times(\mathrm{CVA}+\mathrm{SCVA})$$

where CVA is calculated using the last one-year period and SCVA is the stressed CVA based on a one-year stressed period of credit spreads.

[19] A zero floor is added in order to verify that $\mathrm{PD}_B(t_{i-1},t_i)\geq 0$.

4.2.2.2 Standardized method

In the standardized method (or SM-CVA), the capital charge is equal to:

$$\mathcal{K} = 2.33 \times \sqrt{h} \times \sqrt{\left(\frac{1}{2}\sum_i w_i \Omega_i - w^\star_{\text{index}}\Omega^\star_{\text{index}}\right)^2 + \frac{3}{4}\sum_i w_i^2 \Omega_i^2} \tag{4.19}$$

with:

$$\begin{aligned} \Omega_i &= \mathrm{M}_i \times \mathrm{EAD}_i - \mathrm{M}_i^\star \times H_i^\star \\ \Omega^\star_{\text{index}} &= \mathrm{M}^\star_{\text{index}} \times H^\star_{\text{index}} \end{aligned}$$

In this formula, h is the time horizon (one year), w_i is the weight of the i^{th} counterparty based on its rating, M_i is the effective maturity of the i^{th} netting set, EAD_i is the exposure at default of the i^{th} netting set, $\mathrm{M}_i^\star$ is the maturity adjustment factor for the single name hedge, $H_i^\star$ is the hedging notional of the single name hedge, $w^\star_{\text{index}}$ is the weight of the index hedge, $\mathrm{M}^\star_{\text{index}}$ is the maturity adjustment factor for the index hedge and $H^\star_{\text{index}}$ is the hedging notional of the index hedge. In this formula, EAD_i corresponds to the CCR exposure at default calculated with the CEM or IMM approaches.

Remark 43 *We notice that the Basel Committee recognizes credit hedges (single name CDS, contingent CDS and CDS indexes) for reducing CVA volatility. If there is no hedge, we obtain:*

$$\mathcal{K} = 2.33 \times \sqrt{h} \times \sqrt{\frac{1}{4}\left(\sum_i w_i \times \mathrm{M}_i \times \mathrm{EAD}_i\right)^2 + \frac{3}{4}\sum_i w_i^2 \times \mathrm{M}_i^2 \times \mathrm{EAD}_i^2}$$

The derivation of Equation (4.19) is explained in Pykhtin (2012). We consider a Gaussian random vector $X = (X_1, \ldots, X_n)$ with $X_i \sim \mathcal{N}\left(0, \sigma_i^2\right)$. We assume that the random variables $X_1, \ldots, X_n$ follow a single risk factor model such that the correlations $\rho(X_i, X_j)$ are constant and equal to ρ. We consider another random variable $X_{n+1} \sim \mathcal{N}\left(0, \sigma_{n+1}^2\right)$ such that $\rho(X_i, X_{n+1})$ is also constant and equal to ρ_{n+1}. Let Y be the random variable defined as the sum of X_i's minus X_{n+1}:

$$Y = \sum_{i=1}^n X_i - X_{n+1}$$

It follows that $Y \sim N\left(0, \sigma_Y^2\right)$ with:

$$\sigma_Y^2 = \sum_{i=1}^n \sigma_i^2 + 2\rho \sum_{i=1}^n \sum_{j=1}^i \sigma_i \sigma_j - 2\rho_{n+1}\sigma_{n+1}\sum_{i=1}^n \sigma_i + \sigma_{n+1}^2$$

We finally deduce that:

$$\mathbf{F}_Y^{-1}(\alpha) = \Phi^{-1}(\alpha)\sqrt{\sum_{i=1}^n \sigma_i^2 + 2\rho \sum_{i=1}^n \sum_{j=1}^i \sigma_i \sigma_j - 2\rho_{n+1}\sigma_{n+1}\sum_{i=1}^n \sigma_i + \sigma_{n+1}^2}$$

Equation (4.19) is obtained by setting $\sigma_i = w_i \Omega_i$, $\sigma_{n+1} = w^\star_{\text{index}} \Omega^\star_{\text{index}}$, $\rho = 25\%$, $\rho_{n+1} = 50\%$ and $\alpha = 99\%$. This means that X_i is the CVA net exposure of the i^{th} netting set (including individual hedges) and X_{n+1} is the macro hedge of the CVA based on credit indexes.

4.2.2.3 Basel IV standardized approach (SA-CVA)

4.2.3 Impact of CVA on the banking industry

4.3 Collateral risk

4.4 Exercises

4.4.1 Impact of netting agreements in counterparty credit risk

The table below gives the current mark-to-market of 7 OTC contracts between Bank A and Bank B:

	Equity			Fixed income		FX	
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$
A	$+10$	-5	$+6$	$+17$	-5	-5	$+1$
B	-11	$+6$	-3	-12	$+9$	$+5$	$+1$

The table should be read as follows: Bank A has a mark-to-market equal to 10 for the contract $\mathcal{C}_1$ whereas Bank B has a mark-to-market equal to -11 for the same contract, Bank A has a mark-to-market equal to -5 for the contract $\mathcal{C}_2$ whereas Bank B has a mark-to-market equal to $+6$ for the same contract, etc.

1. (a) Explain why there are differences between the MtM values of a same OTC contract.
   (b) Calculate the exposure at default of Bank A.
   (c) Same question if there is a global netting agreement.
   (d) Same question if the netting agreement only concerns equity products.

2. In the following, we measure the impact of netting agreements on the exposure at default.

   (a) We consider a first OTC contract $\mathcal{C}_1$ between Bank A and Bank B. The mark-to-market $\mathrm{MtM}_1(t)$ of Bank A for the contract $\mathcal{C}_1$ is defined as follows:

   $$\mathrm{MtM}_1(t) = x_1 + \sigma_1 W_1(t)$$

   where $W_1(t)$ is a Brownian motion. Calculate the potential future exposure of Bank A.

   (b) We consider a second OTC contract between Bank A and Bank B. The mark-to-market is also given by the following expression:

   $$\mathrm{MtM}_2(t) = x_2 + \sigma_2 W_2(t)$$

   where $W_2(t)$ is a second Brownian motion that is correlated with $W_1(t)$. Let ρ be this correlation such that $\mathbb{E}[W_1(t) W_2(t)] = \rho t$. Calculate the expected exposure of bank A if there is no netting agreement.

(c) Same question when there is a global netting agreement between Bank A and Bank B.

(d) Comment on these results.

4.4.2 Calculation of the effective expected positive exposure

We denote by $e(t)$ the potential future exposure of an OTC contract with maturity T. The current date is set to $t = 0$.

1. Define the concepts of peak exposure $\text{PE}_\alpha(t)$, maximum peak exposure $\text{MPE}_\alpha(0;t)$, expected exposure $\text{EE}(t)$, expected positive exposure $\text{EPE}(0;t)$, effective expected exposure $\text{EEE}(t)$ and effective expected positive exposure $\text{EEPE}(0;t)$.

2. Calculate these different quantities when the potential future exposure is $e(t) = \sigma\sqrt{t}X$ with $X \sim \mathcal{U}_{[0,1]}$.

3. Same question when $e(t) = \exp\left(\sigma\sqrt{t}X\right)$ with $X \sim \mathcal{N}(0,1)$.

4. Same question when $e(t) = \sigma\left(t^3 - \frac{7}{3}Tt^2 + \frac{4}{3}T^2t\right)X$ with $X \sim \mathcal{U}_{[0,1]}$.

5. Same question when $e(t) = \sigma\sqrt{t}X$ where X is a random variable defined on $[0,1]$ with the following probability density function[20]:

$$f(x) = \frac{x^a}{a+1}$$

6. Comment on these results.

4.4.3 Calculation of the capital charge for counterparty credit risk

We denote by $e(t)$ the potential future exposure of an OTC contract with maturity T. The current date is set to $t = 0$. Let N and σ be the notional and the volatility of the underlying contract. We assume that $e(t) = N\sigma\sqrt{t}X$ with $0 \leq X \leq 1$, $\Pr\{X \leq x\} = x^\gamma$ and $\gamma > 0$.

1. Calculate the peak exposure $\text{PE}_\alpha(t)$, the expected exposure $\text{EE}(t)$ and the effective expected positive exposure $\text{EEPE}(0;t)$.

2. The bank manages the credit risk with the foundation IRB approach and the counterparty credit risk with an internal model. We consider an OTC contract with the following parameters: N is equal to \$3 mn, the maturity T is one year, the volatility σ is set to 20% and γ is estimated at 2.

[20] We assume that $a > 0$.

(a) Calculate the exposure at default EAD knowing that the bank uses the regulatory value for the parameter α.

(b) The default probability of the counterparty is estimated at 1%. Calculate then the capital charge for counterparty credit risk of this OTC contract[21].

4.4.4 Illustration of the wrong-way risk

4.4.5 Counterparty exposure of interest rate swap

4.4.6 Derivation of SA-CCR formulas

4.4.7 Examples of SA-CCR calculation

4.4.8 Calculation of CVA and DVA measures

We consider an OTC contract with maturity T between Bank A and Bank B. We denote by $\mathrm{MtM}(t)$ the risk-free mark-to-market of Bank A. The current date is set to $t = 0$ and we assume that:

$$\mathrm{MtM}(t) = N\sigma\sqrt{t}X$$

where N is the notional of the OTC contract, σ is the volatility of the underlying asset and X is a random variable, which is defined on the support $[-1, 1]$ and whose density function is:

$$f(x) = \frac{1}{2}$$

1. Define the concept of positive exposure $e^+(t)$. Show that the cumulative distribution function $\mathbf{F}_{[0,t]}$ of $e^+(t)$ has the following expression:

$$\mathbf{F}_{[0,t]}(x) = \mathbb{1}\left(0 \leq x \leq \sigma\sqrt{t}\right) \cdot \left(\frac{1}{2} + \frac{x}{2N\sigma\sqrt{t}}\right)$$

   where $\mathbf{F}_{[0,t]}(x) = 0$ if $x \leq 0$ and $\mathbf{F}_{[0,t]}(x) = 1$ if $x \geq \sigma\sqrt{t}$.

2. Deduce the value of the expected positive exposure $\mathrm{EpE}(t)$.

3. We note $\boldsymbol{\mathcal{R}}_B$ the fixed and constant recovery rate of Bank B. Give the mathematical expression of the CVA.

4. By using the definition of the lower incomplete gamma function $\gamma(s, x)$, show that the CVA is equal to:

$$\mathrm{CVA} = \frac{N(1 - \boldsymbol{\mathcal{R}}_B)\sigma\gamma\left(\frac{3}{2}, \lambda_B T\right)}{4\sqrt{\lambda_B}}$$

[21] We will take a value of 70% for the LGD parameter and a value of 20% for the default correlation. We can also use the approximation $\Phi(-1) \simeq 16\%$.

when the default time of Bank B is exponential with parameter λ_B and interest rates are equal to zero.

5. Comment on this result.

6. By assuming that the default time of Bank A is exponential with parameter λ_A, deduce the value of the DVA without additional computations.

Chapter 5

Operational Risk

The integration of operational risk into the Basel II Accord was a long process because of the hostile reaction from the banking sector. At the end of the 1990s, the risk of operational losses was perceived as relatively minor. However, some events had shown that it was not the case. The most famous example was the bankruptcy of the Barings Bank in 1995. The loss of \$1.3 bn was due to a huge position of the trader Nick Leeson in futures contracts without authorization. Other examples included the money laundering in Banco Ambrosiano Vatican Bank (1983), the rogue trading in Sumitomo Bank (1996), the headquarter fire of Crédit Lyonnais (1996), etc. Since the publication of the CP2 in January 2001, the position of banks has significantly changed and operational risk is today perceived as a major risk of the banking industry. Management of operational risk has been strengthened, with the creation of dedicated risk management units, the appointment of compliance officers and the launch of anti-money laundering programs.

5.1 Definition of operational risk

The Basel Committee defines the operational risk in the following way:

> "*Operational risk is defined as the risk of loss resulting from inadequate or failed internal processes, people and systems or from external events. This definition includes legal risk, but excludes strategic and reputational risk*" (BCBS, 2006, page 144).

The operational risk recovers then all the losses of the bank that cannot be attributed to market and credit risk. Nevertheless, losses which result from strategic decisions are not taken into account. An example is the purchase of a software or an information system, which is not relevant for the firm. Losses due to reputational risk are also excluded from the definition of operational risk. They are generally caused by an event, which is related to another risk. The difficulty is to measure the indirect loss of such event in terms of business. For instance, if we consider the diesel emissions scandal of Volkswagen, we can estimate the losses due to the recall of cars, class action lawsuits and potential fines. However, it is impossible to know what will be the impact of this event on the future sales and the market share of Volkswagen.

In order to better understand the concept of operational risk, we give here the loss even type classification adopted by the BCBS:

1. Internal fraud ("*losses due to acts of a type intended to defraud, misappropriate property or circumvent regulations, the law or company policy, excluding diversity/discrimination events, which involves at least one internal party*")
   (a) Unauthorized activity
   (b) Theft and fraud
2. External fraud ("*losses due to acts of a type intended to defraud, misappropriate property or circumvent the law, by a third party*")
   (a) Theft and fraud
   (b) Systems security
3. Employment practices and workplace safety ("*losses arising from acts inconsistent with employment, health or safety laws or agreements, from payment of personal injury claims, or from diversity/discrimination events*")
   (a) Employee relations
   (b) Safe environment
   (c) Diversity & discrimination
4. Clients, products & business practices ("*losses arising from an unintentional or negligent failure to meet a professional obligation to specific clients (including fiduciary and suitability requirements), or from the nature or design of a product*")
   (a) Suitability, disclosure & fiduciary
   (b) Improper business or market practices
   (c) Product flaws
   (d) Selection, sponsorship & exposure
   (e) Advisory activities
5. Damage to physical assets ("*losses arising from loss or damage to physical assets from natural disaster or other events*")
   (a) Disasters and other events

6. Business disruption and system failures ("*losses arising from disruption of business or system failures*")

   (a) Systems

7. Execution, delivery & process management ("*losses from failed transaction processing or process management, from relations with trade counterparties and vendors*")

   (a) Transaction capture, execution & maintenance
   (b) Monitoring and reporting
   (c) Customer intake and documentation
   (d) Customer/client account management
   (e) Trade counterparties
   (f) Vendors & suppliers

This is a long list of loss types, because the banking industry has been a fertile ground for operational risks. We have already cited some well-know operational losses before the crisis. In 2009, the Basel Committee has published the results of a loss data collection exercise. For this LDCE, 119 banks submitted a total of 10.6 million internal losses with an overall loss amount of €59.6 bn. The largest 20 losses represented a total of €17.6 bn. In Table 5.1, we have reported statistics of the loss data, when the loss is larger than €20 000. For each year, we indicate the number n_L of losses, the total loss amount L and the number n_B of reporting banks. Each bank experienced more than 300 losses larger than €20 000 per year on average. We also notice that these losses represented about 90% of the overall loss amount.

TABLE 5.1: Internal losses larger than €20 000 by year

Year	pre 2002	2002	2003	2004	2005	2006	2007
n_L	14 017	10 216	13 691	22 152	33 216	36 386	36 622
L (in € bn)	3.8	12.1	4.6	7.2	9.7	7.4	7.9
n_B	24	35	55	68	108	115	117

Source: BCBS (2009d).

Since 2008, operational risk has dramatically increased. For instance, rogue trading has impacted many banks and the magnitude of these unauthorized trading losses is much higher than before[1]. The Libor interest rate manipulation scandal led to very large fines ($2.5 bn for Deutsche Bank, $1 bn for Rabobank, $545 mn for UBS, etc.). In May 2015, six banks (Bank of America, Barclays, Citigroup, JP Morgan, UBS and RBS) agreed to pay fines totaling

[1] We can cite Société Générale in 2008 ($7.2 bn), Morgan Stanley in 2008 ($9.0 bn), BPCE in 2008 ($1.1 bn), UBS in 2011 ($2 bn) and JPMorgan Chase in 2012 ($5.8 bn).

\$5.6 bn in the case of the forex scandal[2]. The anti-money laundering controls led BNP Paribas to pay a fine of \$8.9 bn in June 2014 to the US federal government. In this context, operational risk, and more specifically compliance and legal risk, is a major concern for banks. It is an expansive risk, because of the direct losses, but also because of the indirect costs induced by the proliferation of internal controls and security infrastructure[3].

Remark 44 *Operational risk is not limited to the banking sector. Other financial sectors have been impacted by such risk. The most famous example is the Ponzi scheme organized by Bernard Madoff, which caused a loss of* \$50 *bn to his investors.*

5.2 Basel approaches for calculating the regulatory capital

In this approach, we present the three approaches described in the Basel II framework in order to calculate the capital charge for operational risk:

1. the basic indicator approach (BIA);
2. the standardized approach (TSA);
3. and advanced measurement approaches (AMA).

We also present the proposed revisions of the Basel Committee to the standardized approach for measuring operational risk capital. Once finalized, the revised standardized approach will replace the current non-model-based approaches, which comprise BIA and TSA.

5.2.1 The basic indicator approach

The basic indicator approach is the simplest method for calculating the operational risk capital requirement. In this case, the capital charge is a fixed percentage of annual gross income:

$$\mathcal{K} = \alpha \times \overline{\mathrm{GI}}$$

[2]The Libor scandal was a series of fraudulent actions connected to the Libor (London Interbank Offered Rate), while the forex scandal concerns several banks, which have manipulated exchange rates via electronic chatrooms in which traders discussed the trades they planned to do.

[3]A typical example of expansive costs is the risk of cyber attack.

where α is set equal to 15% and $\overline{\mathrm{GI}}$ is the average of the positive gross income over the previous three years:

$$\overline{\mathrm{GI}} = \frac{\max\left(\mathrm{GI}_{t-1}, 0\right) + \max\left(\mathrm{GI}_{t-2}, 0\right) + \max\left(\mathrm{GI}_{t-3}, 0\right)}{\sum_{k=1}^{3} \mathbb{1}\left\{\mathrm{GI}_{t-k} > 0\right\}}$$

In this approach, the capital charge is related to the financial results of the bank, but not to its risk exposure.

5.2.2 The standardized approach

The standardized approach is an extended version of the previous method. In this case, the bank are divided into eight business lines, which are given in Table 5.2. The bank then calculates the capital charge for each business lines:

$$\mathcal{K}_{j,t} = \beta_j \times \mathrm{GI}_{j,t}$$

where β_j and $\mathrm{GI}_{j,t}$ are a fixed percentage[4] and the gross income corresponding to the j^{th} business line. The total capital charge is the three-year average of the sum of all capital charges:

$$\mathcal{K} = \frac{1}{3} \sum_{k=1}^{3} \max\left(\sum_{j=1}^{8} \mathcal{K}_{j,t-k}, 0\right)$$

We notice that a negative capital charge in one business line may offset positive capital charges in other business lines. If the values of gross income are all positive, the total capital charge becomes:

$$\begin{aligned} \mathcal{K} &= \frac{1}{3} \sum_{k=1}^{3} \sum_{j=1}^{8} \beta_j \times \mathrm{GI}_{j,t-k} \\ &= \sum_{j=1}^{8} \beta_j \times \overline{\mathrm{GI}}_j \end{aligned}$$

where $\overline{\mathrm{GI}}_j$ is the average gross income over the previous three years of the j^{th} business line.

Example 40 *We consider Bank A, whose activity is mainly driven by retail banking and asset management. We compare it with Bank B, which is more focused on corporate finance. We assume that the two banks are only composed of four business lines: corporate finance, retail banking, agency services and asset management. The gross income expressed in \$ mn for the last three years*

[4]The values taken by the beta coefficient are reported in Table 5.2.

TABLE 5.2: Mapping of business lines for operational risk

Level 1	Level 2	β_j
Corporate Finance†	Corporate Finance Municipal/Government Finance Merchant Banking Advisory Services	18%
Trading & Sales‡	Sales Market Making Proprietary Positions Treasury	18%
Retail Banking	Retail Banking Private Banking Card Services	12%
Commercial Banking♮	Commercial Banking	12%
Payment & Settlement	External Clients	18%
Agency Services	Custody Corporate Agency Corporate Trust	15%
Asset Management	Discretionary Fund Management Non-Discretionary Fund Management	12%
Retail Brokerage	Retail Brokerage	12%

†Mergers and acquisitions, underwriting, securitization, syndications, IPO, debt placements. ‡Buying and selling of securities and derivatives, own position securities, lending and repos, brokerage. ♮Project finance, real estate, export finance, trade finance, factoring, leasing, lending, guarantees, bills of exchange.

is given below:

Business line	*Bank A*			*Bank B*		
	$t-1$	$t-2$	$t-3$	$t-1$	$t-2$	$t-3$
Corporate finance	10	15	−30	200	300	150
Retail banking	250	230	205	50	45	−30
Agency services	10	10	12			
Asset management	70	65	72	12	8	−4

For Bank A, we obtain $\mathrm{GI}_{t-1} = 340$, $\mathrm{GI}_{t-2} = 320$ and $\mathrm{GI}_{t-3} = 259$. The average gross income is then equal to 306.33, implying that the BIA capital charge $\boldsymbol{\mathcal{K}}_A^{\mathrm{BIA}}$ is equal to \$45.95 mn. If we consider Bank B, the required capital $\boldsymbol{\mathcal{K}}_B^{\mathrm{BIA}}$ is lower and equal to \$36.55 mn. In the case of the standardized approach, the beta coefficients are respectively equal to 18%, 12%, 15% and

12%. We deduce that:

$$\begin{aligned}
\mathcal{K}_A^{\mathrm{TSA}} = & \frac{1}{3} \times (\max(18\% \times 10 + 12\% \times 250 + 15\% \times 10 + 12\% \times 70, 0) + \\
& \max(18\% \times 15 + 12\% \times 230 + 15\% \times 10 + 12\% \times 65, 0) + \\
& \max(-18\% \times 30 + 12\% \times 205 + 15\% \times 12 + 12\% \times 72, 0)) \\
= & \$36.98 \text{ mn}
\end{aligned}$$

We also have $\mathcal{K}_B^{\mathrm{TSA}} = \$42.24$ mn. We notice that $\mathcal{K}_A^{\mathrm{BIA}} > \mathcal{K}_A^{\mathrm{TSA}}$ and $\mathcal{K}_B^{\mathrm{BIA}} < \mathcal{K}_B^{\mathrm{TSA}}$. Bank A has a lower capital charge when using TSA instead of BIA, because it is more exposed to low-risk business lines (retail banking and asset management). For Bank B, it is the contrary because its main exposure concerns high-risk business lines (corporate finance). However, if we assume that the gross income of the corporate finance for Bank B at time $t-3$ is equal to -150 instead of $+150$, we obtain $\mathcal{K}_B^{\mathrm{BIA}} = \$46.13$ mn and $\mathcal{K}_B^{\mathrm{TSA}} = \$34.60$ mn. In this case, the TSA approach is favorable, because the gross income at time $t-3$ is negative implying that the capital contribution at time $t-3$ is equal to zero.

Contrary to the basic indicator approach that requires no criteria to be used, banks must satisfy a list of qualifying criteria for the standardized approach. For instance, the board of directors is actively involved in the oversight of the operational risk management framework and each business lines has sufficient resources to manage operational risk. Internationally active banks must also collect operational losses and use this information for taking appropriate action.

5.2.3 Advanced measurement approaches

Like the internal model-based approach for market risk, the AMA method is defined by certain criteria without refereing to a specific statistical model:

- The capital charge should cover the one-year operational loss at the 99.9% confidence level. It corresponds to the sum of expected loss (EL) and unexpected loss (UL).
- The model must be estimated using a minimum five-year observation period of internal loss data, and capture tail loss events by considering for example external loss data when it is needed. It must also include scenario analysis and factors reflecting internal control systems.
- The risk measurement system must be sufficiently granular to capture the main operational risk factors. By default, the operational risk of the bank must be divided into the 8 business lines and the 7 event types. For each cell of the matrix, the model must estimate the loss distribution and may use correlations to perform the aggregation.

- The allocation of economic capital across business lines must create incentives to improve operational risk management.
- The model can incorporate the risk mitigation impact of insurance, which is limited to 20% of the total operational risk capital charge.

The validation of the AMA model does not only concern the measurement aspects, but also the soundness of the entire operational risk management system. This concerns governance of operational risk, dedicated resources, management structure, risk cartography and key risk indicators (KRI), notification and action procedures, emergency and crisis management, business continuity and disaster recovery plans.

In order to better understand the challenges of an internal model, we have reported in Table 5.3 the distribution of annualized loss amounts by business line and event type obtained with the 2008 loss data collection exercise. We first notice an heterogeneity between business lines. For instance, losses were mainly concentrated in the fourth event type (clients, products & business practices) for the corporate finance business line (93.7%) and the seventh event type (execution, delivery & process management) for the payment & settlement business line (76.4%). On overage, these two event types represented more than 75% of the total loss amount. In contrast, fifth and sixth event types (damage to physical assets, business disruption and system failures) had a small contribution close to 1%. We also notice that operational losses mainly affected retail banking, followed by corporate finance and trading & sales. One of the issue is that this picture of operational risk is no longer valid after 2008 with the increase of losses in trading & sales, but also in payment & settlement. The nature of operational risk changes over time, which is a big challenge to build an internal model to calculate the required capital.

TABLE 5.3: Distribution of annualized operational losses (in %)

Business line	Event type							All
	1	2	3	4	5	6	7	
Corporate Finance	0.2	0.1	0.6	93.7	0.0	0.0	5.4	28.0
Trading & Sales	11.0	0.3	2.3	29.0	0.2	1.8	55.3	13.6
Retail Banking	6.3	19.4	9.8	40.4	1.1	1.5	21.4	32.0
Commercial Banking	11.4	15.2	3.1	35.5	0.4	1.7	32.6	7.6
Payment & Settlement	2.8	7.1	0.9	7.3	3.2	2.3	76.4	2.6
Agency Services	1.0	3.2	0.7	36.0	18.2	6.0	35.0	2.6
Asset Management	11.1	1.0	2.5	30.8	0.3	1.5	52.8	2.5
Retail Brokerage	18.1	1.4	6.3	59.5	0.1	0.2	14.4	5.1
Unallocated	6.5	2.8	28.4	28.3	6.5	1.3	26.2	6.0
All	6.1	8.0	6.0	52.4	1.4	1.2	24.9	100.0

Source: BCBS (2009d).

5.2.4 Basel IV approach

5.3 Loss distribution approach

Although the Basel Committee does not advocate any particular method for the AMA framework, the loss distribution approach (LDA) is the recognized standard model for calculating the capital charge. This model is not specific to operational risk because it was developed in the case of the collective risk theory at the beginning of 1900s. However, operational risk presents some characteristics that need to be considered.

5.3.1 Definition

The loss distribution approach is described in Klugman *et al.* (2012) and Frachot *et al.* (2001). Let the operational loss L of the bank be divided into a matrix of homogenous losses:

$$L = \sum_{k=1}^{K} S_k \tag{5.1}$$

where S_k is the sum of losses of the k^{th} cell and K is the number of cells in the matrix. For instance, if we consider the Basel II classification, the mapping matrix contains 56 cells corresponding to the 8 business lines and 7 event types. The loss distribution approach is a method to model the random loss S_k of a particular cell. It assumes that S_k is the random sum of homogeneous individual losses:

$$S_k = \sum_{n=1}^{N_k(t)} X_n^{(k)} \tag{5.2}$$

where $N_k(t)$ is the random number of individual losses for the period $[0, t]$ and $X_n^{(k)}$ is the n^{th} individual loss. For example, if we consider internal fraud in corporate finance, we can write the loss for the next year as follows:

$$S = X_1 + X_2 + \ldots + X_{N(1)}$$

where X_1 is the first observed loss, X_2 is the second observed loss, $X_{N(1)}$ is the last observed loss of the year and $N(1)$ is the number of losses for the next year. We notice that we face two sources of uncertainty:

1. we don't know what will be the magnitude of each loss event (severity risk);
2. and we don't know how many losses will occur in the next year (frequency risk).

In order to simplify the notations, we omit the index k and rewrite the random sum as follows:

$$S = \sum_{n=1}^{N(t)} X_n \tag{5.3}$$

The loss distribution approach is based on the following assumptions:

- The number $N(t)$ of losses follows the loss frequency distribution $\mathbf{P}$. The probability that the number of loss events is equal to n is denoted by $p(n)$.
- The sequence of individual losses X_n is independent and identically distributed (*i.i.d.*). The corresponding probability distribution $\mathbf{F}$ is called the loss severity distribution.
- The number of events is independent from the amount of loss events.

Once the probability distributions $\mathbf{P}$ and $\mathbf{F}$ are chosen, we can determine the probability distribution of the aggregate loss S, which is denoted by $\mathbf{G}$ and is called the compound distribution.

Example 41 *We assume that the number of losses is distributed as follows:*

n	0	1	2	3
$p(n)$	50%	30%	17%	3%

The loss amount can take the values \$100 *and* \$200 *with probabilities* 70% *and* 30%.

To calculate the probability distribution $\mathbf{G}$ of the compound loss, we first define the probability distribution of X_1, $X_1 + X_2$ and $X_1 + X_2 + X_3$, because the maximum number of losses is equal to 3. If there is only one loss, we have $\Pr\{X_1 = 100\} = 70\%$ and $\Pr\{X_1 = 200\} = 30\%$. In the case of two losses, we obtain $\Pr\{X_1 + X_2 = 200\} = 49\%$, $\Pr\{X_1 + X_2 = 300\} = 42\%$ and $\Pr\{X_1 + X_2 = 400\} = 9\%$. Finally, the sum of three losses takes the values 300, 400, 500 and 600 with probabilities 34.3%, 44.1%, 18.9% and 2.7% respectively. We notice that these probabilities are in fact conditional to the number of losses. Using Bayes theorem, we obtain:

$$\Pr\{S = s\} = \sum_n \Pr\left\{\sum\nolimits_{i=1}^n X_i = s \,\middle|\, N(t) = n\right\} \times \Pr\{N(t) = n\}$$

We deduce that:

$$\begin{aligned} \Pr\{S = 0\} &= \Pr\{N(t) = 0\} \\ &= 50\% \end{aligned}$$

and:

$$\begin{aligned}\Pr\{S=100\} &= \Pr\{X_1=100\}\times\Pr\{N(t)=1\}\\ &= 70\%\times 30\%\\ &= 21\%\end{aligned}$$

The compound loss can takes the value 200 in two different ways:

$$\begin{aligned}\Pr\{S=200\} &= \Pr\{X_1=200\}\times\Pr\{N(t)=1\}+\\ &\quad \Pr\{X_1+X_2=200\}\times\Pr\{N(t)=2\}\\ &= 30\%\times 30\%+49\%\times 17\%\\ &= 17.33\%\end{aligned}$$

For the other values of S, we obtain $\Pr\{S=300\}=8.169\%$, $\Pr\{S=400\}=2.853\%$, $\Pr\{S=500\}=0.567\%$ and $\Pr\{S=600\}=0.081\%$.

The previous example shows that the cumulative distribution function of S can be written as[5]:

$$\mathbf{G}(s)=\begin{cases}\sum_{n=1}^{\infty}p(n)\,\mathbf{F}^{n\star}(s) & \text{for} \quad s>0\\ p(0) & \text{for} \quad s=0\end{cases} \tag{5.4}$$

where $\mathbf{F}^{n\star}$ is the n-fold convolution of $\mathbf{F}$ with itself:

$$\mathbf{F}^{n\star}(s)=\Pr\left\{\sum_{i=1}^{n}X_i\leq s\right\} \tag{5.5}$$

In Figure 5.1, we give an example of a continuous compound distribution when the annual number of losses follows the Poisson distribution $\mathcal{P}(50)$ and the individual losses follow the log-normal distribution $\mathcal{LN}(8,5)$. The capital charge, which is also called the capital-at-risk (CaR), corresponds then to the percentile α:

$$\mathrm{CaR}(\alpha)=\mathbf{G}^{-1}(\alpha) \tag{5.6}$$

The regulatory capital is obtained by setting α to 99.9% – $\boldsymbol{\mathcal{K}}=\mathrm{CaR}(99.9\%)$. This capital-at-risk is valid for one cell of the operational risk matrix. Another issue is to calculate the capital-at-risk for the bank as a whole. This requires to define the dependence function between the different compound losses $(S_1,S_2,\ldots,S_K)$. In summary, here are the different steps to implement the loss distribution approach:

- For each cell of the operational risk matrix, we estimate the loss frequency distribution and the loss severity distribution.
- We then calculate the capital-at-risk.
- We define the dependence function between the different cells of the operational risk matrix, and deduce the aggregate capital-at-risk.

[5] When $\mathbf{F}$ is a discrete probability function, it is easy to calculate $\mathbf{F}^{n\star}(s)$ and then deduce $\mathbf{G}(s)$. However, the determination of $\mathbf{G}(s)$ is more difficult in the general case of continuous probability functions. This issue is discussed in Section 5.3.3.

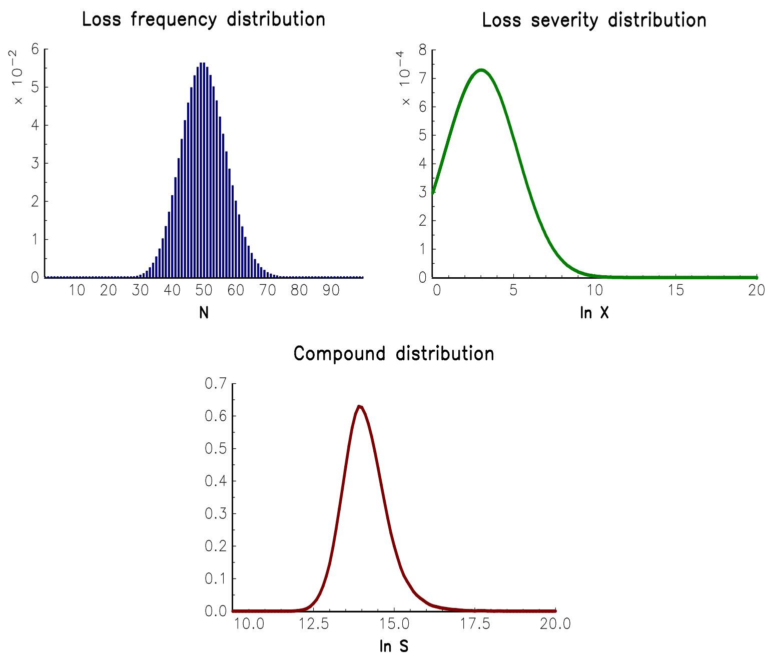

FIGURE 5.1: Compound distribution when $N \sim \mathcal{P}(50)$ and $X \sim \mathcal{LN}(8,5)$

5.3.2 Parametric estimation of F and P

We first consider the estimation of the severity distribution, because we will see that the estimation of the frequency distribution can only be done after this first step.

5.3.2.1 Estimation of the loss severity distribution

We assume that the bank has an internal loss data. We note $\{x_1, \ldots, x_T\}$ the sample collected for a given cell of the operational risk matrix. We consider that the individual losses X follow a given parametric distribution $\mathbf{F}$:

$$X \sim \mathbf{F}(x; \theta)$$

where θ is the vector of parameters to estimate.

In order to be a good candidate for modeling the loss severity, the probability distribution $\mathbf{F}$ must satisfy the following properties: the support of $\mathbf{F}$ is the interval $\mathbb{R}_+$, it is sufficiently flexible to accommodate a wide variety of empirical loss data and it can fit large losses. We list here the cumulative probability distributions that are the most used in operational risk models:

- Gamma $X \sim \mathcal{G}(\alpha, \beta)$

$$\mathbf{F}(x; \theta) = \frac{\gamma(\alpha, \beta x)}{\Gamma(\alpha)}$$

where $\alpha > 0$ and $\beta > 0$.

- Log-gamma $X \sim \mathcal{LG}(\alpha, \beta)$

$$\mathbf{F}(x; \theta) = \frac{\gamma(\alpha, \beta \ln x)}{\Gamma(\alpha)}$$

where $\alpha > 0$ and $\beta > 0$.

- Log-logistic $X \sim \mathcal{LL}(\alpha, \beta)$

$$\begin{aligned} \mathbf{F}(x; \theta) &= \frac{1}{1 + (x/\alpha)^{-\beta}} \\ &= \frac{x^\beta}{\alpha^\beta + x^\beta} \end{aligned}$$

where $\alpha > 0$ and $\beta > 0$.

- Log-normal $X \sim \mathcal{LN}\left(\mu, \sigma^2\right)$

$$\mathbf{F}(x; \theta) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

where $x > 0$ and $\sigma > 0$.

- Generalized extreme value $X \sim \mathcal{GEV}(\mu, \sigma, \xi)$

$$\mathbf{F}(x; \theta) = \exp\left\{-\left[1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

where $x > \mu - \sigma/\xi$, $\sigma > 0$ and $\xi > 0$.

- Pareto $X \sim \mathcal{P}a(\alpha, x_-)$

$$\mathbf{F}(x; \theta) = 1 - \left(\frac{x}{x_-}\right)^{-\alpha}$$

where $x \geq x_-$, $\alpha > 1$ and $x_- > 0$.

The vector of parameters θ can be estimated by the method of maximum likelihood (ML) or the generalized method of moments (GMM). In Chapter 14, we show that the log-likelihood function associated to the sample is:

$$\ell(\theta) = \sum_{i=1}^{T} \ln f(x_i; \theta) \tag{5.7}$$

where $f(x; \theta)$ is the density function. In the case of the GMM, the empirical moments are:

$$\left\{ \begin{array}{l} h_{i,1}(\mu, \sigma) = x_i - \mathbb{E}[X] \\ h_{i,2}(\mu, \sigma) = (x_i - \mathbb{E}[X])^2 - \operatorname{var}(X) \end{array} \right. \tag{5.8}$$

TABLE 5.4: Density function, mean and variance of parametric probability distribution

Distribution	$f(x;\theta)$	$\mathbb{E}[X]$	$\operatorname{var}(X)$
$\mathcal{G}(\alpha,\beta)$	$\dfrac{\beta^{\alpha}x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}$	$\dfrac{\alpha}{\beta}$	$\dfrac{\alpha}{\beta^{2}}$
$\mathcal{LG}(\alpha,\beta)$	$\dfrac{\beta^{\alpha}(\ln x)^{\alpha-1}}{x^{\beta+1}\Gamma(\alpha)}$	$\left(\dfrac{\beta}{\beta-1}\right)^{\alpha}$ if $\beta>1$	$\left(\dfrac{\beta}{\beta-2}\right)^{\alpha}-\left(\dfrac{\beta}{\beta-1}\right)^{2\alpha}$ if $\beta>2$
$\mathcal{LL}(\alpha,\beta)$	$\dfrac{\beta(x/\alpha)^{\beta-1}}{\alpha\left(1+(x/\alpha)^{\beta}\right)^{2}}$	$\dfrac{\alpha\pi}{\beta\sin(\pi/\beta)}$ if $\beta>1$	$\alpha^{2}\left(\dfrac{2\pi}{\beta\sin(2\pi/\beta)}-\dfrac{\pi^{2}}{\beta^{2}\sin^{2}(\pi/\beta)}\right)$ if $\beta>2$
$\mathcal{LN}(\mu,\sigma^{2})$	$\dfrac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\dfrac{1}{2}\left(\dfrac{x-\mu}{\sigma}\right)^{2}\right)$	$\exp\left(\mu+\frac{1}{2}\sigma^{2}\right)$	$\exp\left(2\mu+\sigma^{2}\right)\left(\exp\left(\sigma^{2}\right)-1\right)$
$\mathcal{GEV}(\mu,\sigma,\xi)$	$\dfrac{1}{\sigma}\left[1+\xi\left(\dfrac{x-\mu}{\sigma}\right)\right]^{-(1+1/\xi)}$ $\exp\left\{-\left[1+\xi\left(\dfrac{x-\mu}{\sigma}\right)\right]^{-1/\xi}\right\}$	$\mu+\dfrac{\sigma}{\xi}\left(\Gamma(1-\xi)-1\right)$ if $\xi<1$	$\dfrac{\sigma^{2}}{\xi^{2}}\left(\Gamma(1-2\xi)-\Gamma^{2}(1-\xi)\right)$ if $\xi<\frac{1}{2}$
$\mathcal{P}a(\alpha,x_{-})$	$\dfrac{\alpha x_{-}^{\alpha}}{x^{\alpha+1}}$	$\dfrac{\alpha x_{-}}{\alpha-1}$ if $\alpha>1$	$\dfrac{\alpha x_{-}^{2}}{(\alpha-1)^{2}(\alpha-2)}$ if $\alpha>2$

In Table 5.4, we report the density function $f(x;\theta)$, the mean $\mathbb{E}[X]$ and the variance $\operatorname{var}(X)$ when X follows one of the probability distributions described previously. For instance, if we consider that $X \sim \mathcal{LN}(\mu, \sigma^2)$, it follows that the log-likelihood function is:

$$\ell(\theta) = -\sum_{i=1}^{T} \ln x_i - \frac{T}{2}\ln \sigma^2 - \frac{T}{2}\ln 2\pi - \frac{1}{2}\sum_{i=1}^{T}\left(\frac{\ln x_i - \mu}{\sigma}\right)^2$$

whereas the empirical moments are:

$$\left\{\begin{array}{l} h_{i,1}(\theta) = x_i - e^{\mu + \frac{1}{2}\sigma^2} \\ h_{i,2}(\theta) = \left(x_i - e^{\mu+\frac{1}{2}\sigma^2}\right)^2 - e^{2\mu+\sigma^2}\left(e^{\sigma^2} - 1\right) \end{array}\right.$$

In the case of the log-normal distribution, the vector θ is composed of two parameters μ and σ, implying that two moments are sufficient to define the GMM estimator. This is also the case of other probability distributions, except the GEV distribution that requires specification of three empirical moments[6].

Example 42 *We assume that the individual losses take the following values expressed in thousands of dollars:* 10.1*,* 12.5*,* 14*,* 25*,* 317.3*,* 353*,* 1 200*,* 1 254*,* 52 000 *and* 251 000*.*

Using the method of maximum likelihood, we find that $\hat{\alpha}_{\mathrm{ML}}$ and $\hat{\beta}_{\mathrm{ML}}$ are equal to 15.70 and 1.22 for the log-gamma distribution and 293 721 and 0.51 for the log-logistic distribution. In the case of the log-normal distribution[7], we obtain $\hat{\mu}_{\mathrm{ML}} = 12.89$ and $\hat{\sigma}_{\mathrm{ML}} = 3.35$.

The previous analysis assumes that the sample of operational losses for estimating θ represents a comprehensive and homogenous information of the underlying probability distribution $\mathbf{F}$. In practice, loss data are plagued by various sources of bias. The first issue lies in the data generating processes which underlies the way data have been collected. In almost all cases, loss data have gone through a truncation process by which data are recorded only when their amounts are higher than some thresholds. In practice, banks' internal thresholds are set in order to balance two conflicting wishes: collecting as many data as possible while reducing costs by collecting only significant losses. These thresholds, which are defined by the global risk management policy of the bank, must satisfy some criteria:

---

[6]We can use the moment of order 3, which corresponds to:

$$\mathbb{E}\left[(X - \mathbb{E}[X])^3\right] = \frac{\sigma^3}{\xi^3}\left(\Gamma(1-3\xi) - 3\Gamma(1-2\xi)\Gamma(1-\xi) + 2\Gamma^3(1-\xi)\right)$$

[7]If we consider the generalized method of moments, the estimates are $\hat{\mu}_{\mathrm{GMM}} = 16.26$ and $\hat{\sigma}_{\mathrm{GMM}} = 1.40$.

> "*A bank must have an appropriate de minimis gross loss threshold for internal loss data collection, for example €10 000. The appropriate threshold may vary somewhat between banks, and within a bank across business lines and/or event types. However, particular thresholds should be broadly consistent with those used by peer banks*" (BCBS, 2006, page 153).

The second issue concerns the use of relevant external data, especially when there is reason to believe that the bank is exposed to infrequent, yet potentially severe losses. Typical examples are rogue trading or cyber attacks. If the bank has not yet experienced a large amount of loss due to these events in the past, this does not mean that it will never experience such problems in the future. Therefore, internal loss data must be supplemented by external data from public and/or pooled industry databases. Unfortunately, incorporating external data is rather dangerous and requires careful methodology to avoid the pitfalls regarding data heterogeneity, scaling problems and lack of comparability between too heterogeneous data. Unfortunately, there is no satisfactory solution to deal with these scaling issues, implying that banks use external data by taking into account only reporting biases and a fixed and known threshold[8].

The previous issues imply that loss data for operational risk can not be reduced to the sample of individual losses, but also requires to specify the threshold H_i for each individual loss x_i. The form of operational loss data is then $\{(x_i, H_i), i = 1, \ldots, T\}$, where x_i is the observed value of X knowing that X is larger than the threshold H_i. Reporting thresholds affect severity estimation in the sense that the sample severity distribution (i.e. the severity distribution of reported losses) is different from the "true" one (i.e. the severity distribution one would obtain if all losses were reported). Unfortunately, the true distribution is the most relevant for calculating capital charge. As a consequence, linking the sample distribution to the true one is a necessary task. From a mathematical point of view, the true distribution is the probability distribution of X whereas the sample distribution is the probability distribution of $X \mid X \geq H_i$. We deduce that the sample distribution for a given threshold H is the conditional probability distribution defined as follows:

$$\begin{aligned}
\mathbf{F}^{\star}(x;\theta \mid H) &= \Pr\{X \leq x \mid X \geq H\} \\
&= \frac{\Pr\{X \leq x, X \geq H\}}{\Pr\{X \geq H\}} \\
&= \frac{\Pr\{X \leq x\} - \Pr\{X \leq \min(x, H)\}}{\Pr\{X \geq H\}} \\
&= \mathbb{1}\{x \geq H\} \frac{\mathbf{F}(x;\theta) - \mathbf{F}(H;\theta)}{1 - \mathbf{F}(H;\theta)} \qquad (5.9)
\end{aligned}$$

[8] See Baud *et al.* (2002, 2003) for more advanced techniques based on unknown and stochastic thresholds.

It follows that the density function is:

$$f^\star(x;\theta \mid H) = \mathbb{1}\{x \geq H\} \frac{f(x;\theta)}{1 - \mathbf{F}(H;\theta)}$$

To estimate the vector of parameters θ, we continue to use the method of maximum likelihood or the generalized method of moments by considering the correction due to the truncation of data. For the ML estimator, we have then:

$$\begin{aligned} \ell(\theta) &= \sum_{i=1}^{T} \ln f^\star(x_i;\theta \mid H_i) \\ &= \sum_{i=1}^{T} \ln f(x_i;\theta) + \sum_{i=1}^{T} \ln \mathbb{1}\{x_i \geq H_i\} - \sum_{i=1}^{T} \ln(1 - \mathbf{F}(H_i;\theta)) \end{aligned} \tag{5.10}$$

where H_i is the threshold associated to the i^{th} observation. The correction term $-\sum_{i=1}^{T} \ln(1 - \mathbf{F}(H_i;\theta))$ shows that maximizing a conventional log-likelihood function which ignores data truncation is totally misleading. We also notice that this term vanishes when H_i is equal to zero[9]. For the GMM estimator, the empirical moments become:

$$\begin{cases} h_{i,1}(\theta) = x_i - \mathbb{E}[X \mid X \geq H_i] \\ h_{i,2}(\theta) = (x_i - \mathbb{E}[X \mid X \geq H_i])^2 - \operatorname{var}(X \mid X \geq H_i) \end{cases} \tag{5.11}$$

There is no reason that the conditional moment $\mathbb{E}[X^m \mid X \geq H_i]$ is equal to the unconditional moment $\mathbb{E}[X^m]$. Therefore, the conventional GMM estimator is biased and this is why we have to apply the threshold correction.

If we consider again the log-normal distribution, the expression of the log-likelihood function (5.10) is[10]:

$$\begin{aligned} \ell(\theta) = & -\frac{T}{2}\ln 2\pi - \frac{T}{2}\ln\sigma^2 - \sum_{i=1}^{T}\ln x_i - \frac{1}{2}\sum_{i=1}^{T}\left(\frac{\ln x_i - \mu}{\sigma}\right)^2 - \\ & \sum_{i=1}^{T}\ln\left(1 - \Phi\left(\frac{\ln H_i - \mu}{\sigma}\right)\right) \end{aligned}$$

Let us now calculate the conditional moment $\mu'_m(X) = \mathbb{E}[X^m \mid X \geq H]$. By

[9]Indeed, we have $\mathbf{F}(0;\theta) = 0$ and $\ln(1 - \mathbf{F}(0;\theta)) = 0$.

[10]By construction, the observed value x_i is larger than the threshold H_i, meaning that $\ln \mathbb{1}\{x_i \geq H_i\}$ is equal to 0.

using the notation $\Phi^c(x) = 1 - \Phi((x-\mu)/\sigma)$, we have:

$$\begin{aligned}
\mu'_m(X) =& \frac{1}{\Phi^c(\ln H)} \int_H^\infty \frac{x^m}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right) \mathrm{d}x \\
=& \frac{1}{\Phi^c(\ln H)} \int_{\ln H}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2 + my\right) \mathrm{d}y \\
=& \frac{\exp\left(m\mu + m^2\sigma^2/2\right)}{\Phi^c(\ln H)} \int_{\ln H}^\infty \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{y - \left(\mu + m\sigma^2\right)}{\sigma}\right)^2\right) \mathrm{d}y \\
=& \frac{\Phi^c\left(\ln H - m\sigma^2\right)}{\Phi^c(\ln H)} \exp\left(m\mu + m^2\sigma^2/2\right)
\end{aligned}$$

We deduce that:

$$a(\theta, H) = \mathbb{E}[X \mid X \geq H] = \frac{1 - \Phi\left(\frac{\ln H - \mu - \sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\ln H - \mu}{\sigma}\right)} e^{\mu + \frac{1}{2}\sigma^2}$$

and:

$$b(\theta, H) = \mathbb{E}\left[X^2 \mid X \geq H\right] = \frac{1 - \Phi\left(\frac{\ln H - \mu - 2\sigma^2}{\sigma}\right)}{1 - \Phi\left(\frac{\ln H - \mu}{\sigma}\right)} e^{2\mu + 2\sigma^2}$$

We finally obtain:

$$\left\{\begin{array}{l}
h_{i,1}(\theta) = x_i - a(\theta, H_i) \\
h_{i,2}(\theta) = x_i^2 - 2x_i a(\theta, H_i) + 2a^2(\theta, H_i) - b(\theta, H_i)
\end{array}\right.$$

In order to illustrate the impact of the truncation, we report in Figure 5.2 the cumulative distribution function and the probability density function of $X \mid X > H$ when X follows the log-normal distribution $\mathcal{LN}(8, 5)$. The threshold H is set at \$10 000, meaning that the bank collects operational losses when the amount is larger than this threshold. In the bottom panels of the figure, we indicate the mean and the variance with respect to the threshold H. We notice that data truncation increases the magnitude of the mean and the variance. For instance, when H is set at \$10 000, the conditional mean and variance are multiplied by a factor equal to 3.25 with respect to the unconditional mean and variance.

Example 43 *We consider Example 42 and assume that the losses have been collected using a unique threshold that is equal to* \$5 000*.*

By using the truncation correction, the ML estimates become $\hat{\mu}_{\mathrm{ML}} = 8.00$ and $\hat{\sigma}_{\mathrm{ML}} = 5.71$. In Figure 5.3, we compare the log-normal cumulative distribution function without and with the truncation correction. We notice that the results are very different.

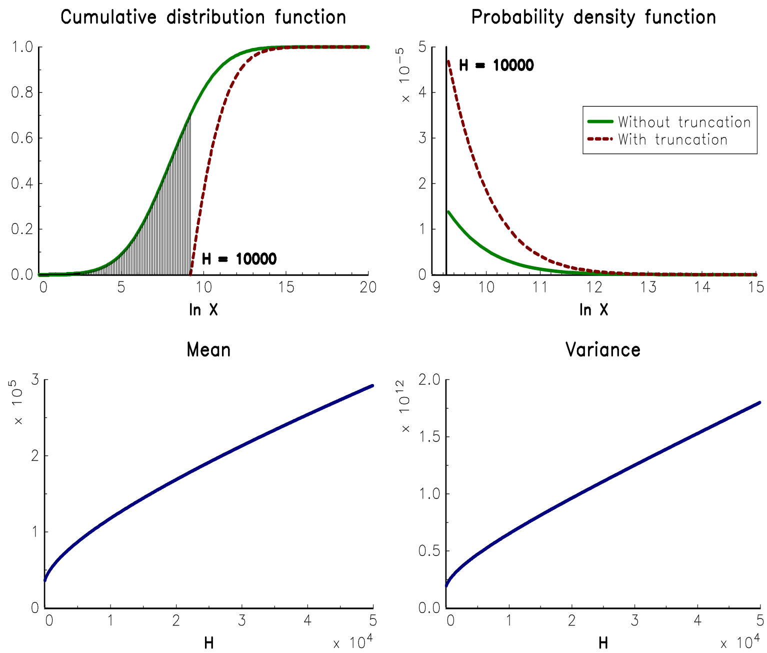

FIGURE 5.2: Impact of the threshold H on the severity distribution

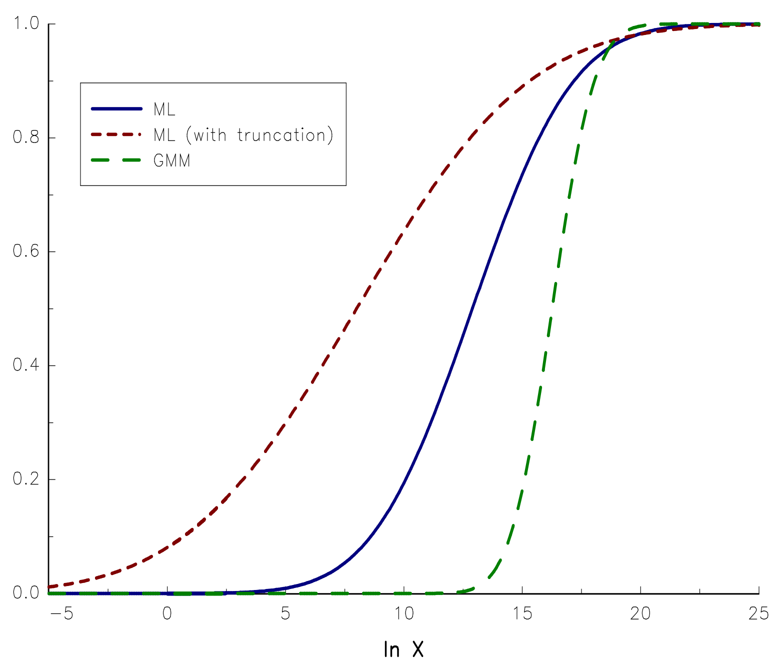

FIGURE 5.3: Comparison of the estimated severity distributions

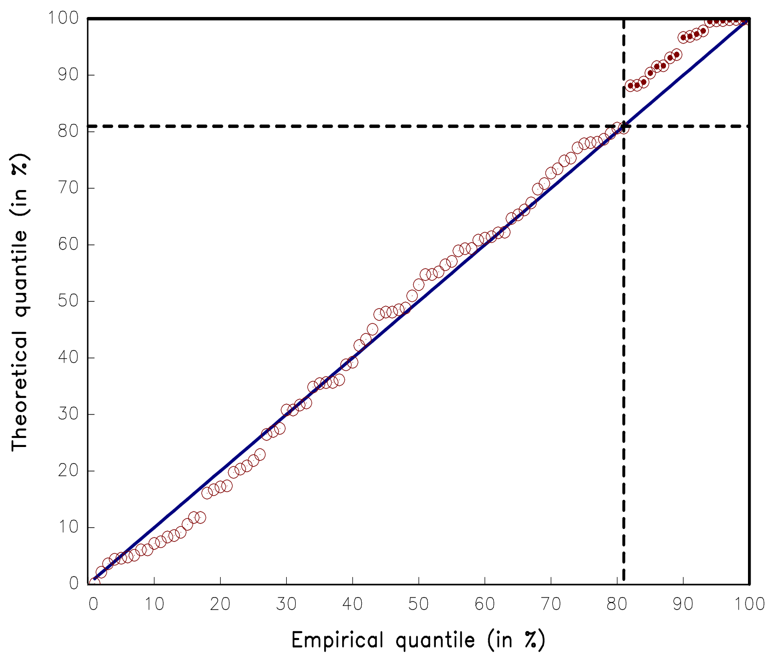

FIGURE 5.4: An example of QQ plot where extreme events are underestimated

The previous example shows that estimating the parameters of the probability distribution is not sufficient to define the severity distribution. Indeed, ML and GMM give two different log-normal probability distributions. The issue is to decide which is the best parametrization. In a similar way, the choice between the several probability families (log-normal, log-gamma, GEV, Pareto, etc.) is an open question. This is why fitting the severity distribution does not reduce to estimate the parameters of a given probability distribution. It must be completed by a second step that consists in selecting the best estimated probability distribution. However, traditional goodness-of-fit tests (Kolmogorov-Smirnov, Anderson-Darling, etc.) are not useful, because they concern the entire probability distribution. In operational risk, extreme events are more relevant. This explains why QQ plots or order statistics are generally used to assess the fitting of the upper tail. A QQ plot represents the quantiles of the empirical distribution against those of the theoretical model. If the statistical model describes perfectly the data, we obtain the diagonal line $y = x$. In Figure 5.4, we show an example of QQ plot. We notice that the theoretical quantiles obtained from the statistical model are in line with those calculated with the empirical data when the quantile is lower than 80%. Otherwise, the theoretical quantiles are above the empirical quantiles, meaning that extreme events are underestimated by the statistical model. We deduce that the body of the distribution is well estimated, but not the upper tail of the distribution. However, medium losses are less important than high losses in operational risk.

5.3.2.2 Estimation of the loss frequency distribution

In order to model the frequency distribution, we have to specify the counting process $N(t)$, which defines the number of losses occurring during the time period $[0, t]$. The number of losses for the time period $[t_1, t_2]$ is then equal to:

$$N(t_1; t_2) = N(t_2) - N(t_1)$$

We generally made the following statements about the stochastic process $N(t)$:

- The distribution of the number of losses $N(t; t+h)$ for each $h > 0$ is independent of t. Moreover, $N(t; t+h)$ is stationary and depends only on the time interval h.
- The random variables $N(t_1; t_2)$ and $N(t_3; t_4)$ are independent if the time intervals $[t_1, t_2]$ and $[t_3, t_4]$ are disjoint.
- No more than one loss may occur at time t.

These simple assumptions define a Poisson process, which satisfies the following properties:

1. There exists a scalar $\lambda > 0$ such that the distribution of $N(t)$ has a Poisson distribution with parameter λt.
2. The duration between two successive losses is *i.i.d.* and follows the exponential distribution $\mathcal{E}(\lambda)$.

Let $p(n)$ be the probability to have n losses. We deduce that:

$$\begin{aligned} p(n) &= \Pr\{N(t) = n\} \\ &= \frac{e^{-\lambda t}(\lambda t)^n}{n!} \end{aligned} \tag{5.12}$$

Without loss of generality, we can fix $t = 1$ because it corresponds to the required one-year time period for calculating the capital charge. In this case, $N(1)$ is simply a Poisson distribution with parameter λ. This probability distribution has a useful property for time aggregation. Indeed, the sum of two independent Poisson variables N_1 and N_2 with parameters λ_1 and λ_2 is also a Poisson variable with parameter $\lambda_1 + \lambda_2$. This property is a direct result of the definition of the Poisson process. In particular, we have:

$$\sum_{k=1}^{K} N\left(\frac{k-1}{K}; \frac{k}{K}\right) = N(1)$$

where $N\left((k-1)/K; k/K\right) \sim \mathcal{P}(\lambda/K)$. This means that we can estimate the frequency distribution at a quarterly or monthly period and convert it to an annual period by simply multiply the quarterly or monthly intensity parameter by 4 or 12.

The estimation of the annual intensity λ can be done using the method of maximum likelihood. In this case, $\hat{\lambda}$ is the mean of the annual number of losses:

$$\hat{\lambda} = \frac{1}{n_y} \sum_{y=1}^{n_y} N_y \tag{5.13}$$

where N_y is the number of losses occurring at year y and n_y is the number of observations. One of the key features of the Poisson distribution is that the variance equals the mean:

$$\lambda = \mathbb{E}\left[N(1)\right] = \operatorname{var}\left(N(1)\right) \tag{5.14}$$

We can use this property to estimate λ by the method of moments. If we consider the first moment, we obtain the ML estimator, whereas we have with the second moment:

$$\hat{\lambda} = \frac{1}{n_y} \sum_{y=1}^{n_y} \left(N_y - \bar{N}\right)^2$$

where $\bar{N}$ is the average number of losses.

Example 44 *We assume that the annual number of losses from 2006 to 2015 is the following:* 57, 62, 45, 24, 82, 36, 98, 75, 76 *and* 45.

The mean is equal to 60 whereas the variance is equal to 474.40. In Figure 5.5, we show the probability function of the Poisson distribution with parameter 60. We notice that the parameter λ is not enough large to reproduce the variance and the range of the sample. However, using the moment estimator based on the variance is completely unrealistic.

When the variance exceeds the mean, we use the negative binomial distribution $\mathcal{NB}(r,p)$, which is defined as follows:

$$\begin{aligned} p(n) &= \binom{r+n-1}{n} (1-p)^r p^n \\ &= \frac{\Gamma(r+n)}{n!\,\Gamma(r)} (1-p)^r p^n \end{aligned}$$

where $r > 0$ and $p \in [0,1]$. The negative binomial distribution can be viewed as the probability distribution of the number of successes in a sequence of *i.i.d.* Bernoulli random variables $\mathcal{B}(p)$ until we get r failures. The negative binomial distribution is then a generalization of the geometric distribution. Concerning the two first moments, we have:

$$\mathbb{E}\left[\mathcal{NB}(r,p)\right] = \frac{pr}{1-p}$$

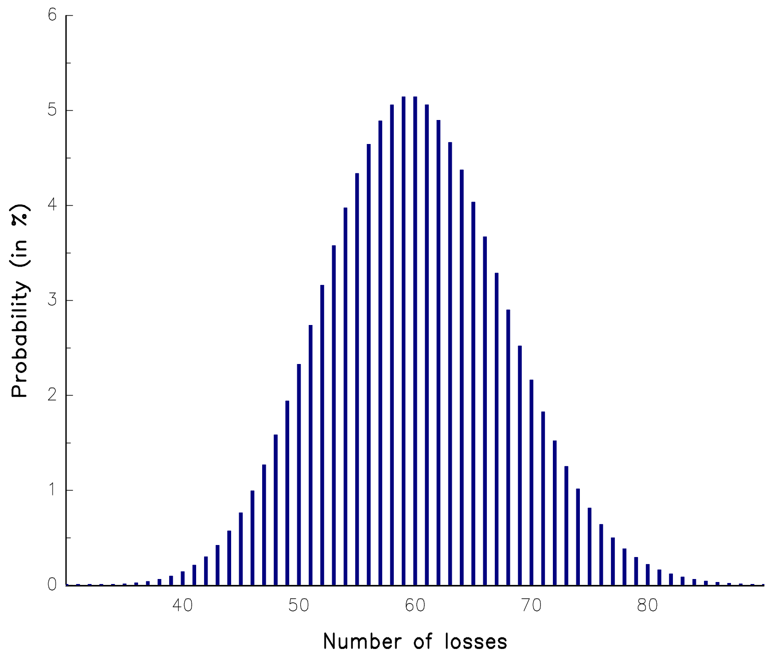

FIGURE 5.5: PMF of the Poisson distribution $\mathcal{P}(60)$

and:

$$\text{var}\left(\mathcal{NB}(r,p)\right) = \frac{pr}{(1-p)^2}$$

We verify that:

$$\begin{aligned}\text{var}\left(\mathcal{NB}(r,p)\right) &= \frac{1}{1-p}\mathbb{E}\left[\mathcal{NB}(r,p)\right] \\ &> \mathbb{E}\left[\mathcal{NB}(r,p)\right]\end{aligned}$$

Remark 45 *The negative binomial distribution corresponds to a Poisson process where the intensity parameter is random and follows a gamma distribution*[11]*:*

$$\mathcal{NB}(r,p) \sim \mathcal{P}(\Lambda) \quad \textit{and} \quad \Lambda \sim \mathcal{G}(\alpha,\beta)$$

where $\alpha = r$ *and* $\beta = (1-p)/p$.

We consider again Example 44 and assume that the number of losses is described by the negative binomial distribution. Using the method of moments,

[11] See Exercise 5.4.6 in page 309.

we obtain the following estimates:

$$\begin{aligned}\hat{r} &= \frac{m^2}{v-m} \\ &= \frac{60^2}{474.40-60} \\ &= 8.6873\end{aligned}$$

and

$$\begin{aligned}\hat{p} &= \frac{v-m}{v} \\ &= \frac{474.40-60}{474.40} \\ &= 0.8735\end{aligned}$$

where m is the mean and v is the variance of the sample. Using these estimates as the starting values of the numerical optimization procedure, the ML estimates are $\hat{r} = 7.7788$ and $\hat{p} = 0.8852$. We report the corresponding probability mass function $p(n)$ in Figure 5.6. We notice that this distribution better describes the sample that the Poisson distribution, because it has a larger support. In fact, we show in Figure 5.6 the probability density function of λ for the two estimated counting processes. For the Poisson distribution, λ is constant and equal to 60, whereas λ has a gamma distribution $\mathcal{G}(7.7788, 0.1296)$ in the case of the negative binomial distribution. The variance of the gamma distribution explains the larger variance of the negative binomial distribution with respect to the Poisson distribution, while we notice that the two distributions have the same mean.

As in the case of the severity distribution, data truncation and reporting bias have an impact of the frequency distribution (Frachot *et al.*, 2006). For instance, if one bank's reporting threshold H is set at a high level, then the average number of reported losses will be low. It does not imply that the bank is allowed to have a lower capital charge than another bank that uses a lower threshold and is otherwise identical to the first one. It simply means that the average number of losses must be corrected for reporting bias as well. It appears that the calibration of the frequency distribution comes as a second step (after having calibrated the severity distribution) because the aforementioned correction needs an estimate of the exceedance probability $\Pr\{X > H\}$ for its calculation. This is rather straightforward: the difference (more precisely the ratio) between the number of reported events and the "true" number of events (which would be obtained if all losses were reported, i.e. with a zero-threshold) corresponds exactly to the probability of one loss being higher than the threshold. This probability is a direct by-product of the severity distribution.

Let $N_H(t)$ be the number of events that are larger than the threshold H.

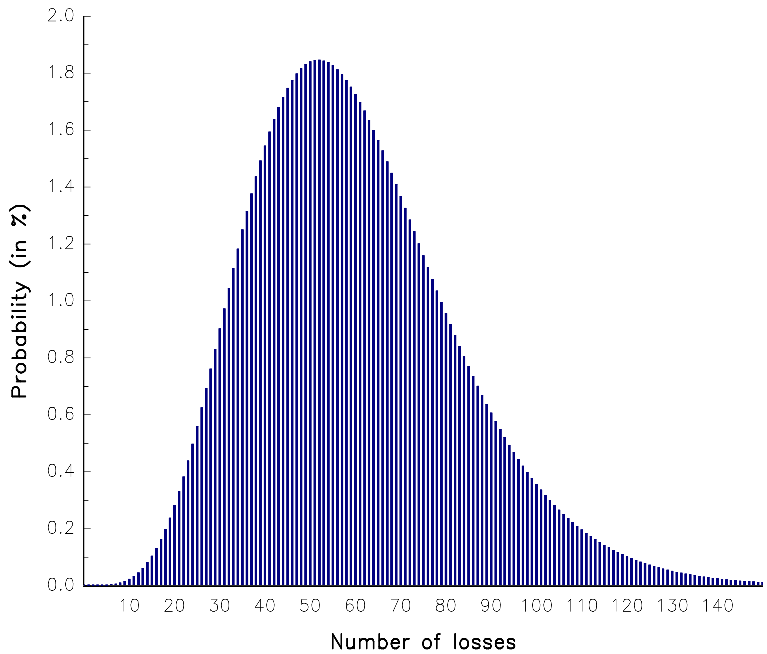

FIGURE 5.6: PMF of the negative binomial distribution

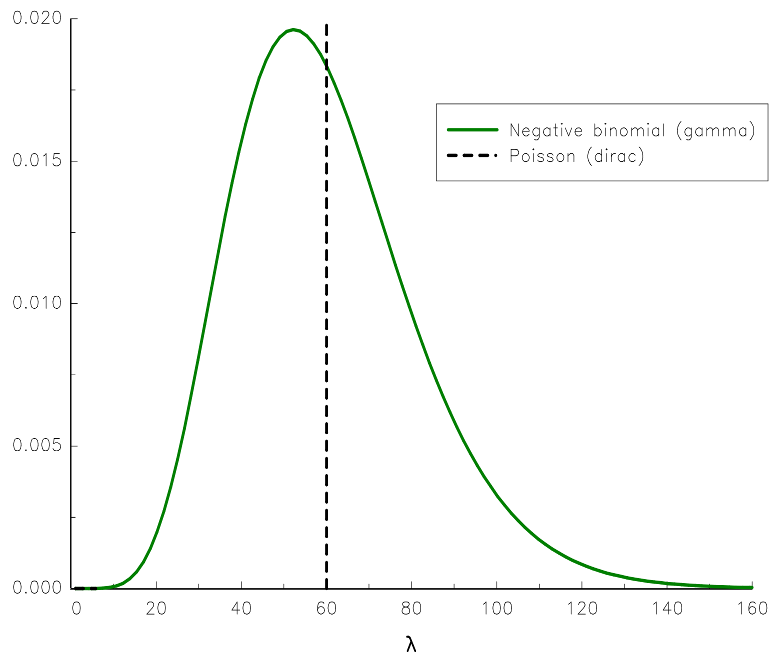

FIGURE 5.7: PDF of the parameter λ

By definition, $N_H(t)$ is the counting process of exceedance events:

$$N_H(t) = \sum_{i=1}^{N(t)} \mathbb{1}\{X_i > H\}$$

It follows that:

$$\begin{aligned}
\mathbb{E}[N_H(t)] &= \mathbb{E}\left[\sum_{i=1}^{N(t)} \mathbb{1}\{X_i > H\}\right] \\
&= \mathbb{E}\left[\left.\sum_{i=1}^{n} \mathbb{1}\{X_i > H\}\right| N(t) = n\right] \\
&= \mathbb{E}[N(t)] \times \mathbb{E}[\mathbb{1}\{X_i > H\}]
\end{aligned}$$

because the random variables $X_1, \ldots, X_n$ are *i.i.d.* and independent from the random number of events $N(t)$. We deduce that:

$$\begin{aligned}
\mathbb{E}[N_H(t)] &= \mathbb{E}[N(t)] \times \Pr\{X_i > H\} \\
&= \mathbb{E}[N(t)] \times (1 - \mathbf{F}(H;\theta))
\end{aligned} \tag{5.15}$$

This latter equation provides information about the transformation of the counting process $N(t)$ into the exceedance process. However, it only concerns the mean and not the distribution itself. One interesting feature of data truncation is when the distribution of the threshold exceedance process belongs to the same distribution class of the counting process. It is the case of the Poisson distribution[12]:

$$\mathbf{P}_H(\lambda) = \mathbf{P}(\lambda_H)$$

Using Equation (5.15), it follows that the Poisson parameter λ_H of the exceedance process is simply the product of the Poisson parameter λ by the exceedance probability $\Pr\{X > H\}$:

$$\lambda_H = \lambda \times (1 - \mathbf{F}(H;\theta))$$

We deduce that the estimator $\hat{\lambda}$ has the following expression:

$$\hat{\lambda} = \frac{\hat{\lambda}_H}{1 - \mathbf{F}\left(H;\hat{\theta}\right)}$$

where $\hat{\lambda}_H$ is the average number of losses that are collected above the threshold H and $\mathbf{F}\left(x;\hat{\theta}\right)$ is the parametric estimate of the severity distribution.

[12]See Exercise 5.4.7 for the proof of this result.

Example 45 *We consider that the bank has collected the loss data from 2006 to 2015 with a threshold of* \$20 000*. For a given event type, the calibrated severity distribution corresponds to a log-normal distribution with parameters* $\hat{\mu} = 7.3$ *and* $\hat{\sigma} = 2.1$*, whereas the annual number of losses is the following:* 23*,* 13*,* 50*,* 12*,* 25*,* 36*,* 48*,* 27*,* 18 *and* 35*.*

Using the Poisson distribution, we obtain $\hat{\lambda}_H = 28.70$. The probability that the loss exceeds the threshold H is equal to:

$$\Pr\{X > 20\,000\} = 1 - \Phi\left(\frac{\ln(20\,000) - 7.3}{2.1}\right) = 10.75\%$$

This means that only 10.75% of losses can be observed when we apply a threshold of \$20 000. We then deduce that the estimate of the Poisson parameter is equal to:

$$\hat{\lambda} = \frac{28.70}{10.75\%} = 266.90$$

On average, there are in fact about 270 loss events per year.

We could discuss whether the previous result remains valid in the case of the negative binomial distribution. If it is the case, then we have:

$$\mathbf{P}_H(r, p) = \mathbf{P}(r_H, p_H)$$

Using Equation (5.15), we deduce that:

$$\frac{p_H r_H}{1 - p_H} = \frac{pr}{1 - p} \times (1 - \mathbf{F}(H; \theta))$$

If we assume that r_H is equal to r, we obtain:

$$p_H = \frac{p(1 - \mathbf{F}(H; \theta))}{1 - p\mathbf{F}(H; \theta)}$$

We verify the following inequality $p \leq p_H \leq 1$. However, this solution is not completely satisfactory as shown in Exercise 5.4.7.

5.3.3 Calculating the capital charge

Once the frequency and severity distributions are calibrated, the computation of the capital charge is straightforward. For that, we can use the Monte Carlo method or different analytical methods. The Monte Carlo method is much more used, because it is more flexible and give better results in the case of low frequency/high severity events. Analytical approaches, which are very popular in insurance, can be used for high frequency/low severity events. One remaining challenge, however, is aggregating the capital charge of the different cells of the mapping matrix. By construction, the loss distribution approach assumes that aggregate losses are independent. Nevertheless, regulation are forcing banks to take into account positive correlation between risk events. The solution is then to consider copula functions.

5.3.3.1 Monte Carlo approach

We remind that the one-year compound loss of a given cell is defined as follows:

$$S = \sum_{i=1}^{N(1)} X_i$$

where $X_i \sim \mathbf{F}$ and $N(1) \sim \mathbf{P}$. The capital-at-risk is then the 99% quantile of the compound loss distribution. To estimate the capital charge by Monte Carlo, we first simulate the annual number of losses from the frequency distribution and then simulate individual losses in order to calculate the compound loss. Finally, the quantile is estimated by order statistics. The algorithm is described below.

Algorithm 1 Compute the capital-at-risk for an operational risk cell

```
Initialize the number of simulations n_S
for j = 1 : n_S do
   Simulate an annual number n of losses from the frequency distribution P
   S_j ← 0
   for i = 1 : n do
      Simulate a loss X_i from the severity distribution F
      S_j = S_j + X_i
   end for
end for
Calculate the order statistics S_{1:n_S}, ..., S_{n_S:n_S}
Deduce the capital-at-risk CaR = S_{α n_S : n_S} with α = 99.9%
return CaR
```

Let us illustrate this algorithm when $N(1) \sim \mathcal{P}(4)$ and $X_i \sim \mathcal{LN}(8,4)$. Using a linear congruential method, the simulated values of $N(1)$ are 3, 4, 1, 2, 3, etc. while the simulated values of X_i are 3388.6, 259.8, 13328.3, 39.7, 1220.8, 1486.4, 15197.1, 3205.3, 5070.4, 84704.1, 64.9, 1237.5, 187073.6, 4757.8, 50.3, 2805.7, etc. For the first simulation, we have three losses and we obtain:

$$S_1 = 3388.6 + 259.8 + 13328.3 = \$16\,976.7$$

For the second simulation, the number of losses is equal to four and the compound loss is equal to:

$$S_2 = 39.7 + 1220.8 + 1486.4 + 15197.1 = \$17\,944.0$$

For the third simulation, we obtain $S_3 = \$3\,205.3$, and so on. Using n_S simulations, the value of the capital charge is estimated with the 99.9% empirical quantile based on order statistics. For instance, Figure 5.8 shows the histogram of 2 000 simulated values of the capital-at-risk estimated with one million simulation. The right value is equal to \$3.24 mn. However, we notice that the variance of the estimator is large. Indeed, the range of the MC estimator is between \$3.10 mn and 3.40 mn in our experiments with one million simulation.

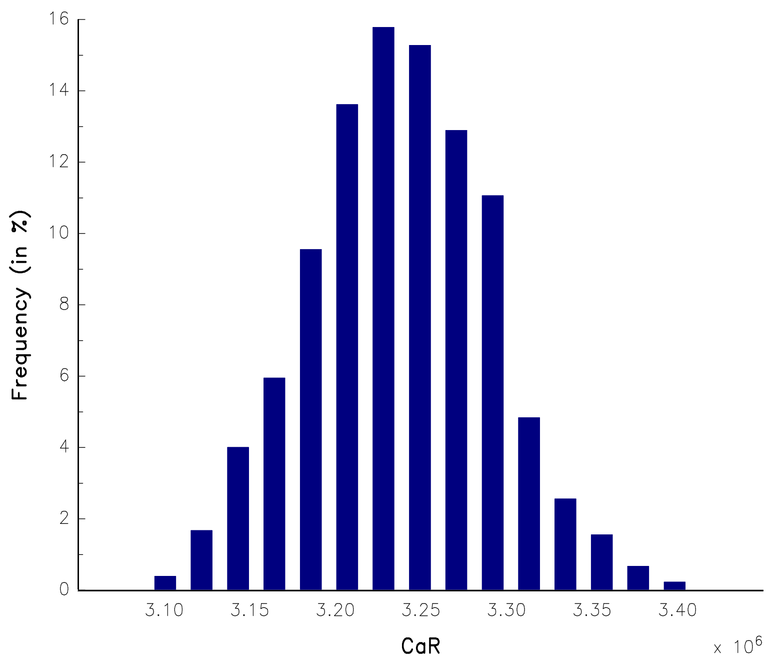

FIGURE 5.8: Histogram of the MC estimator $\widehat{\mathrm{CaR}}$

The estimation of the capital-at-risk with a high accuracy is therefore difficult. The convergence of the Monte Carlo algorithm is low and the estimated quantile can be very far from the true quantile especially when the severity loss distribution is heavy tailed and the confidence level α. That's why it is important to control the accuracy of $\mathbf{G}^{-1}(\alpha)$. This can be done by verifying that the estimated moments are close to the theoretical ones. For the first two central moments, we have[13]:

$$\mathbb{E}[S] = \mathbb{E}[N(1)]\,\mathbb{E}[X_i]$$

and:

$$\operatorname{var}(S) = \mathbb{E}[N(1)]\operatorname{var}(X_i) + \operatorname{var}(N(1))\,\mathbb{E}^2[X_i]$$

To illustrate the convergence problem, we consider the example of the compound Poisson distribution where $N(1) \sim P(10)$ and $X_i \sim \mathcal{LN}\left(5, \sigma^2\right)$. We compute the aggregate loss distribution by the Monte Carlo method for different number n_S of simulations and different runs. To measure the accuracy,

[13]See Exercise 5.4.8 for the derivation of these results and the extension to other moments.

we calculate the ratio between the MC standard deviation $\hat{\sigma}_{n_S}(S)$ and the true value $\sigma(S)$:

$$R(n_s) = \frac{\hat{\sigma}_{n_S}(S)}{\sigma(S)}$$

We notice that the convergence is much more erratic when σ takes a high value (Figure 5.10) than when σ is low (Figure 5.9). When σ takes the value 1, the convergence of the Monte Carlo method is verified with 100 000 simulations. When σ takes the value 2.5, 100 millions of simulations are not sufficient to estimate the second moment, and then the capital-at-risk. Indeed, the occurrence of probability extreme events is generally underestimated. Sometimes, a severe loss is simulated implying a jump in the empirical standard deviation (see Figure 5.10). This is why we need a large number of simulations in order to be confident when estimating the 99.9% capital-at-risk with high severity distributions.

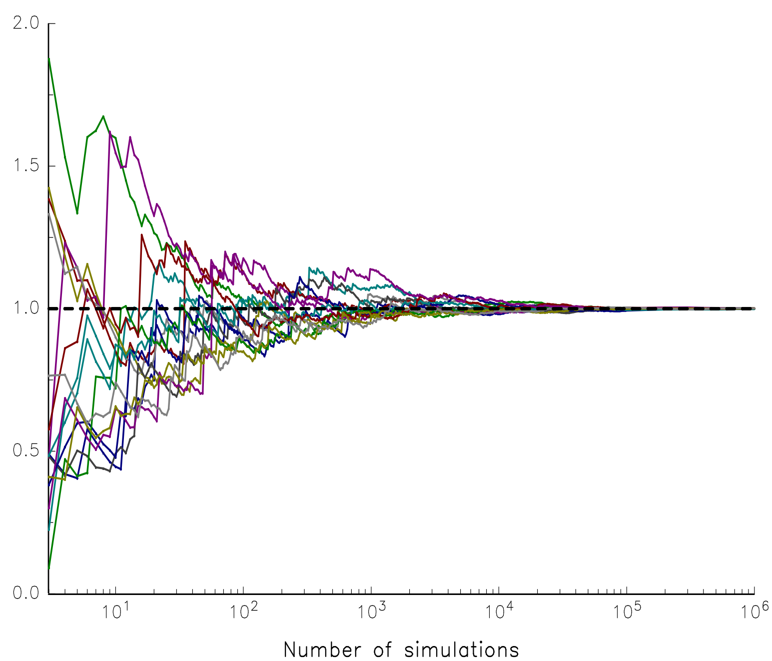

FIGURE 5.9: Convergence of the accuracy ratio $R(n_s)$ when $\sigma = 1$

Remark 46 *With the Monte Carlo approach, we can easily integrate mitigation factors such as insurance coverage. An insurance contract is generally defined by a deductive*[14] *A and the maximum amount B of a loss, which is covered by the insurer. The effective loss $\tilde{X}_i$ suffered by the bank is then the*

[14]It corresponds to the loss amount the bank has to cover by itself.

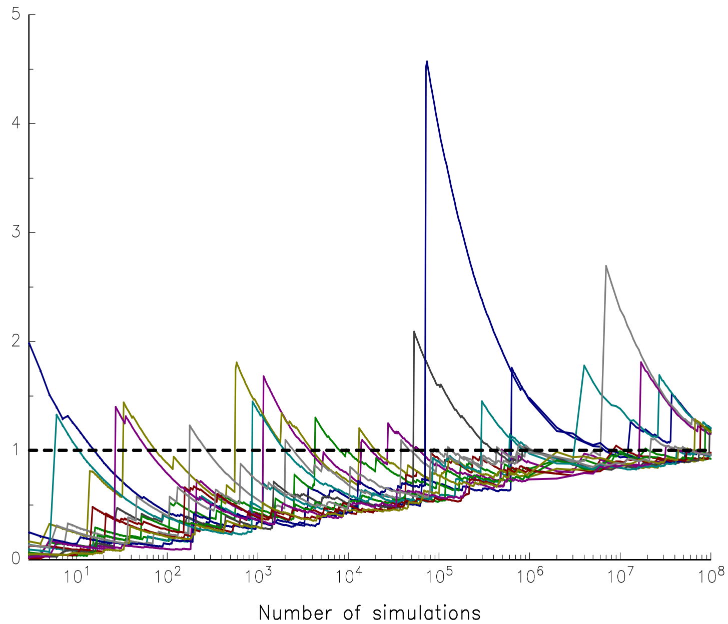

FIGURE 5.10: Convergence of the accuracy ratio $R(n_s)$ when $\sigma = 2.5$

difference between the loss of the event and the amount paid by the insurer:

$$\tilde{X}_i = X_i - \max(\min(X_i, B) - A, 0)$$

The relationship between X_i and $\tilde{X}_i$ is shown in Figure 5.11. In this case, the annual loss of the bank becomes:

$$S = \sum_{i=1}^{N(1)} \tilde{X}_i$$

Taking into account an insurance contract is therefore equivalent to replace X_i by $\tilde{X}_i$ in the Monte Carlo simulations.

5.3.3.2 Analytical approaches

There are three analytical (or semi-analytical) methods to compute the aggregate loss distribution: the solution based on characteristic functions, Panjer recursion and the single loss approximation.

Method of characteristic functions Formally, the characteristic function of the random variable X is defined by:

$$\varphi_X(t) = \mathbb{E}\left[e^{itX}\right]$$

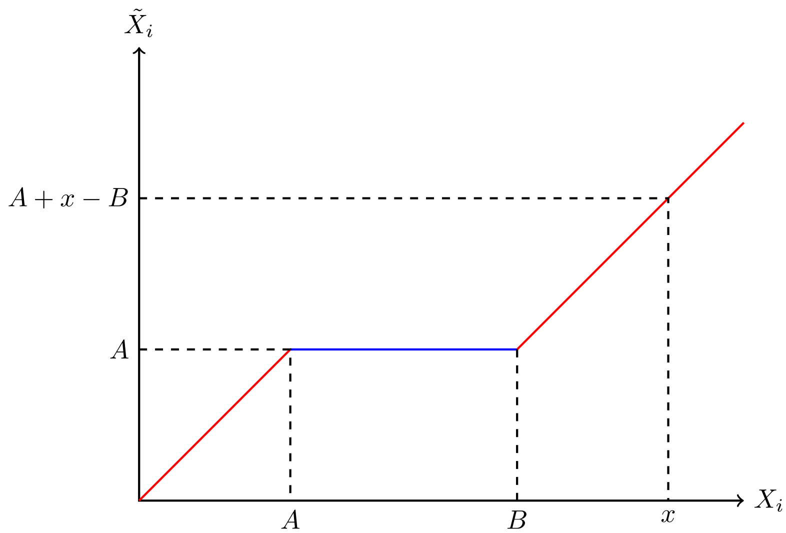

FIGURE 5.11: Impact of an insurance contract in an operational risk loss

If X has a continuous probability distribution $\mathbf{F}$, we obtain:

$$\varphi_X(t) = \int_0^\infty e^{itx}\,\mathrm{d}\mathbf{F}(x)$$

We notice that the characteristic function of the sum of n independent random variables is the product of their characteristic functions:

$$\begin{aligned}
\varphi_{X_1+\ldots+X_n}(t) &= \mathbb{E}\left[e^{it(X_1+X_2+\cdots+X_n)}\right] \\
&= \prod_{i=1}^n \mathbb{E}\left[e^{itX_i}\right] \\
&= \prod_{i=1}^n \varphi_{X_i}(t)
\end{aligned}$$

It comes that the characteristic function of the compound distribution $\mathbf{G}$ is given by:

$$\varphi_S(t) = \sum_{n=0}^\infty p(n)\left(\varphi_X(t)\right)^n = \varphi_{N(1)}\left(\varphi_X(t)\right)$$

where $\varphi_{N(1)}(t)$ is the probability generating function of $N(1)$. For example, if $N(1) \sim \mathcal{P}(\lambda)$, we have:

$$\varphi_{N(1)}(t) = e^{\lambda(t-1)}$$

and:

$$\varphi_S(t) = e^{\lambda(\varphi_X(t)-1)}$$

We finally deduce that S has the probability density function given by the Laplace transform of $\varphi_S(t)$:

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \varphi_S(t) \, \mathrm{d}t$$

Using this expression, we can easily compute the cumulative distribution function and its inverse with the fast fourier transform[15].

Panjer recursive approach PANJER [1981] introduces recursive approaches to compute high-order convolutions. He showed that if the probability mass function of the counting process $N(t)$ satisfies:

$$p(n) = \left(a + \frac{b}{n} \right) p(n-1)$$

where a and b are two scalars, then the following recursion holds:

$$g(x) = p(1) f(x) + \int_0^x \left(a + b\frac{y}{x} \right) f(y) \, g(x-y) \, \mathrm{d}y$$

where $x > 0$. For discrete severity distributions satisfying $f_n = \Pr\{X_i = n\delta\}$ where δ is the monetary unit (e.g. \$10 000), the Panjer recursion becomes:

$$\begin{aligned} g_n &= \Pr\{S = n\delta\} \\ &= \frac{1}{1 - af_0} \sum_{j=1}^{n} \left(a + \frac{bj}{n} \right) f_j g_{n-j} \end{aligned}$$

with:

$$\begin{aligned} g_0 &= \sum_{n=0}^{\infty} p(n) (f_0)^n \\ &= \begin{cases} p(0) e^{bf_0} & \text{if } a = 0 \\ p(0) (1 - af_0)^{-1-b/a} & \text{otherwise} \end{cases} \end{aligned}$$

The capital-at-risk is then equal to:

$$\mathrm{CaR}(\alpha) = n^{\star} \delta$$

where:

$$n^{\star} = \inf \left\{ n : \sum_{j=0}^{n} g_j \geq \alpha \right\}$$

Like the method of characteristic functions, the Panjer recursion is very popular among academics, but produces significant numerical errors in practice

[15] See Exercise 5.4.9 for the derivation of the algorithm and a numerical application.

when applied to operational risk losses. The issue is the support of the compound distribution, whose range can be from zero to several billions[16].

Exercise 46 *We consider the compound Poisson distribution with log-normal losses and different sets of parameters:*

(a) $\lambda = 5$, $\mu = 5$, $\sigma = 1.0$

(b) $\lambda = 5$, $\mu = 5$, $\sigma = 1.5$

(c) $\lambda = 5$, $\mu = 5$, $\sigma = 2.0$

(d) $\lambda = 50$, $\mu = 5$, $\sigma = 2.0$

In order to implement the Panjer recursion, we have to perform a discretization of the severity distribution. Using the central difference approximations, we have:

$$\begin{aligned} f_n &= \Pr\left\{n\delta - \frac{\delta}{2} \leq X_i \leq n\delta + \frac{\delta}{2}\right\} \\ &= \mathbf{F}\left(n\delta + \frac{\delta}{2}\right) - \mathbf{F}\left(n\delta - \frac{\delta}{2}\right) \end{aligned}$$

To initialize the algorithm, we use the convention $f_0 = \mathbf{F}(\delta/2)$. In Figure 5.12, we compare the cumulative distribution function of the aggregate loss obtained with the Panjer recursion and Monte Carlo simulations[17]. We deduce the capital-at-risk for different values of α in Table 5.5. In our case, the Panjer algorithm gives a good approximation, because the support of the distribution is 'bounded '. When the aggregate loss can take very large values, we need a lot of iterations to achieve the convergence[18]. Moreover, we may have underflow in computations because $g_0 \approx 0$.

TABLE 5.5: Comparison of the capital-at-risk calculated with Panjer recursion and Monte Carlo simulations

α	Panjer recursion				Monte Carlo simulations			
	(a)	(b)	(c)	(d)	(a)	(b)	(c)	(d)
90%	2400	4500	11000	91000	2350	4908	11648	93677
95%	2900	6500	19000	120000	2896	6913	19063	123569
99%	4300	13500	52000	231000	4274	13711	51908	233567
99.5%	4900	18000	77000	308000	4958	17844	77754	310172
99.9%	6800	32500	182000	604000	6773	32574	185950	604756

[16]See Exercise 5.4.10 in page 310 for a detailed study of the Panjer recursion and numerical calculations of approximation error.

[17]We use one million of simulations.

[18]In this case, it is not obvious that the Panjer recursion is faster than Monte Carlo simulations.

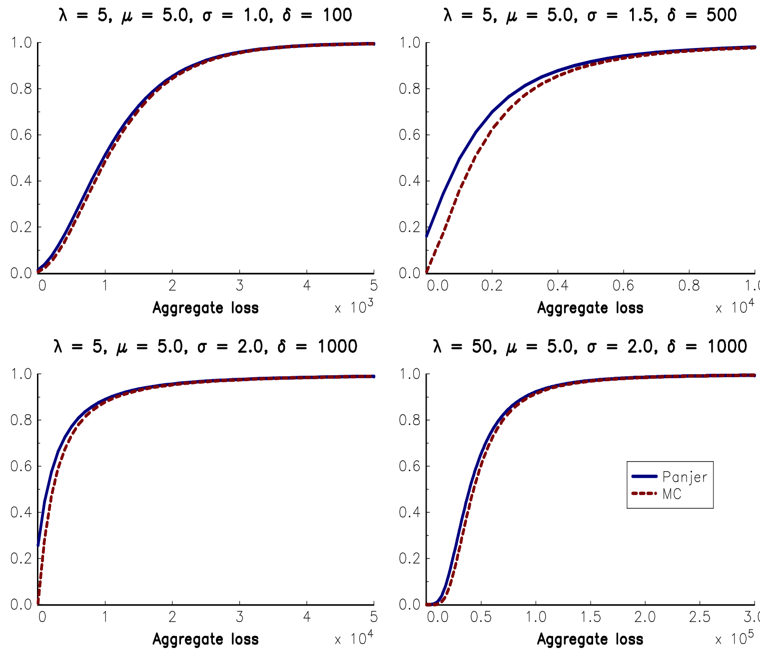

FIGURE 5.12: Comparison between the Panjer and MC compound distributions

Single loss approximation If the severity belongs to the family of subexponential distributions, then Böcker and Klüppelberg (2005) and Böcker and Sprittulla (2006) show that the percentile of the compound distribution can be approximated by the following expression:

$$\mathbf{G}^{-1}(\alpha) \approx (\mathbb{E}[N(1)] - 1)\,\mathbb{E}[X_i] + \mathbf{F}^{-1}\left(1 - \frac{1-\alpha}{\mathbb{E}[N(1)]}\right) \tag{5.16}$$

It follows that the capital-at-risk is the sum of the expected loss and the unexpected loss defined as follows:

$$\begin{aligned} \mathrm{EL} &= \mathbb{E}[N(1)]\,\mathbb{E}[X_i] \\ \mathrm{UL}(\alpha) &= \mathbf{F}^{-1}\left(1 - \frac{1-\alpha}{N(1)}\right) - \mathbb{E}[X_i] \end{aligned}$$

To understand Formula (5.16), we recall subexponential distributions are a special case of heavy-tailed distributions, which satisfy the following property:

$$\lim_{x \to \infty} \frac{\Pr\{X_1 + \cdots + X_n > x\}}{\Pr\{\max(X_1, \ldots, X_n) > x\}} = 1$$

This means that large values of the aggregate loss are dominated by the maximum loss of one event. If we decompose the capital-at-risk as a sum of risk contributions, we obtain:

$$\mathbf{G}^{-1}(\alpha) = \sum_{i=1}^{\mathbb{E}[N(1)]} \mathcal{RC}_i$$

where:

$$\mathcal{RC}_i = \mathbb{E}[X_i] \qquad \text{for } i \neq i^\star$$

and:

$$\mathcal{RC}_{i^\star} = \mathbf{F}^{-1}\left(1 - \frac{1-\alpha}{N(1)}\right)$$

In this model, the capital-at-risk is mainly explained by the single largest loss $i^\star$. If we neglect the small losses, the capital-at-risk at the confidence level α_{CaR} is equivalent to the quantile $\alpha_{\mathrm{Severity}}$ of the loss severity where:

$$\alpha_{\mathrm{Severity}} = 1 - \frac{1-\alpha_{\mathrm{CaR}}}{N(1)}$$

This relationship[19] is shown in Figure and explains why this framework is called the single loss approximation (SLA). For instance, if the annual number of losses is equal to 100 on average, computing the capital-at-risk with a 99.9% confidence level is equivalent to estimate the quantile 99.999% of the loss severity.

The most popular subexponential distributions used in operational risk modeling are the Log-gamma, Log-logistic, Log-normal and Pareto probability distributions (BCBS, 2014f). For instance, if $N(1) \sim \mathcal{P}(\lambda)$ and $X_i \sim \mathcal{LN}\left(\mu, \sigma^2\right)$, we obtain:

$$\mathrm{EL} = \lambda \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

and:

$$\mathrm{UL}(\alpha) = \exp\left(\mu + \sigma\Phi^{-1}\left(1 - \frac{1-\alpha}{\lambda}\right)\right) - \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

In Figure 5.14, we report the results of some experiments for different values of paramors. In the top panels, we assume that $\lambda = 100$, $\mu = 5.0$ and $\sigma = 2.0$ (left panel), and $\lambda = 500$, $\mu = 10.0$ and $\sigma = 2.5$ (right panel). These two examples correspond to medium severity/low frequency and high severity/low frequency events. In these cases, we obtain a good approximation. In the bottom panel, the parameters are $\lambda = 1000$, $\mu = 8.0$ and $\sigma = 1.0$. The approximation does not work very well, because we have a low severity/high frequency events and the risk can then not be explained by an extreme single loss. The underestimation of the capital-at-risk is due to the underestimation of the number of losses. In fact, with loss severity/high frequency events, the

[19] In Chapter 16, we will see that such transformation is common in extreme value theory.

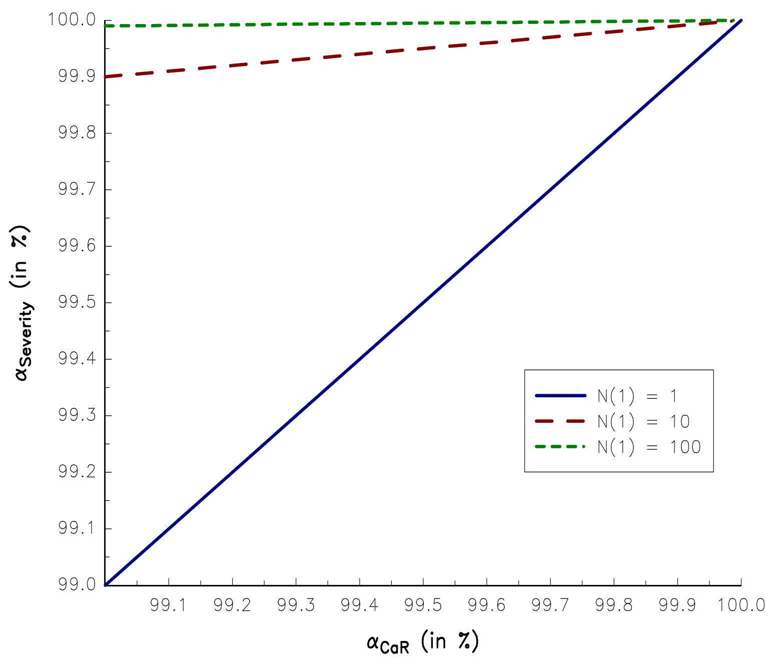

FIGURE 5.13: Relationship between α_{CaR} and α_{Severity}

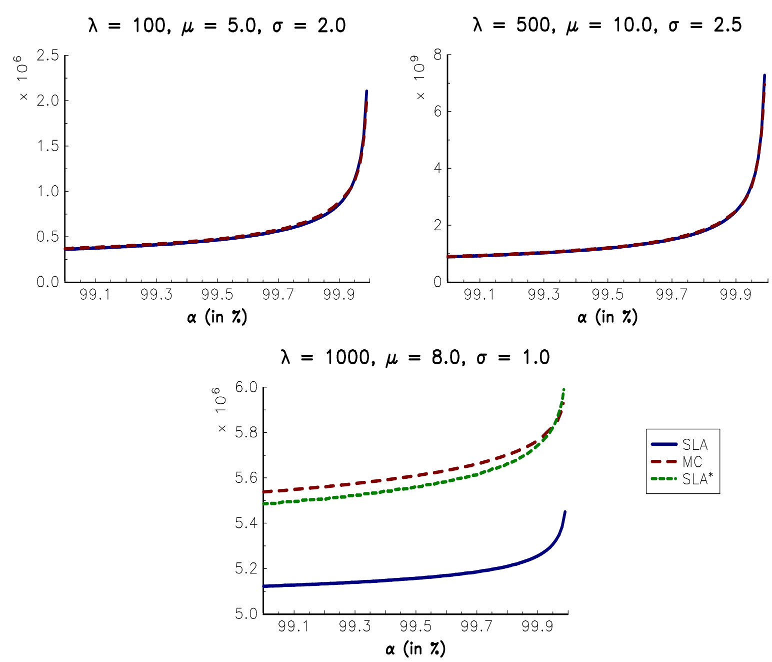

FIGURE 5.14: Numerical illustration of the single loss approximation

risk is not to face a large single loss, but to have a high number of losses in the year. This is why it is better to approximate the capital-at-risk with the following formula:

$$\mathbf{G}^{-1}(\alpha) \approx \left(\mathbf{P}^{-1}(\alpha) - 1\right)\mathbb{E}[X_i] + \mathbf{F}^{-1}\left(1 - \frac{1-\alpha}{\mathbb{E}[N(1)]}\right)$$

where $\mathbf{P}$ is the cumulative distribution function of the counting process $N(1)$. In Figure 5.14, we have also reported this approximation SLA* for the third example. We verify that it gives better results than the classic approximation for high frequency events.

5.3.3.3 Aggregation issues

We remind that the loss at the bank level is equal to:

$$L = \sum_{k=1}^{K} S_k$$

where S_k is the aggregate loss of the k^{th} cell of the mapping matrix. For instance, if the matrix is composed of the eight business lines (BL) and seven even types (ET) of the Basel II classification, we have $L = \sum_{k \in \mathcal{K}} S_k$ where $\mathcal{K} = \{(\mathrm{BL}_{k_1}, \mathrm{ET}_{k_2}), k_1 = 1, \ldots, 8; k_2 = 1, \ldots, 7\}$. Let $\mathrm{CaR}_{k_1,k_2}(\alpha)$ be the capital charge calculated for the business line k_1 and the event type k_2. We have:

$$\mathrm{CaR}_{k_1,k_2}(\alpha) = \mathbf{G}^{-1}_{k_1,k_2}(\alpha)$$

One solution to calculate the capital charge at the bank level is to sum up all the capital charges:

$$\begin{aligned}
\mathrm{CaR}(\alpha) &= \sum_{k_1=1}^{8}\sum_{k_2=1}^{7} \mathrm{CaR}_{k_1,k_2}(\alpha) \\
&= \sum_{k_1=1}^{8}\sum_{k_2=1}^{7} \mathbf{G}^{-1}_{k_1,k_2}(\alpha)
\end{aligned}$$

From a theoretical point of view, this is equivalent to assume that all the aggregate losses S_k are perfectly correlated. This approach is highly conservative and ignores diversification effects between business lines and event types.

Let us consider the two-dimensional case:

$$\begin{aligned}
L &= S_1 + S_2 \\
&= \sum_{i=1}^{N_1} X_i + \sum_{j=1}^{N_2} Y_j
\end{aligned}$$

In order to take into account the dependence between the two aggregate losses

S_1 and S_2, we can assume that frequencies N_1 and N_2 are correlated or severities X_i and Y_j are correlated. Thus, the aggregate loss correlation $\rho(S_1, S_2)$ depends on two key parameters:

- the frequency correlation $\rho(N_1, N_2)$, and
- the severity correlation $\rho(X_i, Y_j)$.

For example, we should observe that, historically, the number of external fraud events is high (respectively low) when the number of internal fraud events is also high (respectively low). Severity correlation is more difficult to justify. In effect, a basic feature of the LDA model requires to assume that individual losses are jointly independent. Therefore it is conceptually difficult to assume simultaneously severity independence within each class of risk and severity correlation between two classes. By assuming that $\rho(X_i, Y_j) = 0$, Frachot *et al.* (2004) find an upper bound of the aggregate loss correlation. We have:

$$\begin{aligned}
\operatorname{cov}(S_1, S_2) &= \mathbb{E}[S_1 S_2] - \mathbb{E}[S_1]\,\mathbb{E}[S_2] \\
&= \mathbb{E}\left[\sum\nolimits_{i=1}^{N_1} X_i \sum\nolimits_{j=1}^{N_2} Y_j\right] - \mathbb{E}\left[\sum\nolimits_{i=1}^{N_1} X_i\right]\mathbb{E}\left[\sum\nolimits_{j=1}^{N_2} Y_j\right] \\
&= \mathbb{E}[N_1 N_2]\,\mathbb{E}[X_i]\,\mathbb{E}[Y_j] - \mathbb{E}[N_1]\,\mathbb{E}[X_i]\,\mathbb{E}[N_2]\,\mathbb{E}[Y_j] \\
&= \left(\mathbb{E}[N_1 N_2] - \mathbb{E}[N_1]\,\mathbb{E}[N_2]\right)\mathbb{E}[X_i]\,\mathbb{E}[Y_j]
\end{aligned}$$

and:

$$\rho(S_1, S_2) = \frac{\left(\mathbb{E}[N_1 N_2] - \mathbb{E}[N_1]\,\mathbb{E}[N_2]\right)\mathbb{E}[X_i]\,\mathbb{E}[Y_j]}{\sqrt{\operatorname{var}(S_1)\operatorname{var}(S_2)}}$$

If we assume that the counting processes N_1 and N_2 are Poisson processes with parameters λ_1 and λ_2, we obtain:

$$\rho(S_1, S_2) = \rho(N_1, N_2)\,\eta(X_i)\,\eta(Y_j)$$

where:

$$\begin{aligned}
\eta(X) &= \frac{\mathbb{E}[X]}{\sqrt{\mathbb{E}[X^2]}} \\
&= \frac{1}{\sqrt{1 + \operatorname{CV}^2(X)}} \leq 1
\end{aligned}$$

Here $\operatorname{CV}(X) = \sigma(X)/\mathbb{E}[X]$ denotes the coefficient of variation of the random variable X. As a result, aggregate loss correlation is always lower than frequency correlation:

$$0 \leq \rho(S_1, S_2) \leq \rho(N_1, N_2) \leq 1$$

We deduce that an upper bound of the aggregate loss correlation is equal to:

$$\rho^+ = \eta(X_i)\,\eta(Y_j)$$

For high severity events, severity independence likely dominates frequency correlation and we obtain $\rho^+ \simeq 0$ because $\eta(X_i) \simeq 0$.

Let us consider the example of log-normal severity distributions. We have:

$$\rho^+ = \exp\left(-\frac{1}{2}\sigma_X^2 - \frac{1}{2}\sigma_Y^2\right)$$

We notice that this function is decreasing with respect to σ_X and σ_Y. Figure 5.15 shows the relationship between σ_X, σ_Y and ρ^+. We verify that ρ^+ is small when σ_X and σ_Y take large values. For instance, if $\sigma_X = \sigma_Y = 2$, the aggregate loss correlation is lower than 2%.

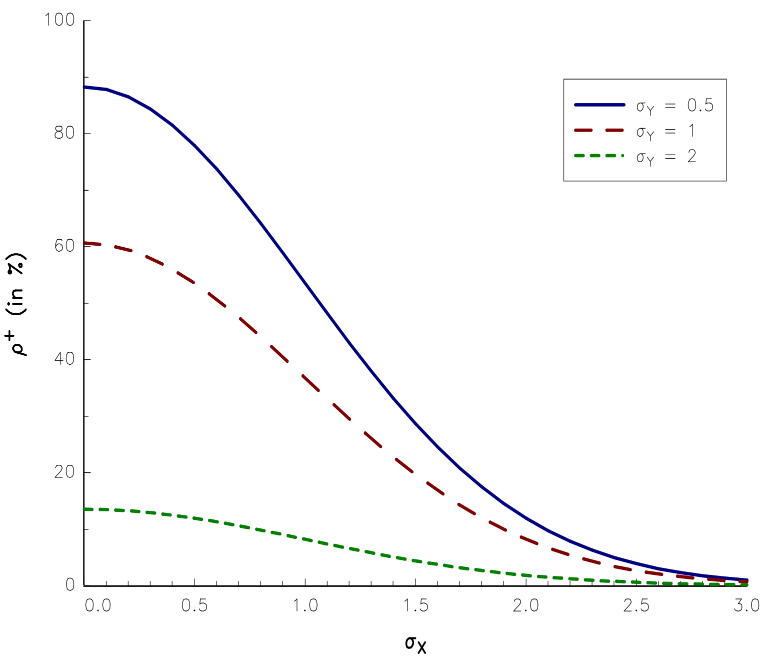

FIGURE 5.15: Upper bound ρ^+ of the aggregate loss correlation

There are two ways to take into account correlations for computing the capital charge of the bank. The first approach is to consider the normal approximation:

$$\mathrm{CaR}(\alpha) = \sum_k \mathrm{EL}_k + \sqrt{\sum_{k,k'} \rho_{k,k'} \times (\mathrm{CaR}_k(\alpha) - \mathrm{EL}_k) \times (\mathrm{CaR}_{k'}(\alpha) - \mathrm{EL}_{k'})}$$

where $\rho_{k,k'}$ is the correlation between the cells k and k' of the mapping matrix. The second approach consists in introducing the dependence between the aggregate losses using a copula function $\mathbf{C}$. The joint distribution of $(S_1, \ldots, S_K)$

has the following form:

$$\Pr\{S_1 \leq s_1, \ldots, S_K \leq s_K\} = \mathbf{C}(\mathbf{G}_1(s_1), \ldots, \mathbf{G}_K(s_K))$$

where $\mathbf{G}_k$ is the cumulative probability distribution of the k^{th} aggregate loss S_k. In this case, the quantile of the random variable $L = \sum_{k=1}^{K} S_k$ is estimated using Monte Carlo simulations. The difficulty comes from the fact that the distributions $\mathbf{G}_k$ have no analytical expression. The solution is then to use the method of empirical distributions, which is presented in page 445.

5.3.4 Incorporating scenario analysis

The concept of scenario analysis should deserve further clarification. Roughly speaking, when we refer to scenario analysis, we want to express the idea that banks' experts and experienced managers have some reliable intuitions on the riskiness of their business and that these intuitions are not entirely reflected in the bank's historical internal data. As a first requirement, we expect that experts should have the opportunity to give their approval to capital charge results. In a second step, one can imagine that experts' intuitions are directly plugged into severity and frequency estimations. Experts' intuition can be captured through scenario building. More precisely, a scenario is given by a potential loss amount and the corresponding probability of occurrence. As an example, an expert may assert that a loss of one million dollars or higher is expected to occur once every (say) 5 years. This is a valuable information in many cases, either when loss data are rare and do not allow for statistically sound results or when historical loss data are not sufficiently forward-looking. In this last case, scenario analysis allows to incorporate external loss data.

In what follows, we show how scenarios can be translated into restrictions on the parameters of frequency and severity distributions. Once these restrictions have been identified, a calibration strategy can be designed where parameters are calibrated by maximizing some standard criterion subject to these constraints. As a result, parameter estimators can be seen as a mixture of the internal data-based estimator and the scenario-based implied estimator.

5.3.4.1 Probability distribution of a given scenario

We assume that the number of losses $N(t)$ is a Poisson process with intensity λ. Let τ_n be the arrival time of the n^{th} loss, i.e.

$$\tau_n = \inf\{t \geq 0 : N(t) = n\}$$

We know that the durations $T_n = \tau_n - \tau_{n-1}$ between two consecutive losses are independent identically distributed exponential random variables with parameter λ. We remind that the losses X_n are *i.i.d.* with distribution $\mathbf{F}$. We note now $T_n(x)$ the duration between two losses exceeding x. It is obvious that the durations are *i.i.d.* It suffice now to characterize $T_1(x)$. By using the fact that a finite sum of exponential times is an Erlang distribution, we have:

$$\begin{aligned}
\Pr\{T_1(x) > t\} &= \sum_{n\geq 1} \Pr\{\tau_n > t; X_1 < x, \ldots, X_{n-1} < x; X_n \geq x\} \\
&= \sum_{n\geq 1} \Pr\{\tau_n > t\} \mathbf{F}(x)^{n-1} (1 - \mathbf{F}(x)) \\
&= \sum_{n\geq 1} \mathbf{F}(x)^{n-1} (1 - \mathbf{F}(x)) \sum_{k=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \\
&= (1 - \mathbf{F}(x)) \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \sum_{n=k}^{\infty} \mathbf{F}(x)^n \\
&= e^{-\lambda t} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} \mathbf{F}(x)^k \\
&= e^{-\lambda(1-\mathbf{F}(x))t}
\end{aligned}$$

We deduce that $T_n(x)$ follows an exponential distribution with parameter $\lambda(x) = \lambda(1 - \mathbf{F}(x))$. The average duration between two losses exceeding x is also the mean of $T_n(x)$:

$$\mathbb{E}[T_n(x)] = \frac{1}{\lambda(1 - \mathbf{F}(x))}$$

Example 47 *We assume that the annual number of losses follows a Poisson parameter with $\lambda = 5$ and the severity of losses are log-normal $\mathcal{LN}(9, 4)$.*

In Figure 5.16, we simulate the corresponding Poisson process $N(t)$ and also the events whose loss is larger than \$20 000 and \$50 000. We then show the exponential distribution[20] of $T_n(x)$.

5.3.4.2 Calibration of a set of scenarios

Let us consider a scenario defined as "a loss of x or higher occurs once every d years". By assuming a compound Poisson distribution with a parametric severity distribution $\mathbf{F}(x;\theta)$, λ is the average number of losses per year, $\lambda(x) = \lambda(1 - \mathbf{F}(x;\theta))$ is the average number of losses higher than x and $1/\lambda(x)$ is the average duration between two losses exceeding x. As a result, for a given scenario (x, d), parameters (λ, θ) are restricted to satisfy:

$$d = \frac{1}{\lambda(1 - \mathbf{F}(x;\theta))}$$

[20]For the parameter $\lambda(x)$, we have:

$$\lambda\left(2 \times 10^4\right) = 5 \times \left(1 - \Phi\left(\frac{\ln\left(2 \times 10^4\right) - 9}{2}\right)\right) = 1.629$$

and $\lambda\left(5 \times 10^4\right) = 0.907$.

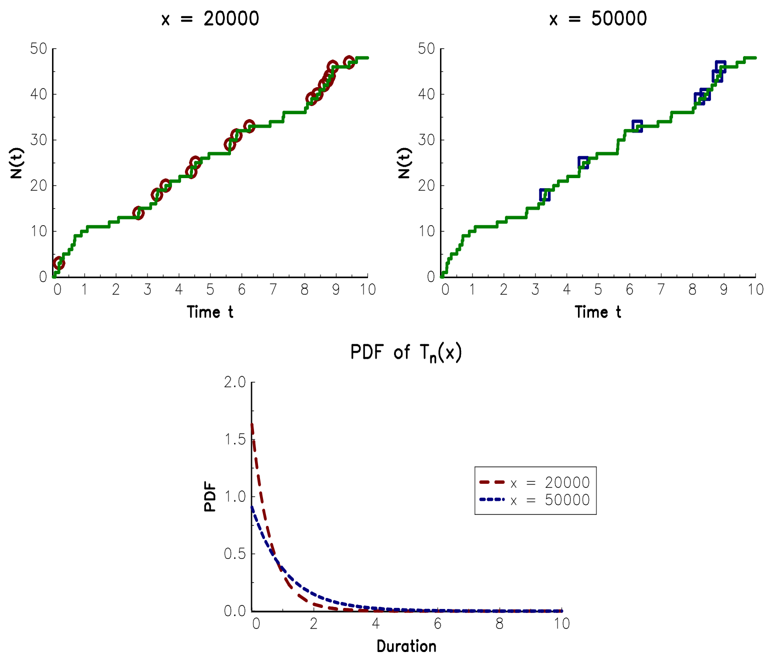

FIGURE 5.16: Simulation of the Poisson process $N(t)$ and peak-over-threshold events

Suppose that we face different scenarios $\{(x_s, d_s), s = 1, \ldots, n_S\}$. We may estimate the implied parameters underlying the expert judgements using the quadratic criterion:

$$\left(\hat{\lambda}, \hat{\theta}\right) = \arg\min \sum_{s=1}^{n_S} w_s \left(d_s - \frac{1}{\lambda \left(1 - \mathbf{F}\left(x_s; \theta\right)\right)}\right)^2$$

where w_s is the weight of the s^{th} scenario. The previous approach belongs to the method of moments. As a result, we can show that the optimal weights w_s correspond to the inverse of the variance of d_s:

$$\begin{aligned} w_s &= \frac{1}{\operatorname{var}\left(d_s\right)} \\ &= \lambda \left(1 - \mathbf{F}\left(x_s; \theta\right)\right) \end{aligned}$$

To solve the previous optimization program, we proceed by iterations. Let $\left(\hat{\lambda}_m, \hat{\theta}_m\right)$ be the solution of this minimization program:

$$\left(\hat{\lambda}_m, \hat{\theta}_m\right) = \arg\min \sum_{j=1}^{p} \hat{\lambda}_{m-1} \left(1 - \mathbf{F}\left(x_s; \hat{\theta}_{m-1}\right)\right) \left(d_s - \frac{1}{\lambda \left(1 - \mathbf{F}\left(x_s; \theta\right)\right)}\right)^2$$

Under some conditions, the estimator $\left(\hat{\lambda}_m, \hat{\theta}_m\right)$ converge to the optimal solution. We also notice that we can simplify the optimization program by using the following approximation:

$$w_s = \frac{1}{\operatorname{var}(d_s)} = \frac{1}{\mathbb{E}[d_s]} \simeq \frac{1}{d_s}$$

Example 48 *We assume that the severity distribution is log-normal and consider the following set of expert's scenarios:*

x_s *(in \$ mn)*	1	2.5	5	7.5	10	20
d_s *(in years)*	1/4	1	3	6	10	40

If $w_s = 1$, we obtain $\hat{\lambda} = 43.400$, $\hat{\mu} = 11.389$ and $\hat{\sigma} = 1.668$ (#1). Using the approximation $w_s \simeq 1/d_s$, the estimates become $\hat{\lambda} = 154.988$, $\hat{\mu} = 10.141$ and $\hat{\sigma} = 1.855$ (#2). Finally, the optimal estimates are $\hat{\lambda} = 148.756$, $\hat{\mu} = 10.181$ and $\hat{\sigma} = 1.849$ (#3). In the table below, we report the estimated values of the duration. We notice that they are close to the expert's scenarios.

x_s (in \$ mn)	1	2.5	5	7.5	10	20
#1	0.316	1.022	2.964	5.941	10.054	39.997
#2	0.271	0.968	2.939	5.973	10.149	39.943
#3	0.272	0.970	2.941	5.974	10.149	39.944

Remark 47 *We can combine internal loss data, expert's scenarios and external loss data*[21] *by maximizing the penalized likelihood:*

$$\begin{aligned} \hat{\theta} = \arg\max \quad & \varpi_{\text{internal}} \ell(\theta) - \\ & \varpi_{\text{expert}} \sum_{s=1}^{n_S} w_s \left(d_s - \frac{1}{\lambda\left(1 - \mathbf{F}(x_s;\theta)\right)}\right)^2 - \\ & \varpi_{\text{external}} \sum_{s=1}^{n_S^\star} w_s^\star \left(d_s^\star - \frac{1}{\lambda\left(1 - \mathbf{F}(x_s^\star;\theta)\right)}\right)^2 \end{aligned}$$

where ϖ_{internal}, ϖ_{expert} *and* ϖ_{external} *are the weights reflecting the confidence placed on internal loss data, expert's scenarios and external loss.*

5.3.5 Stability issue of the LDA model

One of the big issue of AMA (and LDA) models is their stability. It is obvious that the occurrence of a large loss changes dramatically the estimated capital-at-risk as explained by Ames *et al.* (2015):

"*Operational risk is fundamentally different from all other risks*

[21] In this case, each external loss is treated as an expert's scenario.

> *taken on by a bank. It is embedded in every activity and product of an institution, and in contrast to the conventional financial risks (e.g. market, credit) is harder to measure and model, and not straight forwardly eliminated through simple adjustments like selling off a position. While it varies considerably, operational risk tends to represent about 10-30% of the total risk pie, and has grown rapidly since the 2008-09 crisis. It tends to be more fat-tailed than other risks, and the data are poorer. As a result, models are fragile – small changes in the data have dramatic impacts on modeled output – and thus required operational risk capital is unstable*".

In this context, the Basel Committee has decided to review the different measurement approaches to calculate the operational risk capital. In Basel IV, advanced measurement approaches have been dropped. This decision marks a serious setback for operational risk modeling. The LDA model continue to be used by Basel II jurisdictions, and will continue to be used by large international banks, because it is the only way to assess an 'economic' capital using internal loss data. However, solutions for stabilizing the LDA model can only be partial and even hazardous or counter-intuitive, because it ignores the nature of operational risk.

5.4 Exercises

5.4.1 Estimation of the loss severity distribution

We consider a sample of n individual losses $\{x_1, \ldots, x_n\}$. We assume that they can be described by different probability distributions:

(i) X follows a log-normal distribution $\mathcal{LN}\left(\mu, \sigma^2\right)$.

(ii) X follows a Pareto distribution $\mathcal{P}\left(\alpha, x^-\right)$ defined by:

$$\Pr\{X \leq x\} = 1 - \left(\frac{x}{x_-}\right)^{-\alpha}$$

with $x \geq x_-$ and $\alpha > 1$.

(iii) X follows a Gamma distribution $\Gamma(\alpha, \beta)$ defined by:

$$\Pr\{X \leq x\} = \int_0^x \frac{\beta^\alpha t^{\alpha-1} e^{-\beta t}}{\Gamma(\alpha)}\,\mathrm{d}t$$

with $x \geq 0$, $\alpha > 0$ and $\beta > 0$.

(iv) The natural logarithm of the loss X follows a Gamma distribution: $\ln X \sim \Gamma(\alpha; \beta)$.

1. We consider the case *(i)*.

   (a) Show that the probability density function is:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2\right)$$

   (b) Calculate the two first moments of X. Deduce the orthogonal conditions of the generalized method of moments.

   (c) Find the maximum likelihood estimators $\hat{\mu}$ and $\hat{\sigma}$.

2. We consider the case *(ii)*.

   (a) Calculate the two first moments of X. Deduce the GMM conditions for estimating the parameter α.

   (b) Find the maximum likelihood estimator $\hat{\alpha}$.

3. We consider the case *(iii)*. Write the log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$. Deduce the first-order conditions of the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$.

4. We consider the case *(iv)*. Show that the probability density function of X is:

$$f(x) = \frac{\beta^\alpha (\ln x)^{\alpha-1}}{\Gamma(\alpha)\, x^{\beta+1}}$$

   What is the support of this probability density function? Write the log-likelihood function associated to the sample of individual losses $\{x_1, \ldots, x_n\}$.

5. We now assume that the losses $\{x_1, \ldots, x_n\}$ have been collected beyond a threshold H meaning that $X \geq H$.

   (a) What becomes the generalized method of moments in the case *(i)*.

   (b) Calculate the maximum likelihood estimator $\hat{\alpha}$ in the case *(ii)*.

   (c) Write the log-likelihood function in the case *(iii)*.

5.4.2 Estimation of the loss frequency distribution

We consider a dataset of individual losses $\{x_1, \ldots, x_n\}$ corresponding to a sample of T annual loss numbers $\{N_{Y_1}, \ldots, N_{Y_T}\}$. This implies that:

$$\sum_{t=1}^{T} N_{Y_t} = n$$

If we measure the number of losses per quarter $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$, we use the notation:

$$\sum_{t=1}^{4T} N_{Q_t} = n$$

1. We assume that the annual number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Y)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Y$ associated to the sample $\{N_{Y_1}, \ldots, N_{Y_T}\}$.

2. We assume that the quarterly number of losses follows a Poisson distribution $\mathcal{P}(\lambda_Q)$. Calculate the maximum likelihood estimator $\hat{\lambda}_Q$ associated to the sample $\{N_{Q_1}, \ldots, N_{Q_{4T}}\}$.

3. What is the impact of considering a quarterly or annual basis on the computation of the capital charge?

4. What does this result become if we consider a method of moments based on the first moment?

5. Same question if we consider a method of moments based on the second moment.

5.4.3 Using the method of moments in operational risk models

1. Let $N(t)$ be the number of losses for the time interval $[0, t]$. We note $\{N_1, \ldots, N_T\}$ a sample of $N(t)$ and we assume that $N(t)$ follows a Poisson distribution $\mathcal{P}(\lambda)$. We recall that:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

   (a) Calculate the first moment $\mathbb{E}[N(t)]$.

   (b) Show the following result:

$$\mathbb{E}\left[\prod_{i=0}^{m} (N(t) - i)\right] = \lambda^{m+1}$$

   Then deduce the variance of $N(t)$.

   (c) Propose two estimators based on the method of moments.

2. Let S be the random sum:

$$S = \sum_{i=0}^{N(t)} X_i$$

   with $X_i \sim \mathcal{LN}(\mu, \sigma^2)$, $X_i \perp X_j$ and $N(t) \sim \mathcal{P}(\lambda)$.

(a) Calculate the mathematical expectation $\mathbb{E}[S]$.

(b) We recall that:

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n} x_i^2 + \sum_{i \neq j} x_i x_j$$

Show that:

$$\operatorname{var}(S) = \lambda \exp\left(2\mu + 2\sigma^2\right)$$

(c) How can we estimate μ and σ if we have already calibrated λ?

3. We assume that the annual number of losses follows a Poisson distribution $\mathcal{P}(\lambda)$. We also assume as well that the individual losses are independent and follow the Pareto distribution $\mathcal{P}(\alpha, x^{-})$ defined by:

$$\Pr\{X \leq x\} = 1 - \left(\frac{x}{x_-}\right)^{-\alpha}$$

with $x \geq x_-$ and $\alpha > 1$.

(a) Show that the duration between two consecutive losses that are larger than ℓ is an exponential distribution with parameter $\lambda x_-^{\alpha} \ell^{-\alpha}$.

(b) How can we use this result to calibrate experts' scenarios?

5.4.4 Calculation of the Basel II required capital

We consider the simplified balance sheet of a bank, which is described below.

1. In the Excel file, we provide the price evolution of stocks A and B. The trading portfolio consists of of 10 000 shares A and 25 000 shares B. Calculate the daily historical VaR of this portfolio by assuming that the current stock prices are equal to \$105.5 and \$353. Deduce the capital charge for market risk assuming that the VaR has not fundamentally changed during the last 3 months[22].

2. We consider that the credit portfolio of the bank can be summarized by 4 meta-credits whose characteristics are the following:

	Sales	EAD	PD	LGD	M
Bank		\$80 mn	1%	75%	1.0
Corporate	\$500 mn	\$200 mn	5%	60%	2.0
SME	\$30 mn	\$50 mn	2%	40%	4.5
Mortgage		\$50 mn	9%	45%	
Retail		\$100 mn	4%	85%	

Calculate the IRB capital charge for the credit risk.

[22] The multiplication coefficient ξ is set equal to 0.5.

3. We assume that the bank is exposed to a single operational risk. The severity distribution is a log-normal probability distribution $\mathcal{LN}(8, 4)$, whereas the frequency distribution is the following discrete probability distribution:

$$\begin{aligned}\Pr\{N = 5\} &= 60\% \\ \Pr\{N = 10\} &= 40\%\end{aligned}$$

Calculate the AMA capital charge for the operational risk.

4. Deduce the capital charge of the bank and the capital ratio knowing that the capital of the bank is equal to \$70 mn.

5.4.5 Parametric estimation of the loss severity distribution

1. We assume that the severity losses are log-logistic distributed $X_i \sim \mathcal{LL}(\alpha, \beta)$ with:

$$\mathbf{F}(x; \alpha, \beta) = \frac{(x/\alpha)^{\beta}}{1 + (x/\alpha)^{\beta}}$$

   (a) Find the density function.

   (b) Deduce the log-likelihood function of the sample $\{x_1, \ldots, x_n\}$.

   (c) Show that the ML estimators satisfy the following first-order conditions:

$$\left\{\begin{array}{l}\sum_{i=1}^{n} \mathbf{F}\left(x_i; \hat{\alpha}, \hat{\beta}\right) = n/2 \\ \sum_{i=1}^{n} \left(2\mathbf{F}\left(x_i; \hat{\alpha}, \hat{\beta}\right) - 1\right) \ln x_i = n/\hat{\beta}\end{array}\right.$$

   (d) The sample of loss data is 2 918, 740, 3 985, 2 827, 2 839, 6 897, 7 665, 3 766, 3 107 and 3 304. Verify that $\hat{\alpha} = 3\,430.050$ and $\hat{\beta} = 3.315$ are the ML estimates.

   (e) What becomes the log-likelihood function of the sample $\{x_1, \ldots, x_n\}$ if we assume that the losses were collected beyond a threshold H?

5.4.6 Mixed Poisson process

1. We consider the mixed poisson process where $N(t) \sim \mathcal{P}(\Lambda)$ and Λ is a random variable. Show that:

$$\operatorname{var}(N(t)) = \mathbb{E}[N(t)] + \operatorname{var}(\Lambda)$$

2. Deduce that $\operatorname{var}(N(t)) \geq \mathbb{E}[N(t)]$. Determine the probability distribution Λ such that the equality holds. Let $\varphi(n)$ be the following ratio:

$$\varphi(n) = \frac{(n+1)\, p(n+1)}{p(n)}$$

Show that $\varphi(n)$ is constant.

3. We assume that $\Lambda \sim \mathcal{G}(\alpha, \beta)$.
   (a) Calculate $\mathbb{E}[N(t)]$ and $\mathrm{var}(N(t))$.
   (b) Show that $N(t)$ has a negative binomial distribution $\mathcal{NB}(r, p)$. Calculate the parameters r and p with respect to α and β.
   (c) Show that $\varphi(n)$ is an affine function.
4. We assume that $\Lambda \sim \mathcal{E}(\lambda)$.
   (a) Calculate $\mathbb{E}[N(t)]$ and $\mathrm{var}(N(t))$.
   (b) Show that $N(t)$ has a geometric distribution $\mathcal{G}(p)$. Determine the parameter p.

5.4.7 Loss frequency distribution with data truncation

5.4.8 Moments of compound distribution

5.4.9 Characteristic functions and fast Fourier transform

5.4.10 Derivation and implementation of the Panjer recursion

5.4.11 The Böcker-Klüppelberg-Sprittulla approximation formula

5.4.12 Frequency correlation, severity correlation and loss aggregation

5.4.13 Loss aggregation using copula functions

5.4.14 Scenario analysis and stress testing

Chapter 6

Liquidity Risk

6.1 Market liquidity

6.2 Funding liquidity

6.3 Regulation of the liquidity risk

6.4 Exercises

Part II

Risk Management in Other Financial Sectors

Chapter 12

Systemic Risk and Shadow Banking System

The financial crisis of 2008 is above all a crisis of the financial system as a whole. This is why it is called the Global Financial Crisis (GFC) and is different than the previous crises (the Great Depression in the 1930s, the Japan crisis in the early 1990s, the Black Monday of 1987, the 1997 Asian financial crisis, etc.). It is a superposition of the 2007 subprime crisis, affecting primarily the mortgage and credit derivative markets, and a liquidity funding crisis following the demise of Lehman Brothers, which affected the credit market and more broadly the shadow banking system. This crisis was not limited to the banking system, but has affected the different actors of the financial sector, in particular insurance companies, asset managers and of course investors. As we have seen in the previous chapters, this led to a strengthening of financial regulation, and not only on the banking sector. The purpose of new regulations in banks, insurance, asset management, pension funds and organization of the financial market is primarily to improve the rules of each sector, but also to reduce the overall systemic risk of the financial sector. In this context, systemic risk is now certainly the biggest concern of financial regulators and the Financial Stability Board (FSB) was created in April 2009 especially to monitor the stability of the global financial system and to manage the systemic risk[1]. It rapidly became clear that the identification of the systemic risk is a hard task and can only be conducted in a gradual manner. This is why some

[1]The FSB is the successor to the Financial Stability Forum (FSF), which was founded in 1999 by the G7 Finance Ministers and Central Bank Governors. With an expanded membership to the G20 countries, the mandate of the FSB has been reinforced with the creation of three Standing Committees:

- the Standing Committee on Assessment of Vulnerabilities (SCAV), which is the FSB's main mechanism for identifying and assessing risks;
- the Standing Committee on Supervisory and Regulatory Cooperation (SRC), which is charged with undertaking further supervisory analysis or framing a regulatory or supervisory policy response to a material vulnerability identified by SCAV;
- the Standing Committee on Standards Implementation (SCSI), which is responsible for monitoring the implementation of agreed FSB policy initiatives and international standards.

As the Basel Committee on Banking Supervision, the secretariat to the Financial Stability Board is hosted by the Bank for International Settlements and located in Basel.

policy responses are not yet finalized, in particular with the emergence of a shadow banking system, whose borders are not well defined.

12.1 Defining systemic risk

The Financial Stability Board defines systemic events in broad terms:

> "*Systemic event is the disruption to the flow of financial services that is (i) caused by an impairment of all or parts of the financial system and (ii) has the potential to have serious negative consequences on the real economy*" (FSB, 2009, page 6).

This definition focuses on three important points. Firstly, systemic events are associated with negative externalities and moral hazard risk, meaning that every financial institution's incentive is to manage its own risk/return trade-off but not necessarily the implications of its risk on the global financial system. Secondly, a systemic event can cause the impairment of the financial system. Lastly, it implies significant spillovers to the real economy and negative effects on economic welfare.

It is clear that the previous definition may appear too large, but also too restrictive. It may be too large, because it is not precise and many events can be classified as systemic events. It is also too restrictive, because it is difficult to identify the event that lies at the origin of the systemic risk. Most of the times, it is caused by the combination of several events. As noted by Zigrand (2014), systemic risk often refers to exogenous shocks, whereas it can also be generated by endogenous shocks:

> "*Systemic risk comprises the risk* ***to*** *the proper functioning of the system as well as the risk created* ***by*** *the system*" (Zigrand, 2014, page 3).

In fact, there are numerous definitions of systemic risk because it is a multi-faceted concept.

12.1.1 Systemic risk, systematic risk and idiosyncratic risk

In financial theory, systemic and idiosyncratic risks are generally opposed. Systemic risk refers to the system whereas idiosyncratic risk refers to an entity of the system. For instance, the banking system may collapse, because many banks may be affected by a severe common risk factor and may default at the same time. In economics, we generally make the assumption that idiosyncratic and common risk factors are independent. However, there exists some situations where idiosyncratic risk may affect the system itself. It is the case of large institutions, for example the default of big banks. In this situation, systemic risk refers to the propagation of a single bank distressed risk to the other banks.

Let us consider one of the most famous model in finance, which is the capital asset pricing model (CAPM) developed by William Sharpe in 1964. Under some assumptions, he showed that the expected return of asset i is related to the expected return of the market portfolio in the following way:

$$\mathbb{E}[R_i] - r = \beta_i \left(\mathbb{E}\left[R^{\mathrm{mkt}}\right] - r\right) \tag{12.1}$$

where R_i and R^{mkt} are the asset and market returns, r is the risk-free rate and the coefficient β_i is the beta of the asset i with respect to the market portfolio:

$$\beta_i = \frac{\mathrm{cov}\left(R_i, R^{\mathrm{mkt}}\right)}{\sigma^2\left(R^{\mathrm{mkt}}\right)}$$

Contrary to idiosyncratic risks, systematic risk R^{mkt} cannot be diversified, and investors are compensated for taking this risk. This means that the market risk premium is positive ($\mathbb{E}\left[R^{\mathrm{mkt}}\right] - r > 0$) whereas the expected return of idiosyncratic risk is equal to zero. By definition, the idiosyncratic risk of asset i is equal to:

$$\epsilon_i = (R_i - r) - \beta_i \left(\mathbb{E}\left[R^{\mathrm{mkt}}\right] - r\right)$$

with $\mathbb{E}[\epsilon_i] = 0$. As explained above, this idiosyncratic risk is not rewarded because it can be hedged. In this framework, we obtain the one-factor model given by the following equation:

$$R_i = \alpha_i + \beta_i R^{\mathrm{mkt}} + \varepsilon_i \tag{12.2}$$

where $\alpha_i = (1 - \beta_i) r$ and $\varepsilon_i = \epsilon_i - \beta_i \left(R^{\mathrm{mkt}} - \mathbb{E}\left[R^{\mathrm{mkt}}\right]\right)$ is a white noise process[2]. Because ε_i is a new parametrization of the idiosyncratic risk, it is easy to show that this specific factor is independent from the common factor R^{mkt} and the other specific factors ε_j. If we assume that asset returns are normally distributed, we have $R^{\mathrm{mkt}} \sim \mathcal{N}\left(\mathbb{E}\left[R^{\mathrm{mkt}}\right], \sigma^2_{\mathrm{mkt}}\right)$ and:

$$\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, \mathrm{diag}\left(\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_n^2\right)\right)$$

In the capital asset pricing model, it is obvious that the risk of the system $(R_1, \ldots, R_n)$ is due to the common risk factor also called the systematic risk factor. Indeed, a stress $\mathbb{S}$ can only be transmitted to the system by a shock on R^{mkt}:

$$\mathbb{S}\left(R^{\mathrm{mkt}}\right) \implies \mathbb{S}(R_1, \ldots, R_n)$$

[2] ε_i is a new form of the idiosyncratic risk.

This is the traditional form of systemic risk. In the CAPM, idiosyncratic risks are not a source of systemic risk:

$$\mathbb{S}(\varepsilon_i) \not\Longrightarrow \mathbb{S}(R_1, \ldots, R_n)$$

because the specific risk ε_i only affects one component of the system, and not all the components.

In practice, systemic risk can also occur because of an idiosyncratic shock. In this case, we distinguish two different transmission channels:

1. The first channel is the impact of a specific stress on the systematic risk factor:
$$\mathbb{S}(\varepsilon_i) \Longrightarrow \mathbb{S}\left(R^{\mathrm{mkt}}\right) \Longrightarrow \mathbb{S}(R_1, \ldots, R_n)$$
This transmission channel implies that the assumption $\varepsilon_i \perp R^{\mathrm{mkt}}$ is not valid.

2. The second channel is the impact of a specific stress on the other specific risk factors:
$$\mathbb{S}(\varepsilon_i) \Longrightarrow \mathbb{S}(\varepsilon_1, \ldots, \varepsilon_n) \Longrightarrow \mathbb{S}(R_1, \ldots, R_n)$$
This transmission channel implies that the assumption $\varepsilon_i \perp \varepsilon_j$ is not valid.

Traditional financial models (CAPM, APT) fail to capture these two channels, because they neglect some characteristics of systemic factors: the feedback dynamic of specific risks, the possibility of multiple equilibria and the network density.

The distinction between systematic and idiosyncratic shocks is done by De Bandt and Hartmann (2000). However, as noted by Hansen (2012), systematic risks are aggregate risks that cannot be avoided. A clear example is the equity risk premium. In this case, systematic risks are normal and inherent to financial markets and there is no reason to think that we can prevent them. In the systemic risk literature, common or systematic risks reflect another reality. They are abnormal and are viewed as a consequence of simultaneous adverse shocks that affect a large number of system components (De Bandt and Hartmann, 2000). In this case, the goal of supervisory policy is to prevent them, or at least to mitigate them. In practice, it is however difficult to make the distinction between these two concepts of systematic risk. In what follows, we will use the term *systematic market risk* for normal shocks, even if they are severe and we now reserve the term *systematic risk* for abnormal shocks.

12.1.2 Sources of systemic risk

De Bandt and Hartmann (2000) explained that shocks and propagation mechanisms are the two main elements to characterize systemic risk. If we consider our previous analysis, the shock corresponds to the initial stress $\mathcal{S}$ whereas the propagation mechanism indicates the transmission channel $\Longrightarrow$ of this initial shock. It is then useful to classify the several sources of systemic risk depending on the nature of the (systematic) shock or the type of propagation[3].

12.1.2.1 Systematic shocks

Benoit *et al.* (2015) list four main systematic shocks: asset-price bubble risk, correlation risk, leverage risk and tail risk. In what follows, we give their characteristics and some examples. However, even if these risks recover different concepts, they are also highly connected and the boundaries between them are blurred.

Asset-price (or speculative) bubble corresponds to a situation where prices of an asset class rise so sharply that they strongly deviate from their fundamental values[4].The formation of asset bubbles implies that many financial institutions (banks, insurers, asset managers and asset owners) are exposed to the asset class, because they are momentum investors. They prefer to ride the bubble and take advantage of the situation, because being a contrarian investor is a risky strategy[5]. In this context, the probability of crash occurring increases with investors' belief that "*they can sell the asset at an even higher price in the future*" (Brunnermeier and Oehmke, 2013). Examples of speculative bubbles are Japanese asset bubble in the 1980's, the dot.com bubble between 1997 and 2000 and the United States housing bubble before 2007.

Correlation risk means that financial institutions may invest in the same assets at the same time. They are several reasons to this phenomenon. Herd behavior is an important phenomenon in finance (Grinblatt *et al.*, 1995; Wermers, 1999; Acharya and Yorulmazer, 2008). It corresponds to the tendency for mimicking the actions of others. According to Devenow and Welch (1996), "*such herding typically arises either from direct payoff externalities (negative externalities in bank runs; positive externalities in the generation of trading liquidity or in information acquisition), principal-agent problems (based on managerial desire to protect or signal reputation), or informational learning (cascades)*". Another reason that explains correlated investments is the regulation, which may have a high impact on the investment behavior of financial institutions. Examples include the liquidity coverage ratio, national regulations of pension funds, Solvency II, etc. Finally, a third reason is the search of diversification or yield. Indeed, we generally notice a strong enthusiasm for

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[3]Concerning idiosyncratic risks, they are several sources of stress, but they can all be summarized by the default of one system's component.

[4]A bubble can be measured by the price-to-earnings (or P/E) ratio, which is equal to the current share price divided by the earnings per share. For instance, technology stock had an average price-to-earnings ratio larger than 100 in March 2000.

[5]It is extremely difficult for a financial institution to miss the trend from a short-term business perspective and to see the other financial institutions be successful.

an asset class which is is considered as an investment that helps to diversify portfolios or improve their return.

In periods of expansion, we observe an increase of leverage risk, because financial institutions want to benefit from the good times of the business cycle. As the expansion proceeds, investors becomes then more optimistic and the appetite for risky investments and leverage develops[6]. However, a high leverage is an issue in a stressed period, because of the drop of asset prices. Theoretically, the stress $\mathbb{S}$ can not be lower than the opposite of the inverse of the financial institution's leverage ratio $\mathcal{LR}$ in order to maintain its safety:

$$\mathbb{S} \leq -\frac{1}{\mathcal{LR}}$$

For instance, in the case where $\mathcal{LR}$ is equal to 5, the financial institution defaults if the stress is less than -20%. In practice, the stress tolerance depends also on the liquidity constraints. It is then easier to leverage a portfolio in a period of expansion than to deleverage it in a period of crisis, where we generally face liquidity problems. Geanakoplos (2010) explained the downward spiral created by leverage by the amplification mechanism due to the demand of collateral assets[7]. Indeed, decline in asset prices results in asset sales of leveraged investors because of margin call requirements and asset sales results in decline in asset prices. Leverage induces then non-linear and threshold effects that can create systemic risk. The failure of LTCM is a good illustration of leverage risk (Jorion, 2000).

The concept of tail risk suggests that the decline in one asset class is abnormal with respect to the normal risk. This means that the probability to observe a tail event is very small. Generally, the normal risk is measured by the volatility. For instance, an order of magnitude is 20% for the long-term volatility of the equity asset class. The probability to observe an annual drop in equities larger than 40% is equal to 2.3%. An equity market crash can therefore not be assimilated to a tail event. By contrast, an asset class whose volatility is equal to 2.5% will experience a tail risk if the prices are 20% lower than before. In this case, the decrease represents eight times the annual volatility. In Figure 12.1, we have reported these two examples of normal and abnormal risks. When the ratio between the drawdown[8] and the volatility is higher (e.g. larger than 4), this generally indicates the occurrence of a tail risk. The issue with tail risks is that they are rarely observed and financial institutions tend to underestimate them. Acharya *et al.* (2010) even suggested that tail risk investments are sought by financial institutions. Such examples are carry or short volatility strategies. For instance, investing in relatively high credit-quality bonds is typically a tail risk strategy. The rationale is to carry the default risk, to capture the spread and to hope that the default will

[6]This is known as the Minsky's financial instability hypothesis.
[7]see Section 4.3 in page 256.
[8]It is equal to the maximum loss expressed in percent.

never happen. However, the credit crisis in 2007-2008 showed that very low probability events may occur in financial markets.

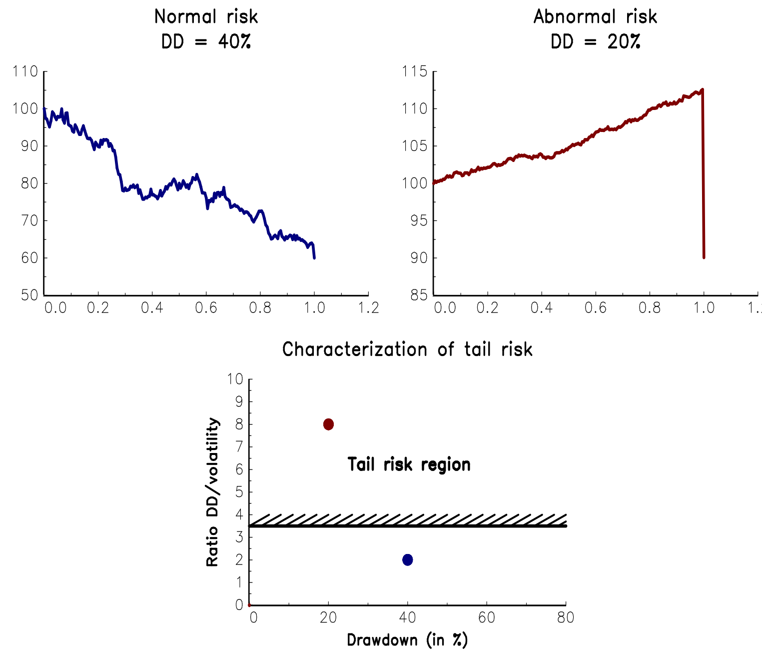

FIGURE 12.1: Illustration of tail risk

The distinction between the four systematic risks is rather artificial and theoretical. In practice, they are highly related. For instance, leverage risk is connected to tail risk. Thus, the carry strategy is generally implemented using leverage. Tail risk is related to bubble risk, which can be partially explained by the correlation risk. In fact, it is extremely difficult to identify a single cause, which defines the zero point of the systemic crisis. Sources of systemic risk are correlated, even between an idiosyncratic event and systematic risks.

12.1.2.2 Propagation mechanisms

As noted by De Bandt and Hartmann (2000), transmission channels of systemic risk are certainly the main element to understand how a systemic crisis happen in an economy. Indeed, propagation mechanisms are more important than the initial (systematic or idiosyncratic) shock, because most of shocks do not produce systemic crisis if they are not spread to the real economy. Among the diversity of propagation mechanisms, academics and regulators have identified three major transmission channels: networks effects, liquidity channel and critical function failure.

Network effects stem from the interconnectedness of financial institutions and can be seen as the system-wide counterpart of an institution's counterparty risk. Network effect is a general term describing the transmission of a systemic shock from one particular entity and market to several entities or markets. In the case of LTCM, systemic risk stemmed from the interconnection between LTCM and the banking system combined with the high leverage strategy pursued by the hedge fund. This created an over sized exposure for the banking system to counterparty credit risk from one single entity. Hence, LTCM's idiosyncratic risk was transferred to the entire financial system and became a source of systemic risk. The early and influential work of Allen and Gale (2000) showed that this source of financial contagion is highly contingent on the network's structure and on the size of the shock. Their model also suggests that a fully connected network might be more resilient than an incomplete network, contradicting the idea that systemic risk increases with average interconnectedness. However, interconnectedness of an individual entity is central to the notion of it being "systemically important". In the banking industry, balance-sheet contagion is an important source of systemic risk and is linked to the counterparty credit risk. The complexity of the banking network can create domino effects and feedback loops, because the failure of one bank is a signal on the health of the other banks. This informational contagion is crucial to understand the freeze of the interbank market during the 2008 financial crisis. Informational contagion is also an important factor of bank runs (Diamond and Dybvig, 1983). However, network effects are not limited to the banking system. Thus, the subprime crisis showed that they concern the different actors of financial system. It was the case with insurance companies[9] and asset managers. In this last case, money market funds (MMF) were notably impacted, forcing some unprecedented measures as the temporary guarantee of money market funds against losses by the U.S. Treasury:

> "*Following the bankruptcy of Lehman Brothers in 2008, a well-known fund – the Reserve Primary Fund – suffered a run due to its holdings of Lehman's commercial paper. This run quickly spread to other funds, triggering investors' redemptions of more than USD 300 billion within a few days of Lehman's bankruptcy. Its consequences appeared so dire to financial stability that the U.S. government decided to intervene by providing unlimited deposit insurance to all money market fund deposits. The intervention was successful in stopping the run but it transferred the entire risk of the USD 3 trillion money market fund industry to the government*" (Kacperczyk and Schnabl, 2013).

Liquidity is another important propagation mechanism of systemic risk. For instance, the Global Financial Crisis can be seen as the superposition of the subprime crisis, affecting primarily the mortgage and credit derivative markets and by extension the global banking system, and a liquidity fund-

[9]The most famous example is the AIG's bailout by the U.S. government in late 2008.

ing crisis following the demise of Lehman Brothers, which affected interbank markets and more broadly the shadow banking system. In this particular case, the liquidity channel caused more stress than the initial systematic event of subprime credit. As shown previously, the concept of liquidity is multi-faceted and recovers various dimensions[10] that are highly connected. In this context, liquidity dry-up events are difficult to predict or anticipate, because they can happen suddenly. This is particular true for the market liquidity with the recent flash crash/rally events[11]. Brunnermeier and Pedersen (2009) demonstrated that a demand shock can create a flight-to-quality environment in which liquidity and loss spirals can arise simply due to funding requirements on speculators such as margin calls and repo haircuts. In some instances, a liquidity dry-up event resulting from a flight-to-quality environment can result in runs, fire sales, and asset liquidations in general transforming the market into a contagion mechanism. This is particularly true if the market size of the early players affected by the shock is large enough to induce a large increase in price pressure. The likelihood and stringency of these spirals is exacerbated by high leverage ratios.

Besides network effects and liquidity-based amplification mechanisms, the third identified transmission channel for systemic risk relates to the specific function a financial institution may come to play in a specific market, either because of its size relative to the market or because of its ownership of a specific skill which makes its services essential to the functioning of that market. De Bandt and Hartmann (2000) identified payment and settlement systems as the main critical function that can generate systemic risk. The development of central counterparties, which is promoted by the recent financial regulation, is a response to mitigate network and counterparty credit risks, but also to strengthen the critical function of clearing systems. Other examples of critical services concern the entire investment chain from the asset manager to the asset owner, for instance securities lending intermediation chains or custody services.

12.1.3 Supervisory policy responses

The strength of the Global Financial Crisis led to massive government interventions around the world to prop up failing financial institutions, seen as "too big too fail". Public concern about the negative externalities of such interventions called pressingly for structural reforms to prevent whenever possible future similar events. The crisis further brought to light, among other key fac-

[10]We recall that the main dimensions are market/funding liquidity, idiosyncratic/systematic liquidity, domestic/global liquidity and inside/outside liquidity (see Chapter 6 in page 311).

[11]Examples are the flash crash of May 6, 2010 (US stock markets), the flash rally of October 15, 2014 (US Treasury bonds), the Swiss Franc move of January 15, 2015 (removal of CHF pleg to EUR) and the market dislocation of August 24, 2015 (stock markets and US ETFs).

tors, the failure of regulation to keep up with the complexity of the activities of global financial institutions. In particular, calls for prudential reforms were made around the world to create mechanisms to monitor, prevent and resolve the liquidation of financial institutions without the need for government intervention. Consequently, a vast program of financial and institutional reforms was undertaken around the world.

12.1.3.1 A new financial regulatory structure

As explained in the introduction of this chapter, the Financial Stability Board is an international oversight institution created in April 2009 to monitor the stability of the global financial system, and not only the activities of banking and insurance industries[12]. Indeed, the 2008 financial crisis also highlighted the increasing reliance of large institutions on the shadow banking system. This refers to the broad range of short-term financing products and activities performed by non-bank actors in the financial markets and therefore historically not subject to the same regulatory supervision as banking activities. This explained that the FSB has also the mandate to oversee the systemic risk induced by shadow banking entities[13]. Besides the analysis of the financial system, the main task of the FSB is the identification of systemically important financial institutions (SIFI). FSB (2010) defines them as institutions whose "*distress or disorderly failure, because of their size, complexity and systemic interconnectedness, would cause significant disruption to the wider financial system and economic activity*". It distinguishes between three types of SIFIs:

1. G-SIBs correspond to global systemically important banks;
2. G-SIIs designate global systemically important insurers;
3. the third category is defined with respect to the two previous ones; it incorporates other SIFIs than banks and insurers (non-bank non-insurer global systemically important financial institutions or NBNI G-SIFIs).

Every year since 2013, the FSB publishes the list of G-SIFIs. In Tables 12.1 and 12.2, we report the 2015 update list of G-SIBs and G-SIIs. At this time, NBNI G-SIFIs are not identified, because the assessment methodology is not achieved[14].

Systemic risk is also monitored at the regional level with the European Systemic Risk Board (ESRB) for the European Union and the Financial Stability Oversight Council (FSOC) for the United States. The ESRB was established

[12]For these two financial sectors, the FSB collaborates with the Basel Committee on Banking Supervision and the International Association of Insurance Supervisors (IAIS).

[13]In this last case, the FSB relies on the works of the International Organization of Securities Commissions (IOSCO).

[14]See the discussion in page 332.

TABLE 12.1: List of global systemically important banks (November 2015)

Agricultural Bank of China	Bank of America	Bank of China
Bank of New York Mellon	Barclays	BNP Paribas
China Construction Bank	Citigroup	Credit Suisse
Deutsche Bank	Goldman Sachs	Crédit Agricole
BPCE	HSBC	ICBC
ING Bank	JP Morgan Chase	Mitsubishi UFJ FG
Mizuho FG	Morgan Stanley	Nordea
Royal Bank of Scotland	Santander	Société Générale
Standard Chartered	State Street	Sumitomo Mitsui FG
UBS	UniCredit	Wells Fargo

Source: FSB (2015), 2015 Update of List of Global Systemically Important Banks.

TABLE 12.2: List of global systemically important insurers (November 2015)

Aegon	Allianz	AIG
Aviva	AXA	MetLife
Ping An Group	Prudential Financial	Prudential plc

Source: FSB (2015), 2015 Update of List of Global Systemically Important Insurers.

on 16 December 2010 and is part of the European System of Financial Supervision (ESFS), the purpose of which is to ensure supervision of the EU financial system[15]. As established under the Dodd-Frank reform (21 July 2010), the FSOC is composed of the Secretary of the Treasury, the Chairman of the Federal Reserve and members of US supervision bodies (CFTC, FDIC, OCC, SEC, etc.).

The Global Financial Crisis had also an impact on the banking supervision structure, in particular in U.S. and Europe. Since 2010, this is the Federal Reserve Board which is in charge to directly supervise large banks and any firm designated as systemically significant by the FSOC (Murphy, 2015). The other banks continue to be supervised by the Federal Deposit Insurance Corporation (FDIC) and the Office of the Comptroller of the Currency (OCC). In Europe, each bank was supervised by its national regulators until the establishment of the Single Supervisory Mechanism (SSM). Starting from 4 November 2014, large European banks are directly supervised by the European Central Bank (ECB), while national supervisors are in a supporting role. This concerns about 120 significant banks and represent 80% of banking assets in the euro

[15]Besides the ESRB, the ESFS comprises the European Banking Authority (EBA), the European Insurance and Occupational Pensions Authority (EIOPA), the European Securities and Markets Authority (ESMA) and the Joint Committee of the European Supervisory Authorities.

area. For each bank regulated by the ECB, a joint supervisory team (JST) is designated. Its main task is to perform the Supervisory Review and Evaluation Process (SREP), propose the supervisory examination programme, implement the approved supervisory decisions and ensure coordination with the on-site inspection teams and liaise with the national supervisors. Public awareness of the systemic risk has also led some countries to reform national supervision structures. For instance in the United Kingdom, the Financial Services Authority (FSA) is replaced in April 2013 by three new supervisory bodies: the Financial Policy Committee (FPC), which is responsible for macro-prudential regulation, the Prudential Regulation Authority (PRA), which is responsible for micro-prudential regulation of financial institutions and the Financial Conduct Authority (FCA), which is responsible for markets regulation.

Remark 48 *The 2008 financial crisis has also impacted other financial sectors than the banking sector, but not to the same degree. Nevertheless, the powers of existing authorities have been expanded in asset management and markets regulation (ESMA, SEC, CFTC). In 2010, the European Insurance and Occupational Pensions Authority (EIOPA) was established in order to ensure a general supervision at the level of the European Union.*

12.1.3.2 A myriad of new standards

Reforms of the financial regulatory framework were also attempted around the world in order to protect the consumers. Thus, the Dodd-Frank Wall Street Reform and Consumer Protection Act was signed into law in the U.S. in July 2010. It is the largest financial regulation overhaul since 1930. Besides the reform of the US financial regulatory structure, it also concerns investment advisers, hedge funds, insurance, central counterparties, credit rating agencies, derivatives, consumer financial protection, mortgages, etc. One of the most famous proposition is the Volcker rule, which prohibits a bank from engaging in proprietary trading and from owning hedge funds and private equity funds. Another controversial proposition is the Lincoln amendment (or swaps push-out rule), which would prohibit federal assistance to swaps entities.

In Europe, directives on the regulations of markets in financial instruments (MiFID 1 and 2) from 2007 to 2014 as well as regulations on packaged retail and insurance-based investment products (PRIIPS) with the introduction of the key information document (KID) in 2014 came to reinforce the regulation and transparency of financial markets and the protection of investors. European Market Infrastructure Regulation (EMIR) is another important European Union regulation, whose aim is to increase the stability of the OTC derivative markets. It introduces reporting obligation for OTC derivatives (trade repositories), clearing obligation for eligible OTC derivatives, independent valuation of OTC derivatives, common rules for central counterparties and post-trading supervisory.

However, the most important reforms concern the banking sector. Many standards of the Basel III Accord are directly related to systemic risk. Capital requirements have been increased to strengthen the safety of banks. The leverage ratio introduces constraints to limit the leverage of banks. The aim of liquidity ratios (LCR and NSFR) is to reduce the liquidity mismatch of banks. Stress testing programs have been highly developed. Another important measure is the designation of systemically important banks[16], which are subject to a capital surcharge ranging from 1% to 4.5%. All these micro-prudential approaches tend to mitigate idiosyncratic factors. However, common factors are also present in the Basel III Accord. Indeed, the Basel Committee has introduced a countercyclical capital buffer in order to increase the capital of banks during excessive credit growth and to limit the impact of common factors on the systemic risk. Another important change is the careful consideration of counterparty credit risk. This includes of course the 1.25 factor to calculate the default correlation $\rho\,(\mathrm{PD})$ in the IRB approach[17], but also the CVA capital charge. The promotion of CCPs since 2010 is also another example to limit network effects and reduce the direct interconnectedness between banks. Last but not least, the stressed VaR of the Basel 2.5 Accord had a strong impact on the capital requirement for market risks.

Remark 49 *Another important reform concerns resolution plans, which describe the banks's strategy for rapid resolution if its financial situation were to deteriorate or if it were to default. In Europe, the Bank Recovery and Resolution Directive (BRRD) applies in all banks and large investment firms since January 2015. In the United States, the orderly liquidation authority (OLA) of the Dodd-Frank Act provides a theoretical framework for bank resolution*[18]*. In Japan, a new resolution regime became effective in March 2014 and ensures a defaulted bank will be resolved via a bridge bank, where certain assets and liabilities are transferred. More recently, the FSB achieves TLAC standard for global systemically important banks. All these initiatives seek to build a framework to resolve a bank failure without public intervention.*

12.2 Systemic risk measurement

They are generally two ways of identify SIFIs. The first one is proposed by supervisors and considers firm-specific information that are linked to the systemic risk, such as the size or the leverage. The second approach has been extensively used by academics and considers market information to measure the impact of the firm-specific default on the entire system.

[16] It concerns both global (G-SIB) and domestic (D-SIB) systemically important banks

[17] See Remark 39 in Page 241.

[18] Bank resolution plans can be found at the following web page: `www.federalreserve.gov/bankinforeg/resolution-plans.htm`.

TABLE 12.3: Scoring system of G-SIBs

	Category		Indicator	Weight
1	Size	1	Total exposures	1/5
2	Interconnectedness	2	Intra-financial system assets	1/15
		3	Intra-financial system liabilities	1/15
		4	Securities outstanding	1/15
3	Substitutability/financial institution infrastructure	5	Payment activity	1/15
		6	Assets under custody	1/15
		7	Underwritten transactions in debt and equity markets	1/15
4	Complexity	8	Notional amount of OTC derivatives	1/15
		9	Trading and AFS securities	1/15
		10	Level 3 assets	1/15
5	Cross-jurisdictional activity	11	Cross-jurisdictional claims	1/10
		12	Cross-jurisdictional liabilities	1/10

12.2.1 The supervisory approach

In what follows, we distinguish between the three categories defined by the FSB: banks, insurers and non-bank non-insurer financial institutions.

12.2.1.1 The G-SIB assessment methodology

In order to measure the systemic risk of a bank, the BCBS (2014g) considers 12 indicators across five large categories. For each indicator, the score of the bank (expressed in basis points) is equal to the bank's indicator value divided by the corresponding sample total[19]

$$\text{Indicator Score} = \frac{\text{Bank Indicator}}{\text{Sample Total}} \times 10^4$$

The indicator scores are then averaged to define the category scores and the final score. The scoring system is summarized in Table 12.3. Each category has a weight of 20% and represents one dimension of systemic risk. The size effect (too big too fail) corresponds to the first category, but is also present in all other categories. Network effects are reflected in Category 2 (interconnectedness) and Category 4 (complexity). The third category measures the degree of critical functions, while the cross-jurisdictional activity tends to identify global banks.

An example of the score computation is given in Table 12.4. It concerns the G-SIB score of BNP Paribas in 2014. Using these figures, the size score is equal to:

$$\text{Score} = \frac{2\,032}{66\,313} = 3.06\%$$

[19]The sample consists of the largest 75 banks defined by the Basel III leverage ratio exposure measure.

TABLE 12.4: An example of calculating the G-SIB score

Category	Indicator	Indicator value(1)	Sample total(1)	Indicator score(2)	Category score(2)
Size	Total exposures	2,032	66,313	306	306
Interconnectedness	Intra-financial system assets	205	7,718	266	370
	Intra-financial system liabilities	435	7,831	556	
	Securities outstanding	314	10,836	290	
Substitutability/financial insitution infrastructure	Payment activity	49,557	1,850,755	268	369
	Assets under custody	4,181	100,012	418	
	Underwritten transactions in debt and equity markets	189	4,487	422	
Complexity	Notional amount of OTC derivatives	39,104	639,988	611	505
	Trading and AFS securities	185	3,311	559	
	Level 3 assets	21	595	346	
Cross-jurisdictional activity	Cross-jurisdictional claims	877	15,801	555	485
	Cross-jurisdictional liabilities	584	14,094	414	
Final score					407

(1)The figures are expressed in billion of EUR.

(2)The figures are expressed in bps.

Source: BCBS (2014), G-SIB Framework: Denominators; BNP Paribas (2014), Disclosure for G-SIBs indicators as of 31 December 2013.

The interconnectedness score is an average of three indicator scores. We obtain:

$$\begin{aligned} \text{Score} &= \frac{1}{3}\left(\frac{205}{7\,718}+\frac{435}{7\,831}+\frac{314}{1\,0836}\right) \\ &= \frac{2.656\%+5.555\%+2.898\%}{3} \\ &= 3.70\% \end{aligned}$$

The final score is an average of the five category scores:

$$\begin{aligned} \text{Score} &= \frac{1}{5}\left(3.06\%+3.70\%+3.69\%+5.05\%+4.85\%\right) \\ &= 4.07\% \end{aligned}$$

Depending on the score value, the bank is then assigned to a specific bucket, which is used to calculate its specific higher loss absorbency (HLA) requirement. The thresholds used to define the buckets are:

1. 130-229 for Bucket 1 (+1.0% CET1);
2. 230-329 for Bucket 2 (+1.5% CET1);
3. 330-429 for Bucket 3 (+2.0% CET1);
4. 430-529 for Bucket 4 (+2.5% CET1);
5. and 530-629 for Bucket 5 (+3.5% CET1).

For instance, the G-SIB score of BNP Paribas was 407 bps. This implies that BNP Paribas belonged to Bucket 3 and the additional buffer was 2% common equity tier 1 at the end of 2014.

In November 2015, the FSB has published the updated list of G-SIBs and the required level of additional loss absorbency. There are no banks in Bucket 5. The two most G-SIBs are HSBC and JPMorgan Chase, which are assigned to Bucket 4 (2.5% of HLA requirement). They are followed by Barclays, BNP Paribas, Citigroup and Deutsche Bank (Bucket 3 and 2.0% of HLA requirement). Bucket 2 is composed of 5 banks (Bank of America, Credit Suisse, Goldman Sachs, Mitsubishi UFJ FG and Morgan Stanley). The 19 remaining banks given in Table 12.1 form Bucket 1.

Remark 50 *The FSB and the BCBS consider a relative measure of the systemic risk. They first select the universe of the* 75 *largest banks and then defines a G-SIB as a bank which has a total score which is higher than the average score*[20]*. This procedure ensures that there are always systemic banks. Indeed, if the score are normally distributed, the number of systemic banks is half the number of banks in the universe. This explains that they found* 30 *G-SIBs among* 75 *banks.*

[20] It is equal to $10^4/75 \approx 133$.

Roncalli and Weisang (2015) reported the average rank correlation (in %) between the five categories for the G-SIBs as of end 2013:

$$\begin{pmatrix} 100.0 & & & & \\ 84.6 & 100.0 & & & \\ 77.7 & 63.3 & 100.0 & & \\ 91.5 & 94.5 & 70.1 & 100.0 & \\ 91.4 & 90.6 & 84.2 & 95.2 & 100.0 \end{pmatrix}$$

We notice the high correlation coefficients[21] between the first (size), second (interconnectedness), fourth (complexity) and fifth categories (cross-jurisdictional activity). This is not surprising that G-SIBs are the largest banks in the world. In fact, the high correlation between the five measures masks the multifaceted reality of systemic risk. This is explained by the homogeneous nature of global systemically important banks in terms of their business model. Indeed, almost all these financial institutions are universal banks mixing both commercial and investment banking.

Besides the HLA requirement, the FSB in consultation with the BCBS has published in November 2015 its proposed minimum standard for "total loss – absorbing capacity" (TLAC). According to FSB (2015d), "*the TLAC standard has been designed so that failing G-SIBs will have sufficient loss-absorbing and recapitalization capacity available in resolution for authorities to implement an orderly resolution that minimizes impacts on financial stability, maintains the continuity of critical functions, and avoids exposing public funds to loss*". In this context, TLAC requirements would be between 8% to 12%. This means that the total capital would be between 19.5% and 25% of RWA for G-SIBs[22] as indicated in Figure 12.2.

12.2.1.2 Identification of G-SIIs

In the case of insurers, the International Association of Insurance Supervisors (IAIS) has developed an approach similar to the Basel Committee's to measure global systemically important insurers (or G-SIIs). The final score is an average of five category scores: size, interconnectedness, substitutability, non-traditional and non-insurance (NTNI) activities and global activity. Contrary to the G-SIB scoring system, the G-SII scoring system does not use an equal weight between the category scores. Thus, a 5% weight is applied to size, substitutability and global activity, whereas interconnectedness and NTNI activities represent respectively 40% and 45% of weighting. In fact, the score highly depends on the banking activities (derivatives trading, short term funding, guarantees,etc.) of the insurance company[23].

[21]The highest correlation is between Category 4 and Category 5 (95.2%) whereas the lowest correlation is between Category 2 and Category 3 (63.3%).

[22]Using Table 1.5 in page 24, we deduce that the total capital is equal to 6% of Tier 1 plus 2% of Tier 2 plus 2.5% of conservation buffer (CB) plus 1% − 3.5% of systemic buffer (HLA) plus 8% − 12% of TLAC.

[23]See IAIS (2013a) in page 20.

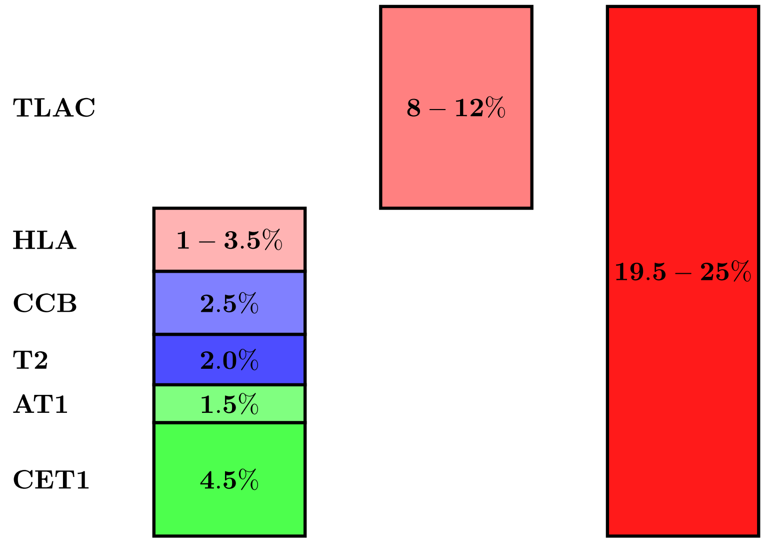

FIGURE 12.2: Impact of the TLAC on capital requirements

12.2.1.3 Extension to NBNI SIFIs

In March 2015, the FSB published a second consultation document, which proposed a methodology for the identification of NBNI SIFIs. The concerned financial sectors were finance companies, market intermediaries, asset managers and their funds. The scoring system was an imitation of the G-SII scoring system with the same 5 categories. As noted by Roncalli and Weisang (2015), this scoring system was not satisfying, because it failed to capture the most important systemic risk of these financial institutions, which is the liquidity risk. Indeed, a large amount of redemptions may create fire sales and affect the liquidity of the underlying market. This liquidity mainly depends on the asset class. For instance, we do not face the same risk when investing in an equity fund and in a bond fund. Finally, the FSB has decided to postpone the assessment framework for NBNI G-SIFIs and to work specifically on financial stability risks from asset management activities.

12.2.2 The academic approach

Academics propose various methods to measure the systemic risk. Even if they are heterogenous, most of them share a common pattern. They are generally based on publicly market data[24]. Among these different approaches, three prominent measures are particularly popular: the marginal expected shortfall

[24]The reason is that academics do not have access to regulatory or private data.

(MES), the delta conditional value-at-risk (Δ CoVaR) and the systemic risk measure (SRISK).

Remark 51 *In what follows, we define the different systemic risk measures and derive their expression in the Gaussian case. Non-Gaussian and non-parametric estimation methods will be presented in Chapter 14.*

12.2.2.1 Marginal expected shortfall

This measure has been proposed by Acharya *et al.* (2010). Let w_i and L_i be the exposure of the system to Institution i and the corresponding normalized random loss. We note $w = (w_1, \ldots, w_n)$ the vector of exposures. The loss of the system is equal to:

$$L(w) = \sum_{i=1}^{n} w_i L_i$$

We recall that the expected shortfall $\mathrm{ES}_\alpha(w)$ with confidence level α is the expected loss conditional that the loss is higher than the value-at-risk $\mathrm{VaR}_\alpha(w)$:

$$\mathrm{ES}_\alpha(w) = \mathbb{E}\left[L \mid L \geq \mathrm{VaR}_\alpha(w)\right]$$

The marginal expected shortfall of Institution i is then equal to:

$$\mathrm{MES}_i = \frac{\partial\, \mathrm{ES}_\alpha(w)}{\partial\, w_i} = \mathbb{E}\left[L_i \mid L \geq \mathrm{VaR}_\alpha(w)\right] \tag{12.3}$$

In the Gaussian case $(L_1, \ldots, L_n) \sim \mathcal{N}(\mu, \Sigma)$, we have found that[25]:

$$\mathrm{MES}_i = \mu_i + \frac{\phi\left(\Phi^{-1}(\alpha)\right)}{(1-\alpha)\sqrt{w^\top \Sigma w}} (\Sigma w)_i$$

Another expression of MES is then:

$$\mathrm{MES}_i = \mu_i + \beta_i(w) \times \left(\mathrm{ES}_\alpha(w) - \mathbb{E}(L)\right) \tag{12.4}$$

where $\beta_i(w)$ is the beta of the institution loss with respect to the total loss:

$$\beta_i(w) = \frac{\mathrm{cov}(L, L_i)}{\sigma^2(L)} = \frac{(\Sigma w)_i}{w^\top \Sigma w}$$

Acharya *et al.* (2010) approximated the MES measure as the expected value of the stock return R_i when the return R^{mkt} of the market portfolio is below the 5% quantile:

$$\mathrm{MES}_i = -\mathbb{E}\left[R_i \mid R^{\mathrm{mkt}} \leq \mathbf{F}^{-1}(5\%)\right]$$

[25] See Equation (2.15) in page 117.

where $\mathbf{F}$ is the cumulative distribution function of the market return R^{mkt}. We have:

$$\mathrm{MES}_i = -\frac{1}{\mathrm{card}\left(\mathbb{T}\right)}\sum_{t\in\mathbb{T}} R_{i,t}$$

where $\mathbb{T}$ represents the set of trading days, which corresponds to the 5% worst days for the market returns. Another way of implementing the MES measure is to specify the components of the system and the confidence level α for defining the conditional expectation. For instance, the system can be defined as a set of the largest banks and w_i is the size of Bank i (measured by the market capitalization or the total amount of assets).

Example 49 *We consider a system composed of 3 banks. The total assets managed by these banks are respectively equal to* \$139, \$75 *and* \$81 *bn. We assume that the annual normalized losses are Gaussian. Their means are equal to zero whereas their standard deviations are set equal to* 10%, 12% *and* 15%. *The correlations are given by the following matrix:*

$$\rho = \left(\begin{array}{rrr} 100\% & & \\ 75\% & 100\% & \\ 82\% & 85\% & 100\% \end{array}\right)$$

By considering a 95% confidence level, the value-at-risk of the system is equal to \$53.86 bn. Using the analytical results given in Section 2.3 in page 111, we deduce that the systemic expected shortfall $\mathrm{ES}_{95\%}$ of the entire system reaches the amount of \$67.55 bn. Finally, we calculate the MES and obtain the values reported in Table 12.5. The MES is expressed in %. This means that if the total assets managed by the first bank increases by \$1 bn, the systemic expected shortfall will increase by \$0.19 bn. In the third column of the table, we have indicated the risk contribution $\mathcal{RC}_i$, which is the product of the size w_i and the marginal expected shortfall MES_i. This quantity is also called the systemic expected shortfall of Institution i:

$$\mathrm{SES}_i = \mathcal{RC}_i = w_i \times \mathrm{MES}_i$$

We have also reported the beta coefficient $\beta_i\left(w\right)$ (expressed in bps). Because we have $\mu_i = 0$, we verify that the marginal expected shortfall is equal to the beta times the systemic expected shortfall.

The marginal expected shortfall can be used to rank the relative systemic risk of a set of financial institutions. For instance, in the previous example, this is the third bank that is the most risky according to the MES. However, the first bank, which has the lowest MES value, has the highest systemic expected shortfall, because its size is larger than the two other banks. This is why we must not confuse the relative (or marginal) risk and the absolute risk.

The marginal expected shortfall has been criticized because it measures the systematic risk of a financial institution, and not necessarily its systemic risk. In Table 12.5, we give the traditional beta coefficient $\beta_i\left(w^\star\right)$, which is

TABLE 12.5: Risk decomposition of the 95% systemic expected shortfall

Bank	w_i (in \$ bn)	MES_i (in %)	SES_i (in \$ bn)	$\beta_i(w)$ (in bps)	$\beta_i(w^\star)$
1	139	19.28	26.80	28.55	0.84
2	75	22.49	16.87	33.29	0.98
3	81	29.48	23.88	43.64	1.29
ES_α			67.55		

calculated with respect to the relative weights $w_i^\star = w_i / \sum_{j=1}^n w_j$. As already shown in Equation (12.4), ranking the financial institutions by their MES is equivalent to rank them by their beta coefficients. In practice, we can nevertheless observe some minor differences because stock returns are not exactly Gaussian.

12.2.2.2 Delta conditional value-at-risk

Adrian and Brunnermeier (2015) define the CoVaR as the value-at-risk of the system conditional on some event $\mathcal{E}_i$ of Institution i:

$$\Pr\left\{L(w) \geq \text{CoVaR}_i(\mathcal{E}_i)\right\} = \alpha$$

Adrian and Brunnermeier determine the risk contribution of Institution i as the difference between the CoVaR conditional on the institution being in distressed situation and the CoVaR conditional on the institution being in normal situation:

$$\Delta\,\text{CoVaR}_i = \text{CoVaR}_i(\mathcal{D}_i = 1) - \text{CoVaR}_i(\mathcal{D}_i = 0)$$

where $\mathcal{D}_i$ indicates if the bank is in distressed situation or not. Adrian and Brunnermeier use the value-at-risk to characterize the distress situation:

$$\mathcal{D}_i = 1 \Leftrightarrow L_i = \text{VaR}_\alpha(L_i)$$

whereas the normal situation corresponds to the case when the loss of Institution i is equal to its median[26]:

$$\mathcal{D}_i = 0 \Leftrightarrow L_i = m(L_i)$$

Finally, we obtain:

$$\Delta\,\text{CoVaR}_i = \text{CoVaR}_i(L_i = \text{VaR}_\alpha(L_i)) - \text{CoVaR}_i(L_i = m(L_i)) \tag{12.5}$$

In the Gaussian case and using the previous notations, we have:

$$\left(\begin{array}{c} L_i \\ L(w) \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_i \\ w^\top \mu \end{array}\right), \left(\begin{array}{cc} \sigma_i^2 & (\Sigma w)_i \\ (\Sigma w)_i & w^\top \Sigma w \end{array}\right)\right)$$

[26] In this case, we have $m(L_i) = \text{VaR}_{50\%}(L_i)$.

We deduce that[27]:

$$L(w) \mid L_i = \ell_i \sim \mathcal{N}\left(\mu(\ell_i), \sigma^2(\ell_i)\right)$$

with:

$$\mu(\ell_i) = w^\top \mu + \frac{(\ell_i - \mu_i)}{\sigma_i^2} (\Sigma w)_i$$

and:

$$\sigma^2(\ell_i) = w^\top \Sigma w - \frac{(\Sigma w)_i^2}{\sigma_i^2}$$

It follows that:

$$\begin{aligned}
\mathrm{CoVaR}_i(L_i = \ell) &= \mu(\ell_i) + \Phi^{-1}(\alpha)\,\sigma(\ell_i) \\
&= w^\top \mu + \frac{(\ell_i - \mu_i)}{\sigma_i^2}(\Sigma w)_i + \Phi^{-1}(\alpha)\sqrt{w^\top \Sigma w - \frac{(\Sigma w)_i^2}{\sigma_i^2}}
\end{aligned}$$

Because $\mathrm{VaR}_\alpha(L_i) = \mu_i + \Phi^{-1}(\alpha)\,\sigma_i$ and $m(L_i) = \mathbb{E}(L_i) = \mu_i$, we obtain:

$$\begin{aligned}
\Delta\,\mathrm{CoVaR}_i &= \mathrm{CoVaR}_i\left(L_i = \mu_i + \Phi^{-1}(\alpha)\,\sigma_i\right) - \mathrm{CoVaR}_i(L_i = \mu_i) \\
&= \Phi^{-1}(\alpha) \times \frac{(\Sigma w)_i}{\sigma_i} \\
&= \Phi^{-1}(\alpha) \times \sum_{j=1}^n w_j \rho_{i,j} \sigma_j
\end{aligned}$$

Another expression of $\Delta\,\mathrm{CoVaR}_i$ is:

$$\Delta\,\mathrm{CoVaR}_i = \Phi^{-1}(\alpha) \times \sigma^2(L) \times \frac{\beta_i(w)}{\sigma_i} \tag{12.6}$$

The Gaussian case highlights different properties of the CoVaR measure:

- If the losses are independent meaning that $\rho_{i,j} = 0$, the Delta CoVaR is the unexpected loss, which is the difference between the nominal value-at-risk and the nominal median (or expected) loss:

$$\begin{aligned}
\Delta\,\mathrm{CoVaR}_i &= \Phi^{-1}(\alpha) \times w_i \times \sigma_i \\
&= w_i \times (\mathrm{VaR}_\alpha(L_i) - m(L_i)) \\
&= w_i \times \mathrm{UL}_\alpha(L_i)
\end{aligned}$$

- If the losses are perfectly dependent meaning that $\rho_{i,j} = 1$, the Delta

[27]We use results of the conditional expectation given in Appendix A.2.2.3 in page 456.

CoVaR is the sum over all the financial institutions of the unexpected losses:

$$\begin{aligned}\Delta\,\mathrm{CoVaR}_i &= \Phi^{-1}(\alpha)\times\sum_{j=1}^{n} w_j\sigma_j \\ &= \sum_{j=1}^{n} w_j \times \mathrm{UL}_\alpha(L_j)\end{aligned}$$

In this case, the Delta CoVaR measure does not depend on the financial institution.

- The sum of all Delta CoVaRs is a weighted average of the unexpected losses:

$$\begin{aligned}\sum_{i=1}^{n}\Delta\,\mathrm{CoVaR}_i &= \Phi^{-1}(\alpha)\times\sum_{i=1}^{n}\sum_{j=1}^{n} w_j\rho_{i,j}\sigma_j \\ &= \Phi^{-1}(\alpha)\times\sum_{j=1}^{n} w_j\sigma_j\sum_{i=1}^{n}\rho_{i,j} \\ &= n\sum_{j=1}^{n}\bar{\rho}_j\times w_j\times\mathrm{UL}_\alpha(L_j)\end{aligned}$$

where $\bar{\rho}_j$ is the average correlation between institution j and the other institutions (including itself). This quantity has no financial interpretation and is not a coherent risk measure, which satisfies the Euler principle.

Remark 52 *In practice, losses are approximated by stock returns. Empirical results show that MES and CoVaR measures may give different rankings. This can be easily explained in the Gaussian case. Indeed, measuring systemic risk with MES is equivalent to analyze the beta of each financial institution whereas the CoVaR approach consists of ranking them by their beta divided by their volatility. If the beta coefficients are very close, the CoVaR ranking will be highly sensitive to the volatility of the financial institution's stock.*

We consider Example 49 and report in Table 12.6 the calculation of the 95% CoVaR measure. If Bank 1 suffers a loss larger than its 95% value-at-risk (\$22.86 bn), it induces a Delta CoVaR of \$50.35 bn. This systemic loss includes the initial loss of Bank 1, but also additional losses of the other banks due to their interconnectedness. We notice that CoVaR and MES produce the same ranking for this example. However, if we define the systemic risk as the additional loss on the other components of the system[28], we find that the stress on Bank 2 induces the largest loss on the other banks[29].

[28]This additional loss is equal to $\mathrm{CoVaR}_i - w_i \times \mathrm{VaR}_\alpha(L_i)$.

[29]The additional loss (expressed in \$ bn) is equal to 27.49 for Bank 1, 34.13 for Bank 2 and 31.33 for Bank 3.

TABLE 12.6: Calculation of the 95% CoVaR measure

Bank	w_i (in \$ bn)	$\text{VaR}_\alpha(L_i)$ (in %)	(in \$ bn)	$\text{CoVaR}_i(\mathcal{E})$ $\mathcal{D}_i = 1$	$\mathcal{D}_i = 0$	$\Delta\,\text{CoVaR}_i$ (in \$ bn)
1	139	16.45	22.86	69.48	19.13	50.35
2	75	19.74	14.80	71.44	22.50	48.94
3	81	24.67	19.98	67.69	16.37	51.32

The dependence function between financial institutions is very important when calculating the CoVaR measure. For instance, we consider again Example 49 with a constant correlation matrix. In Figure 12.3, we represent the relationship between $\Delta\,\text{CoVaR}_i$ and the uniform correlation ρ. When losses are independent, we obtain the value-at-risk of each bank. When losses are comonotonic, $\Delta\,\text{CoVaR}_i$ is the sum of the VaRs. Because losses are perfectly correlated, a stress on one bank is entirely transmitted to the other banks.

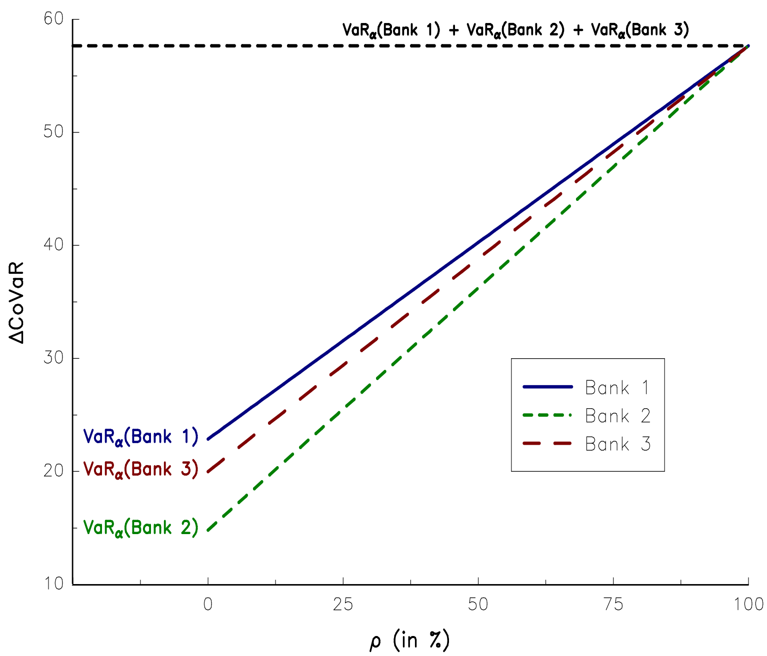

FIGURE 12.3: Impact of the uniform correlation on $\Delta\,\text{CoVaR}_i$

12.2.2.3 Systemic risk measure

Another popular risk measure is the systemic risk measure (SRISK) proposed by Acharya *at al.* (2012), which is a new form of the systemic expected shortfall of Acharya *at al.* (2010) and which was originally developed by Brownlees and Engle (2015) in 2010. Using a stylized balance sheet, the capital shortfall $\mathrm{CS}_{i,t}$ of Institution i at time t is the difference between the required capital $\mathcal{K}_{i,t}$ and the market value of equity $V_{i,t}$:

$$\mathrm{CS}_{i,t} = \mathcal{K}_{i,t} - V_{i,t}$$

We assume that $\mathcal{K}_{i,t}$ is equal to $kA_{i,t}$ where $A_{i,t}$ is the asset value and k is the capital ratio (typically 8% in Basel II). We also have $A_{i,t} = D_{i,t} + V_{i,t}$ where $D_{i,t}$ represents the debt value[30]. We deduce that:

$$\begin{aligned}\mathrm{CS}_{i,t} &= k\left(D_{i,t} + V_{i,t}\right) - V_{i,t} \\ &= kD_{i,t} - (1-k)\, V_{i,t}\end{aligned}$$

We define the capital shortfall of the system as the total amount of capital shortfall $\mathrm{CS}_{i,t}$:

$$\mathrm{CS}_t = \sum_{i=1}^{n} \mathrm{CS}_{i,t}$$

Acharya *et al.* (2012) define the amount of systemic risk as the expected value of the capital shortfall conditional to a systemic stress $\mathbb{S}$:

$$\begin{aligned}\mathrm{SRISK}_t &= \mathbb{E}\left[\left.\mathrm{CS}_{t+1}\right|\mathbb{S}\right] \\ &= \mathbb{E}\left[\left.\sum_{i=1}^{n} \mathrm{CS}_{i,t+1}\right|\mathbb{S}\right] \\ &= \sum_{i=1}^{n} k\mathbb{E}\left[\left.D_{i,t+1}\right|\mathbb{S}\right] - (1-k)\,\mathbb{E}\left[\left.V_{i,t+1}\right|\mathbb{S}\right]\end{aligned}$$

They also assume that $\mathbb{E}\left[\left.D_{i,t+1}\right|\mathbb{S}\right] \approx D_{i,t}$ and:

$$\mathbb{E}\left[\left.V_{i,t+1}\right|\mathbb{S}\right] = \left(1 - \mathrm{MES}_{i,t}\right) V_{i,t}$$

where $\mathrm{MES}_{i,t}$ is the marginal expected shortfall conditional to the systemic risk $\mathbb{S}$. By using the leverage ratio $\mathcal{LR}_{i,t}$ defined as the asset value divided by the market value of equity:

$$\mathcal{LR}_{i,t} = \frac{A_{i,t}}{V_{i,t}} = 1 + \frac{D_{i,t}}{V_{i,t}},$$

they finally obtain the following expression of the systemic risk[31]:

$$\mathrm{SRISK}_t = \sum_{i=1}^{n} \left(k\left(\mathcal{LR}_{i,t} - 1\right) - (1-k)\left(1 - \mathrm{MES}_{i,t}\right)\right) \times V_{i,t}$$

[30]Here, we assume that the bank capital is equal to the market value, which is not the case in practice.

[31]We have $D_{i,t} = \left(\mathcal{LR}_{i,t} - 1\right) V_{i,t}$.

We notice that the systemic risk can be decomposed as the sum of the risk contributions $\mathrm{SRISK}_{i,t}$. We have:

$$\mathrm{SRISK}_{i,t} = \vartheta_{i,t} \times V_{i,t} \tag{12.7}$$

with:

$$\vartheta_{i,t} = k\mathcal{LR}_{i,t} + (1-k)\,\mathrm{MES}_{i,t} - 1 \tag{12.8}$$

In these two formulas, k and $\mathrm{MES}_{i,t}$ are expressed in % while $\mathrm{SRISK}_{i,t}$ and $V_{i,t}$ are measured in \$. $\mathrm{SRISK}_{i,t}$ is then a linear function of the market capitalization $V_{i,t}$, which is a proxy of the capital in this model. The scaling factor $\vartheta_{i,t}$ depends on 4 parameters:

1. k is the capital ratio. In the model, we have $\mathcal{K}_{i,t} = kA_{i,t}$ whereas the capital $\mathcal{K}_{i,t}$ is equal to $k\,\mathrm{RWA}_{i,t}$ in Basel Accords. Under some assumptions, k can be set equal to 8% in the Basel I or Basel II framework. For Basel III and Basel IV, we must use a higher value, especially for SIFIs.

2. $\mathcal{LR}_{i,t}$ is the leverage ratio of Institution i. The higher the leverage ratio, the higher the systemic risk.

3. The systemic risk is an increasing function of the marginal expected shortfall. Because we have $\mathrm{MES}_{i,t} \in [0,1]$, we deduce that:

$$(k\mathcal{LR}_{i,t} - 1) \times V_{i,t} \leq \mathrm{SRISK}_{i,t} \leq k\,(\mathcal{LR}_{i,t} - 1) \times V_{i,t}$$

   A high value of the MES decreases the market value of equity, and then the absorbency capacity of systemic losses.

4. The marginal expected shortfall depends on the stress scenario. In the different publications on the SRISK measure, the stress $\mathbb{S}$ generally corresponds to a 40% drop of the equity market:

$$\mathrm{MES}_{i,t} = -\mathbb{E}\left[R_{i,t+1} \mid R_{t+1}^{\mathrm{mkt}} \leq -40\%\right]$$

Example 50 *We consider a universe of 4 banks, whose characteristics are given in the table below*[32]*:*

Bank	$V_{i,t}$	$\mathcal{LR}_{i,t}$	μ_i	σ_i	$\rho_{i,\mathrm{mkt}}$
1	57	23	0%	25%	70%
2	65	28	0%	24%	75%
3	91	13	0%	22%	68%
4	120	20	0%	20%	65%

We assume that the expected return μ_{mkt} *and the volatility* σ_{mkt} *of the equity market are* 0% *and* 17%*.*

[32] The market capitalization $V_{i,t}$ is expressed in \$ bn.

Using the conditional expectation formula, we have:

$$\mathbb{E}\left[R_{i,t+1} \mid R_{t+1}^{\text{mkt}} = \mathbb{S}\right] = \mu_i + \rho_{i,\text{mkt}} \frac{(\mathbb{S} - \mu_{\text{mkt}})}{\sigma_{\text{mkt}}} \sigma_i$$

We can then calculate the marginal expected shortfall and deduce the scaling factor and the systemic risk contribution thanks to Equations (12.7) and (12.8). Results are given in Table 12.7. In this example, the main contributors are Bank 2 because of its high leverage ratio followed by Bank 4 because of its high market capitalization. In Table 12.8, we show how the SRISK measure changes with respect to the stress $\mathbb{S}$.

TABLE 12.7: Calculation of the SRISK measure ($\mathbb{S} = -40\%$)

Bank	$\text{MES}_{i,t}$ (in %)	$\vartheta_{i,t}$	$\text{SRISK}_{i,t}$ (in \$ bn)	$\text{SRISK}_{i,t}$ (in %)
1	41.18	1.22	69.47	22.11
2	42.35	1.63	105.93	33.70
3	35.20	0.36	33.11	10.54
4	30.59	0.88	105.77	33.65

TABLE 12.8: Impact of the stress $\mathbb{S}$ on SRISK

Bank	$\mathbb{S} = -20\%$		$\mathbb{S} = -40\%$		$\mathbb{S} = -60\%$	
	(in \$ bn)	(in %)	(in \$ bn)	(in %)	(in \$ bn)	(in %)
1	58.7	22.6	69.5	22.1	80.3	21.7
2	93.3	36.0	105.9	33.7	118.6	32.1
3	18.4	7.1	33.1	10.5	47.8	13.0
4	88.9	34.3	105.8	33.7	122.7	33.2

According to Acharya *et al.* (2012), the most important SIFIs in the United States were Bank of America, JP Morgan Chase, Citigroup and Goldman Sachs in 2012. They also noticed that 4 insurance companies were also in the top 10 (MetLife, Prudential Financial, AIG and Hertford Financial). Engle *et al.* (2015) conducted the same exercise on European institutions with the same methodology. They found that the five most important SIFIs in Europe were Deutsche Bank, Crédit Agricole, Barclays, Royal Bank of Scotland and BNP Paribas. Curiously, HSBC was only ranked at the 15$^{\text{th}}$ place and the first insurance company (AXA) was 16$^{\text{th}}$. This ranking system is updated in a daily basis by the Volatility Institute at New York University[33]. In Tables 12.9, 12.10 and 12.11, we report the 10 largest systemic risk contributions by regions at the end of November 2015. The ranking within a region seems to be coherent, but the difference in the magnitude of SRISK between American, European and Asian financial institutions is an issue.

[33] The internet web page is `vlab.stern.nyu.edu`.

TABLE 12.9: Systemic risk contributions in America (2015-11-27)

Rank	Institution	$\text{SRISK}_{i,t}$ (in \$ bn)	(in %)	$\text{MES}_{i,t}$ (in %)	$\mathcal{LR}_{i,t}$
1	Bank of America	49.7	10.75	2.75	11.42
2	Citigroup	44.0	9.52	3.23	10.83
3	JP Morgan Chase	42.6	9.22	3.09	9.74
4	Prudential Financial	37.6	8.13	3.07	19.64
5	MetLife	33.9	7.33	2.85	15.40
6	Morgan Stanley	28.6	6.20	3.50	12.60
7	Banco do Brasil	24.1	5.22	4.00	29.45
8	Goldman Sachs	20.3	4.38	3.21	10.51
9	Manulife Financial	20.1	4.36	3.43	15.04
10	Power Corp of Canada	16.2	3.50	2.82	26.81

TABLE 12.10: Systemic risk contributions in Europe (2015-11-27)

Rank	Institution	$\text{SRISK}_{i,t}$ (in \$ bn)	(in %)	$\text{MES}_{i,t}$ (in %)	$\mathcal{LR}_{i,t}$
1	BNP Paribas	94.1	8.63	3.42	33.41
2	Crédit Agricole	88.1	8.09	4.22	59.34
3	Barclays	86.3	7.92	4.31	36.60
4	Deutsche Bank	86.1	7.90	4.32	53.61
5	Société Générale	61.3	5.63	3.85	38.74
6	Royal Bank of Scotland	39.5	3.63	3.15	24.23
7	Banco Santander	38.3	3.51	3.79	18.57
8	HSBC	34.5	3.16	2.49	15.96
9	UniCredit	33.1	3.04	3.58	27.21
10	London Stock Exchange	31.3	2.87	2.90	52.67

TABLE 12.11: Systemic risk contributions in Asia (2015-11-27)

Rank	Institution	$\text{SRISK}_{i,t}$ (in \$ bn)	(in %)	$\text{MES}_{i,t}$ (in %)	$\mathcal{LR}_{i,t}$
1	Mitsubishi UFJ FG	121.5	9.45	2.41	24.80
2	China Construction Bank	117.3	9.12	2.61	17.01
3	Bank of China	94.5	7.35	2.53	15.21
4	Mizuho FG	93.7	7.29	2.10	31.84
5	Agricultural Bank of China	92.0	7.16	0.66	19.20
6	Sumitomo Mitsui FG	85.7	6.67	2.71	26.99
7	ICBC	58.4	4.54	0.84	13.80
8	Bank of Communications	45.0	3.50	2.47	16.89
9	Industrial Bank	29.4	2.29	1.38	17.94
10	National Australia Bank	27.4	2.13	3.27	13.48

Remark 53 *The main drawback of the model is that* $\mathrm{SRISK}_{i,t}$ *is very sensitive to the market capitalization with two effects. The direct effect (*$\mathrm{SRISK}_{i,t} = \vartheta_{i,t} \times V_{i,t}$*) implies that the systemic risk is reduced when the equity market is stressed, whereas the indirect effect due to the leverage ratio increases the systemic risk. When we analyze simultaneous the two effects, the first effect is greater. However, we generally observe an increase of the SRISK, because the marginal expected shortfall is much higher in crisis periods.*

12.2.2.4 Network measures

- Billio *et al.* (2012).
- Cont *et al.* (2013).
- Acemoglu *et al.* (2015).

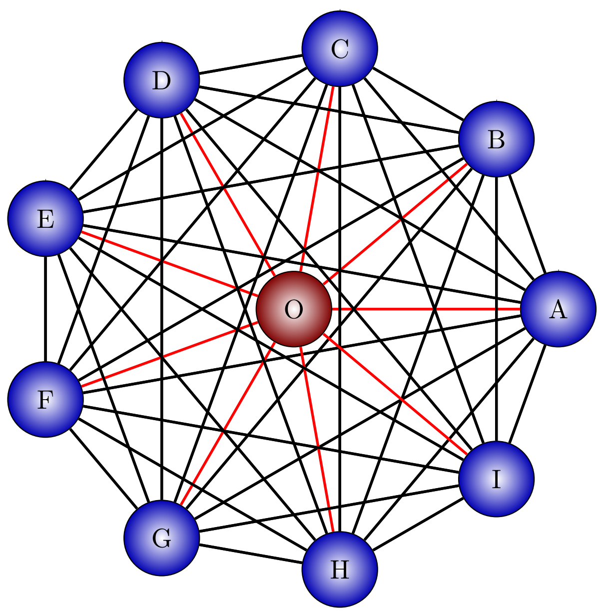

FIGURE 12.4: A completely connected network

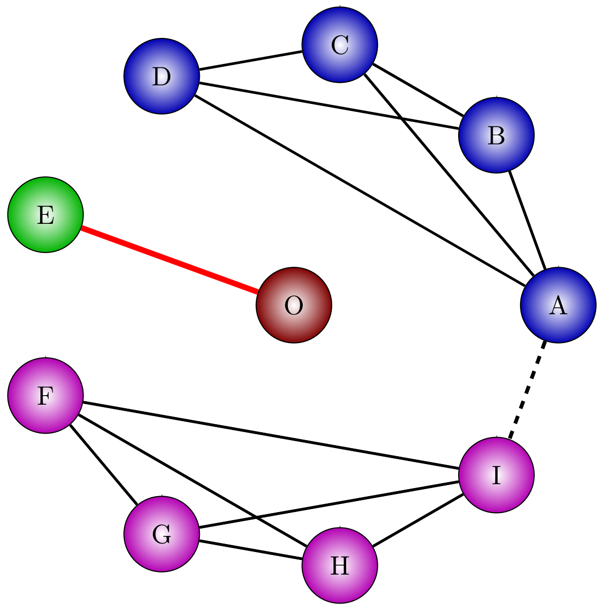

FIGURE 12.5: A sparse network

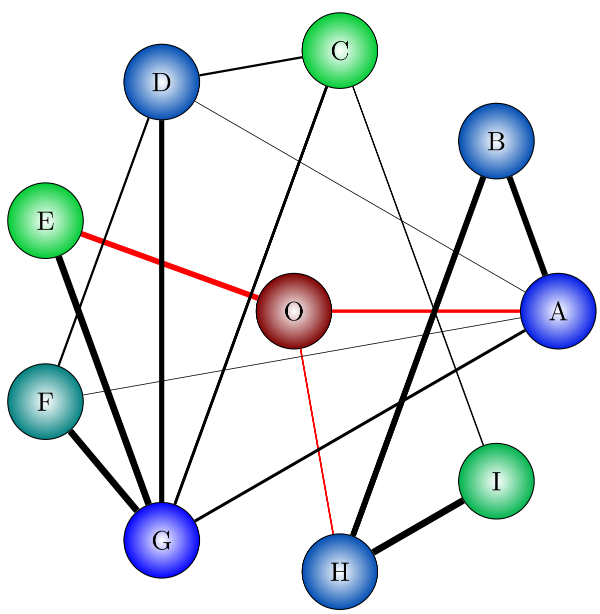

FIGURE 12.6: A partially dense network

12.3 Shadow banking system

This section on the shadow banking has been included in this chapter together with systemic risk, because we will see that they are highly connected.

12.3.1 Definition

The shadow banking system (SBS) can be defined as financial entities or activities involved in credit intermediation outside the regular banking system (FSB, 2011; IMF, 2014). This non-bank credit intermediation complements banking credit, but is not subject to the same regulatory framework. Another important difference is that "*shadow banks are financial intermediaries that conduct maturity, credit, and liquidity transformation without access to central bank liquidity or public sector credit guarantees*" (Pozsar *et al.*, 2013. In this context, shadow banks can raise similar systemic risk issues than regular banks in terms of liquidity, leverage and asset-liability mismatch risks.

However, the main characteristic of shadow banking risk is certainly the high interconnectedness within shadow banks and with the banking system. If we describe the shadow banking system in terms of financial entities, it includes finance companies, broker-dealers and asset managers, whose activities are essential for the functioning of the banking system. If we focus on instruments, the shadow banking corresponds to short-term debt securities that are critical for banks' funding. In particular, this concerns money and repo markets. These linkages between the two systems can then create spillover risks, because stress in the shadow banking system may be transmitted to the rest of the financial system (IMF, 2014). For instance, run risk in shadow banking is certainly the main source of spillover effects and the highest concern of systemic risk. The case of money market funds during the 2008 financial crisis is a good example of the participation of the shadow banking to systemic risk. This dramatic episode also highlights agency and moral hazard problems. Credit risk transfer using asset-backed commercial paper (ABCP) and structured investment vehicles (SIV) is not always transparent for investors of money market funds. This opacity risk increases redemption risk during periods of stress (IMF, 2014). This led the Federal Reserve to introduce the ABCP money market mutual fund liquidity facility (AMLF) between September 2008 and February 2010 in order to support MMFs.

Concepts of shadow banking and NBNI SIFI are very close. To date, the focus was more on financial entities that can be assimilated to shadow banks or systemic institutions. More recently, we observe a refocusing on instruments and activities. These two approaches go together when measuring the shadow banking.

12.3.2 Measuring the shadow banking

The FSB (2015e) define two measures of the shadow banking system. The broad measure considers all the assets of non-bank financial institutions, while the narrow measure only considers the assets that are part of the credit intermediation chain.

12.3.2.1 The broad measure

The broad measure corresponds to the amount of financial assets held by insurance companies, pension funds and other financial intermediaries (OFI). OFIs comprise all financial institutions that are not central banks (CB), banks, insurance companies (IC), pension funds (PF), public financial institutions (PFI) or financial auxiliaries (FA). This broad measure is also called the MUNFI[34] measure of assets. Table 12.12 shows the amount of assets managed by financial institutions and listed in the 2015 monitoring exercise[35]. Assets rose from \$126.6 tn in 2002 to \$316.1 tn in 2014. This growth is explained by an increase in all financial sectors. In 2014, the MUNFI measure is equal to \$137.0 tn with the following repartition: \$28.0 tn for insurance companies (20.4%), \$29.2 tn for pension funds (21.3%) and \$79.8 tn for other financial intermediaries (58.3%). The MUNFI measure is then comparable to banks' assets, which are equal to \$142.2 tn in 2014.

TABLE 12.12: Assets of financial institutions (in \$ tn)

Year	CB	Banks	PFI FA	IC	PF	OFI	Total	MUNFI	(in %)
2002	4.8	53.8	11.5	14.5	12.0	30.0	126.6	56.5	44.6%
2003	5.6	67.0	12.5	17.5	13.8	36.9	153.2	68.2	44.5%
2004	6.5	78.5	12.6	19.8	15.4	42.9	175.6	78.1	44.5%
2005	7.1	79.5	12.0	20.0	16.4	46.3	181.4	82.7	45.6%
2006	8.0	92.7	12.0	22.6	18.3	56.1	209.6	96.9	46.2%
2007	10.4	113.8	12.9	25.1	19.7	66.7	248.6	111.4	44.8%
2008	14.6	123.3	14.0	21.2	19.0	60.6	252.8	100.9	39.9%
2009	15.0	124.0	14.7	23.3	21.7	64.1	262.8	109.1	41.5%
2010	16.7	130.3	14.9	24.8	24.2	68.3	279.0	117.2	42.0%
2011	20.4	140.2	14.9	25.4	24.8	68.2	293.9	118.4	40.3%
2012	22.4	145.2	14.5	27.1	26.8	73.0	309.0	126.9	41.1%
2013	23.0	144.4	14.0	27.9	28.4	78.2	315.9	134.5	42.6%
2014	23.3	142.2	13.7	28.0	29.2	79.8	316.1	137.0	43.3%

Source: FSB (2015e), Shadow Banking Monitoring Dataset 2015.

[34]MUNFI is the acronym of "monitoring universe of non-bank financial intermediation".
[35]This exercise covers 26 countries, including for instance BRICS, Japan, the Euro area, the United Sates and the United Kingdom.

Financial assets managed by OFIs are under the scrutiny of the FSB, which has adopted the following classification: money market funds (MMF), finance companies (FC), structured finance vehicles (SFV), hedge funds (HF), other investment funds (OIF), broker-dealers (BD), real estate investment trusts (REIT) and trust companies (TC). Table 12.13 gives the repartition of assets by categories. We can now decompose the amount of $79.8 tn assets reached in 2014 by category of OFIs. 39.8% of these assets concern other investment funds, that is equity funds, fixed income funds and multi-asset funds. Broker-dealers is an important category of OFIs as they represent 11.8% of assets. It is followed by structured finance vehicles (6.8%) and money market funds (5.8%). We also notice the low level of HF assets, which is due to the fact that many hedge funds are not domiciliated in the list of 26 countries that participated to the FSB report.

TABLE 12.13: Assets of OFIs (in $ tn)

Year	MMF	FC	SFV	HF	OIF	BD	REIT	TC	Other
2002	3.1	2.4	2.4	0.0	5.2	3.2	0.1	0.1	13.4
2003	3.3	2.7	2.7	0.0	6.7	3.9	0.4	0.1	17.0
2004	3.4	2.9	3.3	0.0	8.1	4.7	0.5	0.1	19.9
2005	3.5	2.8	4.1	0.0	9.5	5.1	0.6	0.1	20.5
2006	4.0	3.0	5.0	0.0	11.6	5.8	0.8	0.1	25.8
2007	5.1	3.1	6.3	0.0	13.8	6.7	0.8	0.2	30.7
2008	5.9	3.7	6.1	0.2	15.2	9.2	0.7	0.1	19.4
2009	5.5	3.5	8.8	0.2	19.8	7.9	0.8	0.2	17.4
2010	4.7	3.7	7.4	0.2	22.5	8.6	1.4	0.7	19.1
2011	4.4	3.6	7.0	0.2	21.8	9.0	1.6	1.0	19.5
2012	4.4	3.4	6.5	0.3	25.5	9.2	1.8	1.5	20.5
2013	4.4	3.2	6.0	0.3	29.9	9.1	1.8	2.2	21.3
2014	4.6	3.1	5.4	0.5	31.8	9.4	1.9	2.7	20.4

Source: FSB (2015e), Shadow Banking Monitoring Dataset 2015.

The broad measure suffers from one major shortcoming, because it is an entity-based measure and not an asset-based measure. It then includes both shadow banking assets and other assets. This is particular true for equity assets, which are not shadow banking assets[36]. In this context, the FSB has developed more relevant measures, but with less participating countries. In Figure 12.7, we have reported the credit assets calculated by the FSB for 11 countries[37] and the Euro area. In 2014, the credit intermediation by banks was equal to $77 tn. It has declined these last years, principally because of the

[36]This concerns for instance equity mutual funds and long/short equity hedge funds.

[37]They are Australia, Brazil, Chile, Hong Kong, India, Indonesia, Japan, Mexico, Switzerland, United Kingdom and United States.

Euro area and Japan. At the same time, credit assets by insurance companies and pension funds (ICPF) were equal to \$19 tn, whereas the credit intermediation by OFIs peaked at \$29 tn. The FSB proposes a sub-decomposition of these credit assets by reporting the lending assets (loans and receivables). The difference between credit and lending assets is essentially composed of investments in debt securities. This decomposition is shown in Figure 12.7. We notice that loans are the main component of banks' credit assets (76.8%), whereas they represent a small part of the credit intermediation by ICPFs (12%). For OFIs, loans explain 41% of credit assets, but we observe differences between OFIs' sectors. Finance companies and broker-dealers are the main contributors of lending by OFIs.

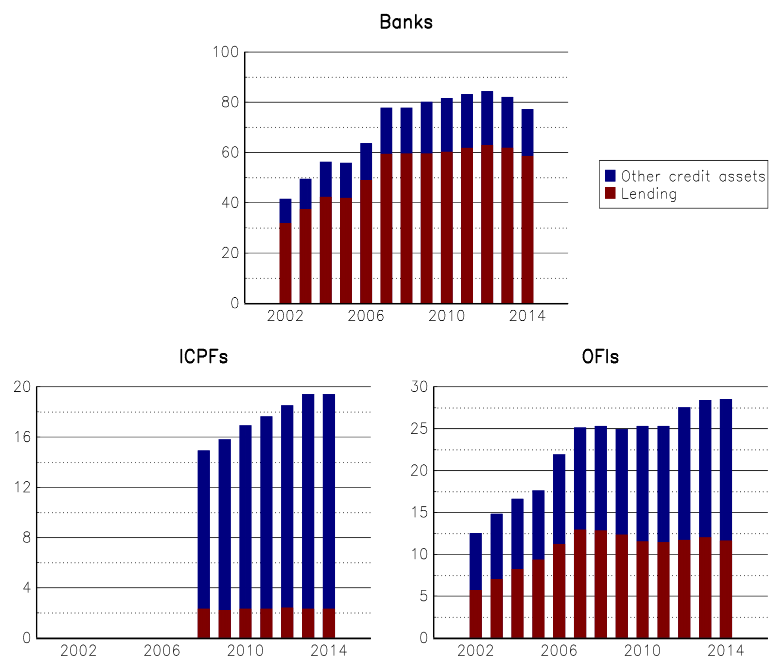

FIGURE 12.7: Credit assets (in \$ tn)

Source: FSB (2015e), Shadow Banking Monitoring Dataset 2015.

12.3.2.2 The narrow measure

Since this year, the FSB produces a more relevant measure of the shadow banking system, which is called the narrow measure. The narrow measure is based on the classification of the shadow banking system by economic functions given in Table 12.14.

The first economic function is related to redemption risks and concerns forced liquidations in an hostile environment. For instance, the lack of liquid-

TABLE 12.14: Classification of the shadow banking system by economic functions

Economic Function	Definition	Typical entity types
EF1	Management of collective investment vehicles with features that are susceptible to runs	Fixed-income funds, mixed funds, credit hedge funds, real estate funds
EF2	Loan provision that is dependent on short-term funding	Finance companies, leasing, factoring and consumer credit companies
EF3	Intermediation of market activities that is dependent on short-term funding or on secured funding of client assets	Broker-dealers
EF4	Facilitation of credit creation	Credit insurance companies, financial guarantors, monolines
EF5	Securitization-based credit intermediation and funding of financial entities	Securitization vehicles

Source: FSB (2015e), Shadow Banking Monitoring Dataset 2015.

ity of some fixed-income instruments implies a premium for the first investors who unwind their positions on money market and bond funds. In this case, one can observe a run on such funds exactly like a bank run because investors lose confidence in such products and do not want to be the lasts to move. Run risk can then be transmitted to the entire asset class. This risk mainly concerns collective investment vehicles, whose underlying assets face liquidity issues (fixed income, real estate). The second and fourth economic functions concern lending and credit that are conducted outside of the banking system. The third economic function is related to market intermediation on short-term funding. This includes securities broking services for market making activities and prime brokerage services to hedge funds. Finally, the last economic function corresponds to credit securitization.

The FSB uses these five economic functions in order to calculate the narrow measure defined in Figure 12.8. They consider that pension funds and insurance companies are not participating to the narrow shadow banking system except credit insurance companies. Nevertheless, this last category represents less than \$200 bn, implying that the narrow measure concerns principally OFIs. Each OFIs is classified or not among the five economic functions by the FSB. For instance, equity funds, closed-end funds without leverage and equity REITs are excluded from the shadow banking estimate. Finally, the FSB also removes entities that are subsidiaries of a banking group and consolidated at the group level for prudential purposes[38].

Narrow measure
=
Broad measure
−
Pension funds
−
Insurance companies
+
Credit insurance companies
−
OFIs that are not classified into the five economic functions
−
OFIs that are consolidated into a banking group for prudential purposes

FIGURE 12.8: Calculation of the shadow banking narrow measure

In Table 12.15, we report the size of the narrow shadow banking and compare it with assets of banks and OFIs. The narrow measure represents 53% of total assets managed by OFIs[39]. These shadow banking assets are located in developed countries, in particular in the United States, United Kingdom, Ireland, Germany, Japan and France (see Figure 12.9). If we analyze the assets with respect to economic functions, EF1 and EF3 represent 60% and 11% of the assets, meaning that the shadow banking system involves in the first instance mutual funds that are exposed to run risks and broker-dealers activities.

TABLE 12.15: Size of the narrow shadow banking (in \$ tn)

Year	2010	2011	2012	2013	2014
Banks	121.2	131.0	136.5	136.7	135.1
OFIs	58.2	58.8	62.8	66.8	68.1
Shadow banking	31.3	31.8	33.8	34.8	35.9

Source: FSB (2015e) & author's calculation.

The FSB (2015e) provides also network measures between the banking system and OFIs. For that, it estimates the aggregate balance sheet bilateral exposure between the two sectors by considering netting exposures within banking groups that are prudentially consolidated:

[38]This category represents almost 15% of OFIs' assets.

[39]There is a discrepancy between the numbers calculated in the narrow measure and the broad measure. This is explained by the fact that the Euro area is replaced by a set of 6 countries (France, Germany, Ireland, Italy, Netherlands and Spain) to define the narrow shadow banking, because data are not available at the Euro area level. In particular, data from Luxembourg are missing, which has a significant impact.

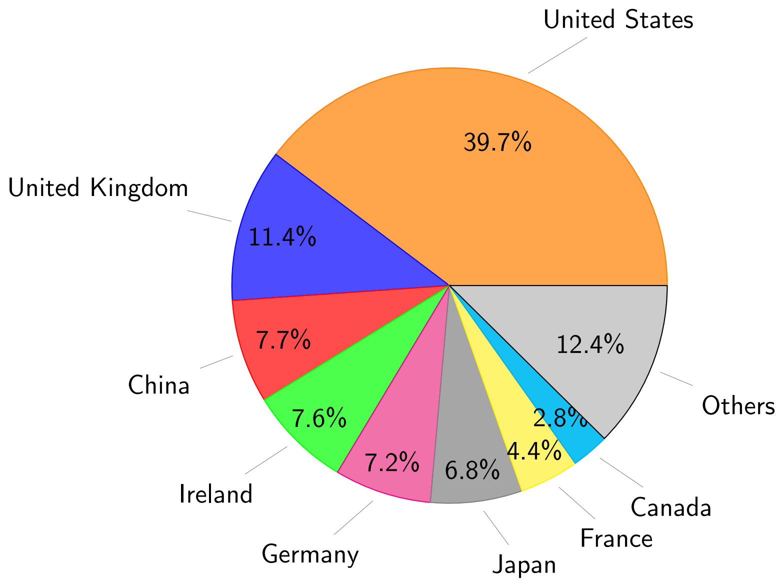

FIGURE 12.9: Breakdown by country of shadow banking assets (2014)

- Assets of banks to OFIs includes loans to institutions, fixed-income securities, reverse repos and investment in money market funds and other investment funds.
- Liabilities of banks to OFIs consists of uninsured bank deposits (e.g. certificates of deposit, notes and commercial paper), reverse repos and other short-term debt instruments.

Linkages between banks and OFIs are represented in Figure 12.10. These linkages measure the interconnectedness between a set $i \in \mathcal{I}$ of banks and a set $j \in \mathcal{J}$ of OFIs. Let A_{Bank_i} and A_{OFI_j} be the total amount of assets managed by Bank i and OFI j. We note $A_{\mathrm{Bank}_i \to \mathrm{OFI}_j}$ and $L_{\mathrm{Bank}_i \to \mathrm{OFI}_j}$ the assets and liabilities of Bank i to OFI j, and $A_{\mathrm{OFI}_j \to \mathrm{Bank}_i}$ and $L_{\mathrm{OFI}_j \to \mathrm{Bank}_i}$ the assets and liabilities of OFI j to Bank i. By construction, we have $A_{\mathrm{Bank}_i \to \mathrm{OFI}_j} = L_{\mathrm{OFI}_j \to \mathrm{Bank}_i}$ and $L_{\mathrm{Bank}_i \to \mathrm{OFI}_j} = A_{\mathrm{OFI}_j \to \mathrm{Bank}_i}$. In the bottom panel, we have represented the linkage from the bank's perspective. In this case, the credit and funding risks of Bank i are equal to:

$$\mathcal{R}_{\mathrm{Bank}_i}^{(\mathrm{credit})} = \frac{A_{\mathrm{Bank}_i \to \mathrm{OFIs}}}{A_{\mathrm{Bank}_i}}$$

and:

$$\mathcal{R}_{\mathrm{Bank}_i}^{(\mathrm{funding})} = \frac{L_{\mathrm{Bank}_i \to \mathrm{OFIs}}}{A_{\mathrm{Bank}_i}}$$

where the aggregate measures are equal to $A_{\mathrm{Bank}_i \to \mathrm{OFIs}} = \sum_{j \in \mathcal{J}} A_{\mathrm{Bank}_i \to \mathrm{OFI}_j}$

and $L_{\text{Bank}_i \to \text{OFIs}} = \sum_{j \in \mathcal{J}} L_{\text{Bank}_i \to \text{OFI}_j}$. In the same way, we can calculate the interconnectedness from the OFI's viewpoint as shown in the top panel. As above, we define the credit and funding risks of OFI j in the following way:

$$\mathcal{R}_{\text{OFI}_j}^{(\text{credit})} = \frac{A_{\text{OFI}_j \to \text{Banks}}}{A_{\text{OFI}_j}}$$

and:

$$\mathcal{R}_{\text{OFI}_j}^{(\text{funding})} = \frac{L_{\text{OFI}_j \to \text{Banks}}}{A_{\text{OFI}_j}}$$

with $A_{\text{OFI}_j \to \text{Banks}} = \sum_{i \in \mathcal{I}} A_{\text{OFI}_j \to \text{Bank}_i}$ and $L_{\text{OFI}_j \to \text{Banks}} = \sum_{i \in \mathcal{I}} L_{\text{OFI}_j \to \text{Bank}_i}$. Using a subset of banks and OFIs, the FSB (2015e) found the following average interconnectedness ratios at the end of 2014:

Ratio	$\mathcal{R}_{\text{Bank}_i}^{(\text{credit})}$	$\mathcal{R}_{\text{Bank}_i}^{(\text{funding})}$	$\mathcal{R}_{\text{OFI}_j}^{(\text{credit})}$	$\mathcal{R}_{\text{OFI}_j}^{(\text{funding})}$
Average	6.5%	8.1%	9.6%	7.9%

This means that 8.1% of bank's funding depends on the shadow banking system, while the credit risk of banks to OFIs is lower and equal to 6.5% of bank's assets. We also notice that about 8% of OFIs' assets are provided by banks, while investments of banks into OFIs reaches 9.6%. These figures give an overview of the linkages between banking and OFIs sectors. In practice, the interconnectedness is stronger because these ratios were calculated by netting exposures within banking groups. It is obvious that linkages are higher in these entities.

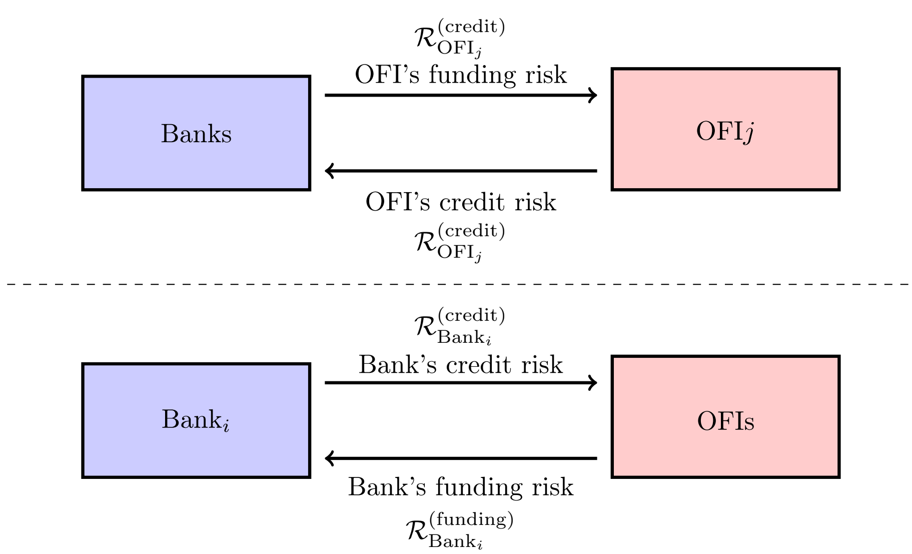

FIGURE 12.10: Interconnectedness between banks and OFIs

12.3.3 Regulatory developments of shadow banking

The road map for regulating shadow banking, which is presented in FSB (2013), focuses on four key principles:

- measurement and analysis of the shadow banking;
- mitigation of interconnectedness risk between banks and shadow banking entities;
- reduction of the run risk posed by money market funds;
- and improvement of transparency in securitization and more generally in complex shadow banking activities.

12.3.3.1 Data gaps

As seen in the previous section, analyzing the shadow banking system is a big challenge, because it is extremely difficult to measure it. In order to address this issue, FSB and IMF are in charge of the implementation of the G-20 data gaps initiative (DGI). DGI is not specific to shadow banking, but is a more ambitious program for monitoring the systemic risk of the global financial system[40]. However, it is obvious that shadow banking begin to be an important component of DGI. This concerns in particular short-term debt instruments, bonds, securitization and repo markets. Trade repositories, which collect data at the transaction level, complete regulatory reportings to understand shadow banking. They already exist for some OTC instruments in EU and US, but they will certainly be expanded to other markets (e.g. collateralized transactions). Simultaneously, supervisory authorities have strengthened regulatory reportings. However, the level of transparency in the shadow banking had still not reach this in banks. Some shadow banking sectors, in particular asset management and pension funds, should then expect new reporting requirements.

12.3.3.2 Mitigation of interconnectedness risk

The BCBS (2013c) has introduced new capital requirements for banks' equity investments in funds that are held in the banking book. They concern investment funds, mutual funds, hedge funds and private equity funds. The framework includes three methods to calculate the capital charge: the fall-back approach (FBA), the mandate-based approach (MBA) and the look-through approach (LTA). In this latter approach, the bank determines the risk-weighted assets of the underlying exposures of the fund. This approach is less conservative than the two others, but requires the full transparency on the portfolio holdings. Under the fall-back approach, the risk weight is equal

[40]For instance, DGI concerns financial soundness indicators (FSI), CDS and securities statistics, banking statistics, public sector debt, real estate prices, etc.

to 1 250% whatever the risk of the fund. According to the BCBS (2013c), the hierarchy in terms of risk sensitivity between the three approaches was introduced to promote "*due diligence by banks and transparent reporting by the funds in which they invest*". This framework had a significant impact on investment policy of banks and has reduced investments in equity funds and hedge funds.

The BCBS (2014c) has developed new standards for measuring and controlling large exposures to single counterparties. This concerns different levels of aggregation from the legal entity to consolidated groups. The large exposures framework is applicable to all international banks, and implies that the exposure of a bank to a consolidated group must be lower than 25% of the bank capital. This figure is reduced to 15% for systemic banks. This framework penalizes then banking groups, which have shadow banking activities (insurance, asset management, brokerage, etc.).

Remark 54 *There are some current proposals to limit exposures to shadow banking entities. For instance, the EBA issued a consultation paper in March 2015 and proposed that each bank must identify, monitor and control its individual exposures to all money market funds, alternative investment funds that fall under the AIFM directive and unregulated funds. In this proposal, UCITS non MMFs (including bond, equity and mixed funds) are then excluded from this shadow banking framework.*

12.3.3.3 Money market funds

Money market funds are under the scrutiny of regulatory authorities since the September 2008 run in the United States. The International Organization of Securities Commissions (2012a) recalled that the systemic risk of these funds is explained by three factors:

1. the illusory perception that MMFs don't have market and credit risks and benefit from capital protection;
2. the first mover advantage, which is pervasive during periods of market distress;
3. and the discrepancy between the published NAV and the asset value.

In order to mitigate these risks, the IOSCO (2012a) proposed several recommendations concerning the management of MMFs. In particular, they should be explicitly defined, investment universe should be restricted to high quality money market and low-duration fixed-income instruments, and they should be priced with the fair value approach. Moreover, MMFs that maintain a stable NAV (e.g. 1$ per share) should be converted into floating NAV.

In September 2015, the IOSCO reviewed the implementation progress made by 31 jurisdictions in adopting regulation and policies of MMFs. In particular, this review concerns the five largest jurisdictions (U.S., Ireland, China, France and Luxembourg), which together account for 90% of global assets under management in MMFs. It appears that only the U.S. reported having final implementation measures in all recommendations, while China and Europe are in the process of finalizing relevant reforms.

In July 2014, the U.S. Securities and Exchange Commission adopted final rules for the reform of MMFs. In particular, institutional MMFs will be required to trade at floating NAV. Moreover, all MMFs may impose liquidity fees and redemption gates during periods of stress. In Europe, a proposal for the regulation on MMFs is still in discussion. In China, two consultation papers regarding MMFs have been issued in May 2015. They propose to redefine the investment universe, improve the liquidity management and introduce new rules on valuation practices.

12.3.3.4 Complex shadow banking activities

We list here some supervisory initiatives related to some shadow banking activities:

- In 2011, the European Union has adopted the Alternative Investment Fund Managers Directive (AIFM), which complements the UCITS directive for asset managers and applies to hedge fund managers, private equity fund managers and real estate fund managers. In particular, it imposes reporting requirements and defines the AIFM passport.
- In a similar way to MMFs, the IOSCO (2012) published recommendations to improve incentive alignments in securitization, in particular by including issuer risk retention.
- According to IMF (2014), Nomura and Daiwa, which are the two largest securities brokerage in Japan, are now subject to Basel III capital requirements and bank-like prudential supervision.
- New regulation proposals on securities financing transactions (SFT) have been done by the European Commission. They concern reporting, transparency and collateral reuse of SFT activities (repo market, securities lending).

These examples show that the regulation of the shadow banking is in progress and non-bank financial institutions should expect to be better controlled in the future.

12.4 Exercises

Part III

Mathematical and Statistical Tools

Chapter 15

Copulas and Dependence

One of the main challenges in risk management is the aggregation of individual risks. This problem can be easily solved or at least move the issue aside by assuming that the random variables modeling individual risks are independent or are only dependent by means of a common factor. The problem becomes much more involved when one wants to model fully dependent random variables. Again a classic solution is to assume that the vector of individual risks follows a multivariate Normal distribution. However, all risks are not likely to be well described by a Gaussian random vector, and the Normal distribution may fail to catch some features of the dependence between individual risks.

Copula functions are a statistical tool to solve the previous issue. A copula function is nothing else but the joint distribution of a vector of uniform random variables. Since it is always possible to map any random vector into a vector of uniform random variables, we are able to split the marginals and the dependence between the random variables. Therefore, a copula function represents the statistical dependence between random variables, and generalizes the concept of correlation when the random vector is not Gaussian.

15.1 Canonical representation of multivariate distributions

The concept of copula has been introduced by Sklar in 1959. During a long time, only a small number of people have used copula functions, more in the field of mathematics than this of statistics. The publication of Genest and MacKay (1986b) in the *American Statistician* marks a breakdown and opens areas of study in empirical modeling, statistics and econometrics. In what follows, we intensively use the materials developed in the books of Joe (1997) and Nelsen (2006).

15.1.1 Sklar's theorem

Nelsen (2006) defines a bi-dimensional copula (or a 2-copula) as a function $\mathbf{C}$ which satisfies the following properties:

1. $\operatorname{Dom} \mathbf{C} = [0,1] \times [0,1]$;
2. $\mathbf{C}(0,u) = \mathbf{C}(u,0) = 0$ and $\mathbf{C}(1,u) = \mathbf{C}(u,1) = u$ for all u in $[0,1]$;
3. $\mathbf{C}$ is 2-increasing:

$$\mathbf{C}(v_1, v_2) - \mathbf{C}(v_1, u_2) - \mathbf{C}(u_1, v_2) + \mathbf{C}(u_1, u_2) \geq 0$$

   for all $(u_1, u_2) \in [0,1]^2$, $(v_1, v_2) \in [0,1]^2$ such that $0 \leq u_1 \leq v_1 \leq 1$ and $0 \leq u_2 \leq v_2 \leq 1$.

This definition means that $\mathbf{C}$ is a cumulative distribution function with uniform marginals:

$$\mathbf{C}(u_1, u_2) = \Pr\{U_1 \leq u_1, U_2 \leq u_2\}$$

where U_1 and U_2 are two uniform random variables.

Example 51 *Let us consider the function* $\mathbf{C}^{\perp}(u_1, u_2) = u_1 u_2$. *We have* $\mathbf{C}^{\perp}(0,u) = \mathbf{C}^{\perp}(u,0) = 0$ *and* $\mathbf{C}^{\perp}(1,u) = \mathbf{C}^{\perp}(u,1) = u$. *When* $v_2 - u_2 \geq 0$ *and* $v_1 \geq u_1$, *it follows that* $v_1(v_2 - u_2) \geq u_1(v_2 - u_2)$ *and* $v_1 v_2 + u_1 u_2 - u_1 v_2 - v_1 u_2 \geq 0$. *We deduce that* $\mathbf{C}^{\perp}$ *is a copula function. It is called the product copula.*

Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be any two univariate distributions. It is obvious that $\mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$ is a probability distribution with marginals $\mathbf{F}_1$ and $\mathbf{F}_2$. Indeed, $u_i = \mathbf{F}_i(x_i)$ defines a uniform transformation ($u_i \in [0,1]$). Moreover, we verify that $\mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(\infty)) = \mathbf{C}(\mathbf{F}_1(x_1), 1) = \mathbf{F}_1(x_1)$. Copulas are then a powerful tool to build a multivariate probability distribution when the marginals are given. Conversely, Sklar proved in 1959 that any bivariate distribution $\mathbf{F}$ admits such a representation:

$$\mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \tag{15.1}$$

and that the copula $\mathbf{C}$ is unique provided the marginals are continuous. This result is important, because we can associate to each bivariate distribution a copula function. We then obtain a canonical representation of a bivariate probability distribution: on one side, we have the marginals or the univariate directions $\mathbf{F}_1$ and $\mathbf{F}_2$; on the other side, we have the copula $\mathbf{C}$ that links these marginals and gives the dependence between the unidimensional directions.

Example 52 *The Gumbel logistic distribution is the function* $\mathbf{F}(x_1, x_2) = \left(1 + e^{-x_1} + e^{-x_2}\right)^{-1}$ *defined on* $\mathbb{R}^2$. *We notice that the marginals are* $\mathbf{F}_1(x_1) \equiv \mathbf{F}(x_1, \infty) = \left(1 + e^{-x_1}\right)^{-1}$ *and* $\mathbf{F}_2(x_2) \equiv \left(1 + e^{-x_2}\right)^{-1}$. *The quantile functions are then* $\mathbf{F}_1^{-1}(u_1) = \ln u_1 - \ln(1 - u_1)$ *and* $\mathbf{F}_2^{-1}(u_2) = \ln u_2 - \ln(1 - u_2)$. *We finally deduce that:*

$$\mathbf{C}(u_1, u_2) = \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right) = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$$

is the Gumbel logistic copula.

15.1.2 Expression of the copula density

If the joint distribution function $\mathbf{F}(x_1, x_2)$ is absolutely continuous, we obtain:

$$\begin{aligned} f(x_1, x_2) &= \partial_{1,2}\,\mathbf{F}(x_1, x_2) \\ &= \partial_{1,2}\,\mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \\ &= c(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)) \times f_1(x_1) \times f_2(x_2) \end{aligned} \tag{15.2}$$

where $f(x_1, x_2)$ is the joint probability density function, f_1 and f_2 are the marginal densities and c is the copula density:

$$c(u_1, u_2) = \partial_{1,2}\,\mathbf{C}(u_1, u_2)$$

We notice that the condition $\mathbf{C}(v_1, v_2) - \mathbf{C}(v_1, u_2) - \mathbf{C}(u_1, v_2) + \mathbf{C}(u_1, u_2) \geq 0$ is then equivalent to $\partial_{1,2}\,\mathbf{C}(u_1, u_2) \geq 0$ when the copula density exists.

Example 53 *In the case of the Gumbel logistic copula, we obtain* $c(u_1, u_2) = 2u_1u_2/(u_1 + u_2 - u_1u_2)^3$. *We easily verify the 2-increasing property.*

From Equation (15.2), we deduce that:

$$c(u_1, u_2) = \frac{f\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)}{f_1\left(\mathbf{F}_1^{-1}(u_1)\right) \times f_2\left(\mathbf{F}_2^{-1}(u_2)\right)} \tag{15.3}$$

We obtain a second canonical representation based on density functions. For some copulas, there is no explicit analytical formula. This is the case of the normal copula, which is equal to $\mathbf{C}(u_1, u_2; \rho) = \Phi\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \rho\right)$. Using Equation (15.3), we can however characterize its density function:

$$\begin{aligned} c(u_1, u_2; \rho) &= \frac{2\pi\left(1-\rho^2\right)^{-1/2}\exp\left(-\frac{1}{2(1-\rho^2)}\left(x_1^2 + x_2^2 - 2\rho x_1 x_2\right)\right)}{(2\pi)^{-1/2}\exp\left(-\frac{1}{2}x_1^2\right) \times (2\pi)^{-1/2}\exp\left(-\frac{1}{2}x_2^2\right)} \\ &= \frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\frac{\left(x_1^2 + x_2^2 - 2\rho x_1 x_2\right)}{(1-\rho^2)} + \frac{1}{2}\left(x_1^2 + x_2^2\right)\right) \end{aligned}$$

where $x_1 = \mathbf{F}_1^{-1}(u_1)$ and $x_2 = \mathbf{F}_2^{-1}(u_2)$. It is then easy to generate bivariate non-normal distributions.

Example 54 *In Figure 15.1, we have built a bivariate probability distribution by considering that the marginals are an inverse Gaussian distribution and a Beta distribution. The copula function corresponds to the normal copula such that its Kendall's tau is equal to* 50%.

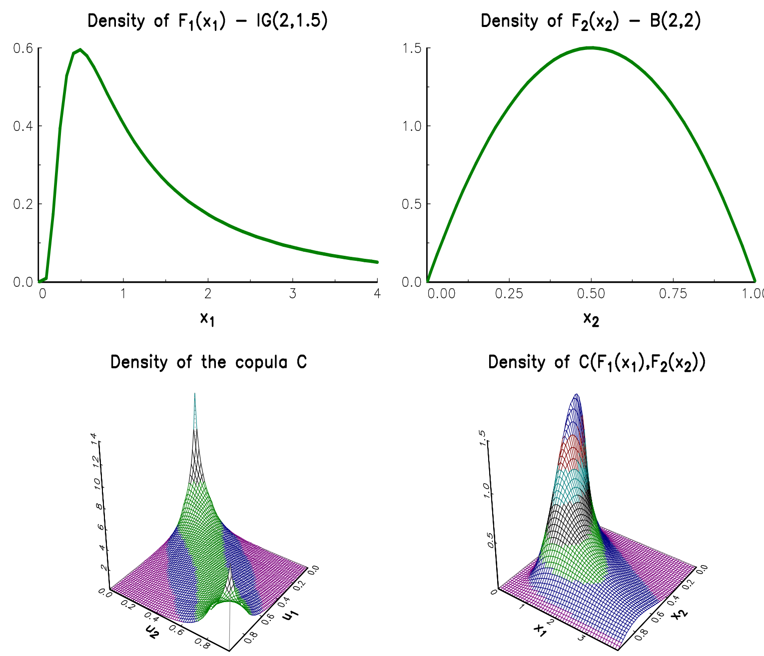

FIGURE 15.1: Example of a bivariate probability distribution with given marginals

15.1.3 Fréchet classes

The goal of Fréchet classes is to study of the structure of the class of distribution with given marginals. These later can be unidimensional, multidimensional or conditional. Let us consider the bivariate distribution functions $\mathbf{F}_{12}$ and $\mathbf{F}_{23}$. The Fréchet class $\mathcal{F}(\mathbf{F}_{12}, \mathbf{F}_{23})$ is the set of trivariate probability distributions that are compatible with the two bivariate marginals $\mathbf{F}_{12}$ and $\mathbf{F}_{23}$. In these lecture notes, we restrict our focus on the Fréchet class $\mathcal{F}(\mathbf{F}_1, \ldots, \mathbf{F}_n)$ with univariate marginals.

15.1.3.1 The bivariate case

Let us first consider the bivariate case. The distribution function $\mathbf{F}$ belongs to the Fréchet class $(\mathbf{F}_1, \mathbf{F}_2)$ and we note $\mathbf{F} \in \mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ if an only if the margins of $\mathbf{F}$ are $\mathbf{F}_1$ and $\mathbf{F}_2$, meaning that $\mathbf{F}(x_1, \infty) = \mathbf{F}_1(x_1)$ and $\mathbf{F}(\infty, x_2) = \mathbf{F}_2(x_2)$. Characterizing the Fréchet class $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ is then equivalent to find the set $\mathcal{C}$ of copula functions:

$$\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2) = \{\mathbf{F} : \mathbf{F}(x_1, x_2) = \mathbf{C}(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)), \mathbf{C} \in \mathcal{C}\}$$

This problem is then not dependent of the marginals $\mathbf{F}_1$ and $\mathbf{F}_2$.

We can show that the extremal distribution functions $\mathbf{F}^-$ and $\mathbf{F}^+$ of the

Fréchet class $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ are:

$$\mathbf{F}^-(x_1, x_2) = \max(\mathbf{F}_1(x_1) + \mathbf{F}_2(x_2) - 1, 0)$$

and:

$$\mathbf{F}^+(x_1, x_2) = \min(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2))$$

$\mathbf{F}^-$ and $\mathbf{F}^+$ are called the Fréchet lower and upper bounds. We deduce that the corresponding copula functions are:

$$\mathbf{C}^-(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$$

and:

$$\mathbf{C}^+(u_1, u_2) = \min(u_1, u_2)$$

Example 55 *We consider the Fréchet class $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$ with $\mathbf{F}_1 = \mathbf{F}_2 = \mathcal{N}(0,1)$. We know that the bivariate normal distribution with correlation ρ belongs to $\mathcal{F}(\mathbf{F}_1, \mathbf{F}_2)$. Nevertheless, a lot of bivariate non-normal distributions are also in this Fréchet class. For instance, this is the case of this probability distribution:*

$$\mathbf{F}(x_1, x_2) = \frac{\Phi(x_1)\Phi(x_2)}{\Phi(x_1) + \Phi(x_2) - \Phi(x_1)\Phi(x_2)}$$

We can also show that[1]*:*

$$\mathbf{F}^-(x_1, x_2) := \Phi(x_1, x_2; -1) = \max(\Phi(x_1) + \Phi(x_2) - 1, 0)$$

and:

$$\mathbf{F}^+(x_1, x_2) := \Phi(x_1, x_2; +1) = \min(\Phi(x_1), \Phi(x_2))$$

Therefore, the bounds of the Fréchet class $\mathcal{F}(\mathcal{N}(0,1), \mathcal{N}(0,1))$ correspond to the bivariate normal distribution, whose correlation is respectively equal to -1 and $+1$.

15.1.3.2 The multivariate case

The extension of bivariate copulas to multivariate copulas is straightforward. Thus, the canonical decomposition of a multivariate distribution function is:

$$\mathbf{F}(x_1, \ldots, x_n) = \mathbf{C}(\mathbf{F}_1(x_1), \ldots, \mathbf{F}_n(x_n))$$

We note $\mathbf{C}_{\mathcal{E}}$ the sub-copula of $\mathbf{C}$ such that arguments that are not in the set $\mathcal{E}$ are equal to 1. For instance, with a dimension of 4, we have $\mathbf{C}_{12}(u, v) = \mathbf{C}(u, v, 1, 1)$ and $\mathbf{C}_{124}(u, v, w) = \mathbf{C}(u, v, 1, w)$. Let us consider the 2-copulas

[1] We remind that:

$$\Phi(x_1, x_2; \rho) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \phi(y_1, y_2; \rho)\, \mathrm{d}y_1\, \mathrm{d}y_2$$

$\mathbf{C}_1$ and $\mathbf{C}_2$. It seems logical to build a copula of higher dimension with copulas of lower dimensions. In fact, the function $\mathbf{C}_1(u_1, \mathbf{C}_2(u_2, u_3))$ is not a copula in most cases (Quesada Molina and Rodriguez Lallena, 1994). For instance, we have:

$$\begin{aligned} \mathbf{C}^-\left(u_1, \mathbf{C}^-(u_2, u_3)\right) &= \max\left(u_1 + \max\left(u_2 + u_3 - 1, 0\right) - 1, 0\right) \\ &= \max\left(u_1 + u_2 + u_3 - 2, 0\right) \\ &= \mathbf{C}^-(u_1, u_2, u_3) \end{aligned}$$

However, the function $\mathbf{C}^-(u_1, u_2, u_3)$ is not a copula.

In the multivariate case, we define:

$$\mathbf{C}^-(u_1, \ldots, u_n) = \max\left(\sum_{i=1}^n u_i - n + 1, 0\right)$$

and:

$$\mathbf{C}^+(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$$

As discussed above, we can show that $\mathbf{C}^+$ is a copula, but $\mathbf{C}^-$ does not belong to the set $\mathcal{C}$. Nevertheless, $\mathbf{C}^-$ is the best-possible bound, meaning that for all $(u_1, \ldots, u_n) \in [0,1]^n$, there is a copula that coincide with $\mathbf{C}^-$ (Nelsen, 2006). This implies that $\mathcal{F}(\mathbf{F}_1, \ldots, \mathbf{F}_n)$ has a minimal distribution function if and only if $\max\left(\sum_{i=1}^n \mathbf{F}_i(x_i) - n + 1, 0\right)$ is a probability distribution (Dall'Aglio, 1972).

15.1.3.3 Concordance ordering

Using the result of the previous paragraph, we have:

$$\mathbf{C}^-(u_1, u_2) \leq \mathbf{C}(u_1, u_2) \leq \mathbf{C}^+(u_1, u_2)$$

for all $\mathbf{C} \in \mathcal{C}$. For a given value $\alpha \in [0,1]$, the level curves of $\mathbf{C}$ are then in the triangle defined as follows:

$$\{(u_1, u_2) : \max(u_1 + u_2 - 1, 0) \leq \alpha, \min(u_1, u_2) \geq \alpha\}$$

An illustration is shown in Figure 15.2. In the multidimensional case, the region becomes a n-volume.

We now introduce a stochastic ordering on copulas. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be two copula functions. We say that the copula $\mathbf{C}_1$ is smaller than the copula $\mathbf{C}_2$ and we note $\mathbf{C}_1 \prec \mathbf{C}_2$ if we verify that $\mathbf{C}_1(u_1, u_2) \leq \mathbf{C}_2(u_1, u_2)$ for all $(u_1, u_2) \in [0,1]^2$. This stochastic ordering is called the concordance ordering and may be viewed as the first order of the stochastic dominance on probability distributions.

Example 56 *This ordering is partial because we can not compare all the copulas. Let us consider the cubic copula defined by* $\mathbf{C}(u_1, u_2; \theta) = u_1 u_2 +$

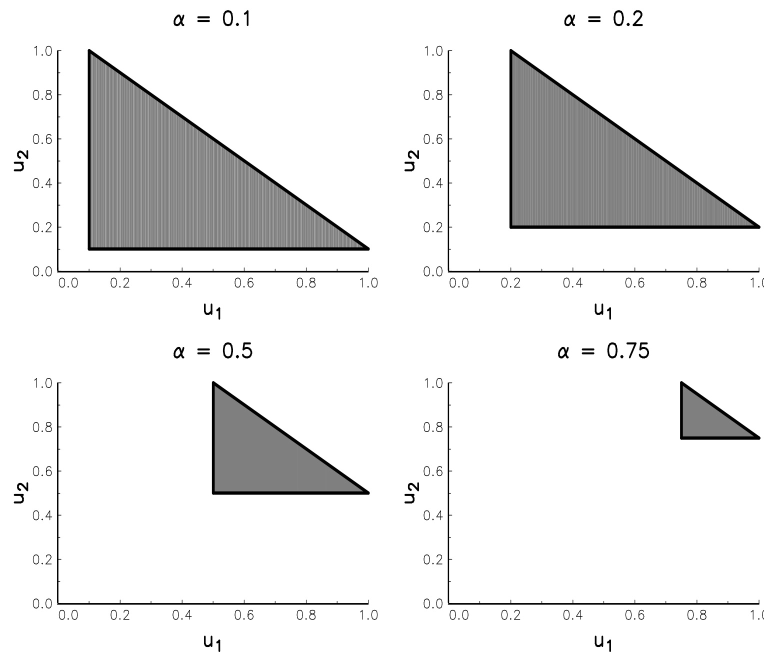

FIGURE 15.2: The triangle region of the contour lines $\mathbf{C}(u_1, u_2) = \alpha$

$\theta\,[u(u-1)(2u-1)]\,[v(v-1)(2v-1)]$ *where* $\theta \in [-1, 2]$. *If we compare it to the product copula* $\mathbf{C}^{\perp}$, *we have:*

$$\mathbf{C}\left(\frac{3}{4}, \frac{3}{4}; 1\right) = 0.5712 \geq \mathbf{C}^{\perp}\left(\frac{3}{4}, \frac{3}{4}\right) = 0.5625$$

but:

$$\mathbf{C}\left(\frac{3}{4}, \frac{1}{4}; 1\right) = 0.1787 \leq \mathbf{C}^{\perp}\left(\frac{3}{4}, \frac{1}{4}\right) = 0.1875$$

Using the Fréchet bounds, we have always $\mathbf{C}^{-} \prec \mathbf{C}^{\perp} \prec \mathbf{C}^{+}$. A copula $\mathbf{C}$ has a positive quadrant dependence (PQD) if it satisfies the inequality $\mathbf{C}^{\perp} \prec \mathbf{C} \prec \mathbf{C}^{+}$. In a similar way, $\mathbf{C}$ has a negative quadrant dependence (NQD) if it satisfies the inequality $\mathbf{C}^{-} \prec \mathbf{C} \prec \mathbf{C}^{\perp}$. As it is a partial ordering, there exists copula functions $\mathbf{C}$ such that $\mathbf{C} \not\succ \mathbf{C}^{\perp}$ and $\mathbf{C} \not\prec \mathbf{C}^{\perp}$. A copula function may then have a dependence structure that is neither positive or negative. This is the case of the cubic copula given in the previous example. In Figure 15.3, we report the cumulative distribution function (above panel) and its contour lines (right panel) of the three copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$, which plays an important role to understand the dependance between unidimensional risks.

Let $\mathbf{C}_{\theta}(u_1, u_2) = \mathbf{C}(u_1, u_2; \theta)$ be a family of copula functions that depends

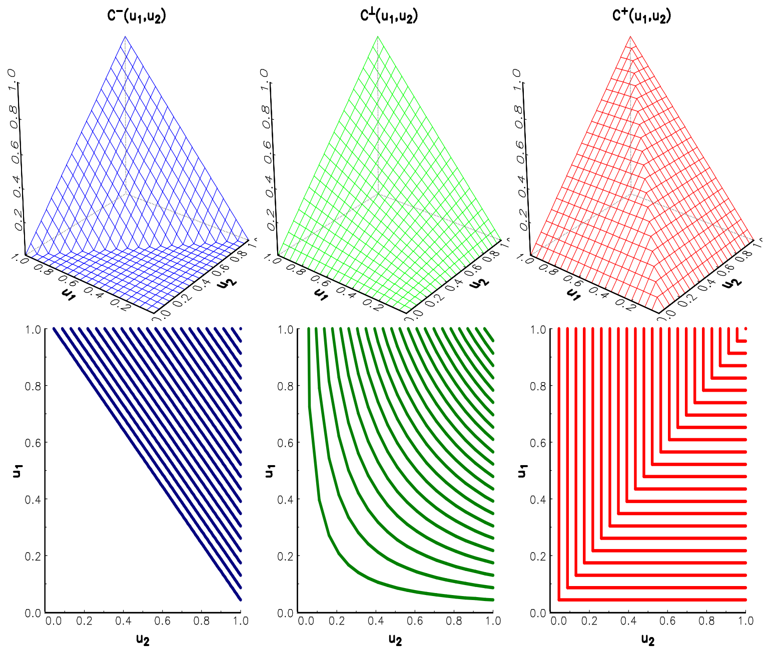

FIGURE 15.3: The three copula functions $\mathbf{C}^{-}$, $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$

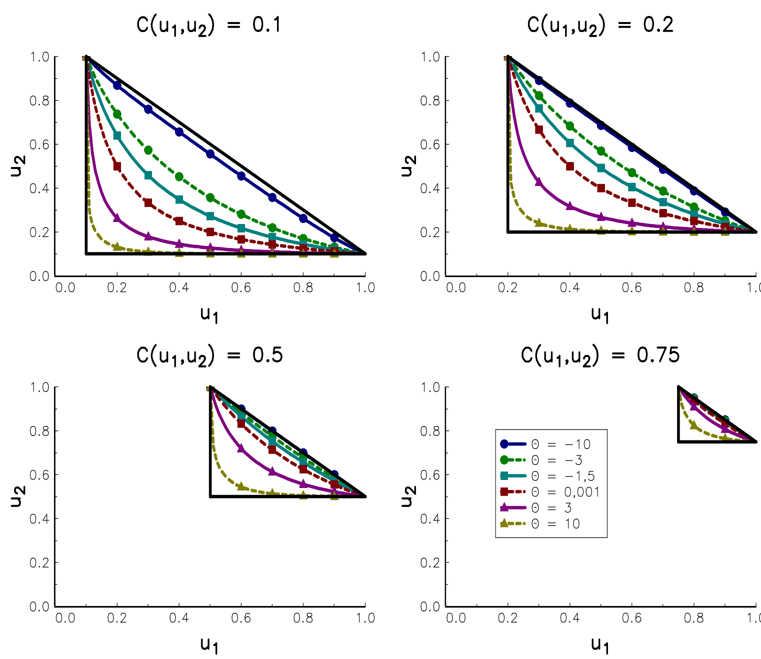

FIGURE 15.4: Concordance ordering of the Frank copula

on the parameter θ. The copula family $\{\mathbf{C}_\theta\}$ is totally ordered if, for all $\theta_2 \geq \theta_1$, $\mathbf{C}_{\theta_2} \succ \mathbf{C}_{\theta_1}$ (positively ordered) or $\mathbf{C}_{\theta_2} \prec \mathbf{C}_{\theta_1}$ (negatively ordered). For instance,the Frank copula defined by:

$$\mathbf{C}(u_1, u_2; \theta) = -\frac{1}{\theta} \ln \left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1} \right)$$

with $\theta \in \mathbb{R}$ is a positively ordered family (Figure 15.4).

Example 57 *Let us consider the copula function $\mathbf{C}_\theta = \theta\,\mathbf{C}^- + (1-\theta)\,\mathbf{C}^+$ where $0 \leq \theta \leq 1$. This copula is a convex sum of the extremal copulas $\mathbf{C}^-$ and $\mathbf{C}^+$. When $\theta_2 \geq \theta_1$, we have:*

$$\begin{aligned}
\mathbf{C}_{\theta_2}(u_1, u_2) &= \theta_2 \mathbf{C}^-(u_1, u_2) + (1-\theta_2)\,\mathbf{C}^+(u_1, u_2) \\
&= \mathbf{C}_{\theta_1}(u_1, u_2) - (\theta_2 - \theta_1)\left(\mathbf{C}^+(u_1, u_2) - \mathbf{C}^-(u_1, u_2)\right) \\
&\leq \mathbf{C}_{\theta_1}(u_1, u_2)
\end{aligned}$$

We deduce that $\mathbf{C}_{\theta_2} \prec \mathbf{C}_{\theta_1}$. This copula family is negatively ordered.

15.2 Copula functions and random vectors

Let $X = (X_1, X_2)$ be a random vector with distribution $\mathbf{F}$. We define the copula of (X_1, X_2) by the copula of $\mathbf{F}$:

$$\mathbf{F}(x_1, x_2) = \mathbf{C}\langle X_1, X_2\rangle\left(\mathbf{F}_1(x_1), \mathbf{F}_2(x_2)\right)$$

In what follows, we give the main results on the dependence of the random vector X found in Deheuvels (1978), Schweizer and Wolff (1981), and Nelsen (2006).

15.2.1 Countermonotonicity, comonotonicity and scale invariance property

We give here a probabilistic interpretation of the three copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$:

- X_1 and X_2 are countermonotonic – or $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ – if there exists a random variable X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where f_1 and f_2 are respectively decreasing and increasing functions[2];
- X_1 and X_2 are independent if the dependence function is the product copula $\mathbf{C}^\perp$;

[2] We also have $X_2 = f(X_1)$ where $f = f_2 \circ f_1^{-1}$ is a decreasing function.

- X_1 are X_2 are comonotonic – or $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$ – if there exists a random variable X such that $X_1 = f_1(X)$ and $X_2 = f_2(X)$ where f_1 and f_2 are both increasing functions[3].

Let us consider a uniform random vector (U_1, U_2). We have $U_2 = 1 - U_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ and $U_2 = U_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$. In the case of a standardized Gaussian random vector, we obtain $X_2 = -X_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ and $X_2 = X_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$. If the marginals are log-normal, it follows that $X_2 = X_1^{-1}$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^-$ and $X_2 = X_1$ when $\mathbf{C}\langle X_1, X_2\rangle = \mathbf{C}^+$. For these three examples, we verify that X_2 is a decreasing (resp. increasing) function of X_1 if the copula function $\mathbf{C}\langle X_1, X_2\rangle$ is $\mathbf{C}^-$ (resp. $\mathbf{C}^+$). The concepts of counter- and comonotonicity concepts generalize the cases where the linear correlation of a Gaussian vector is equal to -1 or $+1$. Indeed, $\mathbf{C}^-$ and $\mathbf{C}^+$ define respectively perfect negative and positive dependence.

We now give one of the most important theorem on copulas. Let (X_1, X_2) be a random vectors, whose copula is $\mathbf{C}\langle X_1, X_2\rangle$. If h_1 and h_2 are two increasing on $\operatorname{Im} X_1$ and $\operatorname{Im} X_2$, then we have:

$$\mathbf{C}\langle h_1(X_1), h_2(X_2)\rangle = \mathbf{C}\langle X_1, X_2\rangle$$

This means that copula functions are invariant under strictly increasing transformations of the random variables. To prove this theorem, we note $\mathbf{F}$ and $\mathbf{G}$ the probability distributions of the random vectors (X_1, X_2) and $(Y_1, Y_2) = (h_1(X_1), h_2(X_2))$. The marginals of $\mathbf{G}$ are:

$$\begin{aligned}
\mathbf{G}_1(y_1) &= \Pr\{Y_1 \leq y_1\} \\
&= \Pr\{h_1(X_1) \leq y_1\} \\
&= \Pr\left\{X_1 \leq h_1^{-1}(y_1)\right\} \quad \text{(because } h_1 \text{ is strictly increasing)} \\
&= \mathbf{F}_1\left(h_1^{-1}(y_1)\right)
\end{aligned}$$

and $\mathbf{G}_2(y_2) = \mathbf{F}_2\left(h_2^{-1}(y_2)\right)$. We deduce that $\mathbf{G}_1^{-1}(u_1) = h_1\left(\mathbf{F}_1^{-1}(u_1)\right)$ and $\mathbf{G}_2^{-1}(u_2) = h_2\left(\mathbf{F}_2^{-1}(u_2)\right)$. By definition, we have:

$$\mathbf{C}\langle Y_1, Y_2\rangle(u_1, u_2) = \mathbf{G}\left(\mathbf{G}_1^{-1}(u_1), \mathbf{G}_2^{-1}(u_2)\right)$$

Moreover, it follows that:

$$\begin{aligned}
\mathbf{G}\left(\mathbf{G}_1^{-1}(u_1), \mathbf{G}_2^{-1}(u_2)\right) &= \Pr\left\{Y_1 \leq \mathbf{G}_1^{-1}(u_1), Y_2 \leq \mathbf{G}_2^{-1}(u_2)\right\} \\
&= \Pr\left\{h_1(X_1) \leq \mathbf{G}_1^{-1}(u_1), h_2(X_2) \leq \mathbf{G}_2^{-1}(u_2)\right\} \\
&= \Pr\left\{X_1 \leq h_1^{-1}\left(\mathbf{G}_1^{-1}(u_1)\right), X_2 \leq h_2^{-1}\left(\mathbf{G}_2^{-1}(u_2)\right)\right\} \\
&= \Pr\left\{X_1 \leq \mathbf{F}_1^{-1}(u_1), X_2 \leq \mathbf{F}_2^{-1}(u_2)\right\} \\
&= \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)
\end{aligned}$$

[3] In this case, $X_2 = f(X_1)$ where $f = f_2 \circ f_1^{-1}$ is an increasing function.

Because we have $\mathbf{C}\langle X_1, X_2\rangle(u_1, u_2) = \mathbf{F}\left(\mathbf{F}_1^{-1}(u_1), \mathbf{F}_2^{-1}(u_2)\right)$, we deduce that $\mathbf{C}\langle Y_1, Y_2\rangle = \mathbf{C}\langle X_1, X_2\rangle$.

Example 58 *If X_1 and X_2 are two positive random variables, the previous theorem implies that:*

$$\begin{aligned}\mathbf{C}\langle X_1, X_2\rangle &= \mathbf{C}\langle \ln X_1, X_2\rangle \\ &= \mathbf{C}\langle \ln X_1, \ln X_2\rangle \\ &= \mathbf{C}\langle X_1, \exp X_2\rangle \\ &= \mathbf{C}\left\langle \sqrt{X_1}, \exp X_2\right\rangle\end{aligned}$$

Applying an increasing transformation does not change the copula function, only the marginals. Thus, the copula of the multivariate log-normal distribution is the same than the copula of the multivariate normal distribution.

The scale invariance property is perhaps not surprising if we consider the canonical decomposition of the bivariate probability distribution. Indeed, the copula $\mathbf{C}\langle U_1, U_2\rangle$ is equal to the copula $\mathbf{C}\langle X_1, X_2\rangle$ where $U_1 = \mathbf{F}_1(X_1)$ and $U_2 = \mathbf{F}_2(X_2)$. In some sense, Sklar's theorem is an application of the scale invariance property by considering $h_1(x_1) = \mathbf{F}_1(x_1)$ and $h_2(x_2) = \mathbf{F}_2(x_2)$.

Example 59 *We assume that $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$. If the copula of (X_1, X_2) is $\mathbf{C}^-$, we have $U_2 = 1 - U_1$. This implies that:*

$$\begin{aligned}\Phi\left(\frac{X_2 - \mu_2}{\sigma_2}\right) &= 1 - \Phi\left(\frac{X_1 - \mu_1}{\sigma_1}\right) \\ &= \Phi\left(-\frac{X_1 - \mu_1}{\sigma_1}\right)\end{aligned}$$

We deduce that X_1 and X_2 are countermonotonic if:

$$X_2 = \mu_2 - \frac{\sigma_2}{\sigma_1}(X_1 - \mu_1)$$

By applying the same reasoning to the copula function $\mathbf{C}^+$, we show that X_1 and X_2 are comonotonic if:

$$X_2 = \mu_2 + \frac{\sigma_2}{\sigma_1}(X_1 - \mu_1)$$

We now consider the log-normal random variables $Y_1 = \exp(X_1)$ and $Y_2 = \exp(X_2)$. For the countermonotonicity case, we obtain:

$$\ln Y_2 = \mu_2 - \frac{\sigma_2}{\sigma_1}(\ln Y_1 - \mu_1)$$

or:

$$Y_2 = e^{\mu_2 + \frac{\sigma_2}{\sigma_1}\mu_1} Y_1^{-\sigma_2/\sigma_1}$$

For the comonotonicity case, the relationship becomes:

$$Y_2 = e^{\mu_2 - \frac{\sigma_2}{\sigma_1}\mu_1} Y_1^{\sigma_2/\sigma_1}$$

If we assume that $\mu_1 = \mu_2$ and $\sigma_1 = \sigma_2$, the log-normal random variables Y_1 and Y_2 are countermonotonic if $Y_2 = Y_1^{-1}$ and comonotonic if $Y_2 = Y_1$.

15.2.2 Dependence measures

We can interpret the copula function $\mathbf{C}\langle X_1, X_2\rangle$ as a standardization of the joint distribution after eliminating the effects of marginals . Indeed, it is a comprehensive statistic of the dependence function between X_1 and X_2. Therefore, a non-comprehensive statistic will be a dependence measure if it can be expressed using $\mathbf{C}\langle X_1, X_2\rangle$.

15.2.2.1 Concordance measures

Following Nelsen (2006), a numeric measure m of association between X_1 and X_2 is a measure if concordance if it satisfies the following properties:

1. $-1 = m\langle X, -X\rangle \le m\langle \mathbf{C}\rangle \le m\langle X, X\rangle = 1$;
2. $m\left\langle \mathbf{C}^{\perp}\right\rangle = 0$;
3. $m\langle -X_1, X_2\rangle = m\langle X_1, -X_2\rangle = -m\langle X_1, X_2\rangle$;
4. if $\mathbf{C}_1 \prec \mathbf{C}_2$, then $m\langle \mathbf{C}_1\rangle \le m\langle \mathbf{C}_2\rangle$;

Using this last property, we have:

$$\mathbf{C} \prec \mathbf{C}^{\perp} \Longrightarrow m\langle \mathbf{C}\rangle < 0$$

and:

$$\mathbf{C} \succ \mathbf{C}^{\perp} \Longrightarrow m\langle \mathbf{C}\rangle > 0$$

The concordance measure can then be viewed as a generalization of the linear correlation when the dependence function is not normal. Indeed, a positive quadrant dependence (PQD) copula will have a positive concordance measure whereas a negative quadrant dependence (NQD) copula will have a negative concordance measure. Moreover, the bounds -1 and $+1$ are reached when the copula function is countermonotonic and comonotonic.

Among the several concordance measures, we find Kendall's tau and Spearman's rho, which play an important role in nonparametric statistics. Let us consider a sample of n observations $\{(x_1, y_1), \ldots, (x_n, y_n)\}$ of the random vector (X, Y). Kendall's tau is the probability of concordance – $(X_i - X_j)(Y_i - Y_j) > 0$ – minus the probability of discordance – $(X_i - X_j)(Y_i - Y_j) < 0$:

$$\tau = \Pr\{(X_i - X_j)(Y_i - Y_j) > 0\} - \Pr\{(X_i - X_j)(Y_i - Y_j) < 0\}$$

Spearman's rho is the linear correlation of the rank statistics $(X_{i:n}, Y_{i:n})$. We can also show that Spearman's rho has the following expression

$$\varrho = \frac{\operatorname{cov}\left(\mathbf{F}_X\left(X\right), \mathbf{F}_Y\left(Y\right)\right)}{\sigma\left(\mathbf{F}_X\left(X\right)\right)\sigma\left(\mathbf{F}_Y\left(Y\right)\right)}$$

Schweizer and Wolff (1981) showed that Kendall's tau and Spearman's rho are concordance measures and have the following expressions:

$$\begin{aligned}
\tau &= 4\iint_{[0,1]^2} \mathbf{C}\left(u_1, u_2\right) \,\mathrm{d}\mathbf{C}\left(u_1, u_2\right) - 1 \\
\varrho &= 12\iint_{[0,1]^2} u_1 u_2 \,\mathrm{d}\mathbf{C}\left(u_1, u_2\right) - 3
\end{aligned}$$

From a numerical point of view, the following formulas should be preferred (Nelsen, 2006):

$$\begin{aligned}
\tau &= 1 - 4\iint_{[0,1]^2} \partial_{u_1}\mathbf{C}\left(u_1, u_2\right) \,\partial_{u_2}\mathbf{C}\left(u_1, u_2\right) \,\mathrm{d}u_1 \,\mathrm{d}u_2 \\
\varrho &= 12\iint_{[0,1]^2} \mathbf{C}\left(u_1, u_2\right) \,\mathrm{d}u_1 \,\mathrm{d}u_2 - 3
\end{aligned}$$

For some copulas, we have analytical formulas. For instance, we have:

Copula	ϱ	τ
Normal	$6\pi^{-1}\arcsin\left(\rho/2\right)$	$2\pi^{-1}\arcsin\left(\rho\right)$
Gumbel	✓	$\left(\theta - 1\right)/\theta$
FGM	$\theta/3$	$2\theta/9$
Frank	$1 - 12\theta^{-1}\left(\mathbf{D}_1\left(\theta\right) - \mathbf{D}_2\left(\theta\right)\right)$	$1 - 4\theta^{-1}\left(1 - \mathbf{D}_1\left(\theta\right)\right)$

where $\mathbf{D}_k\left(x\right)$ is the Debye function. The Gumbel (or Gumbel-Hougaard) copula is equal to:

$$\mathbf{C}\left(u_1, u_2; \theta\right) = \exp\left(-\left[\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right]^{1/\theta}\right)$$

for $\theta \geq 1$, whereas the expression of the Farlie-Gumbel-Morgenstern (or FGM) copula is:

$$\mathbf{C}\left(u_1, u_2; \theta\right) = u_1 u_2 \left(1 + \theta\left(1 - u_1\right)\left(1 - u_2\right)\right)$$

for $-1 \leq \theta \leq 1$.

For illustration, we report in Figures 15.5, 15.6 and 15.7 the level curves of several density functions built with normal, Frank and Gumbel copulas. In order to compare them, the parameter of each copula is calibrated such that Kendall's tau is equal to 50%. This means that these 12 distributions functions have the same dependence with respect to Kendall's tau. However, the dependence is different from one figure to another, because their copula

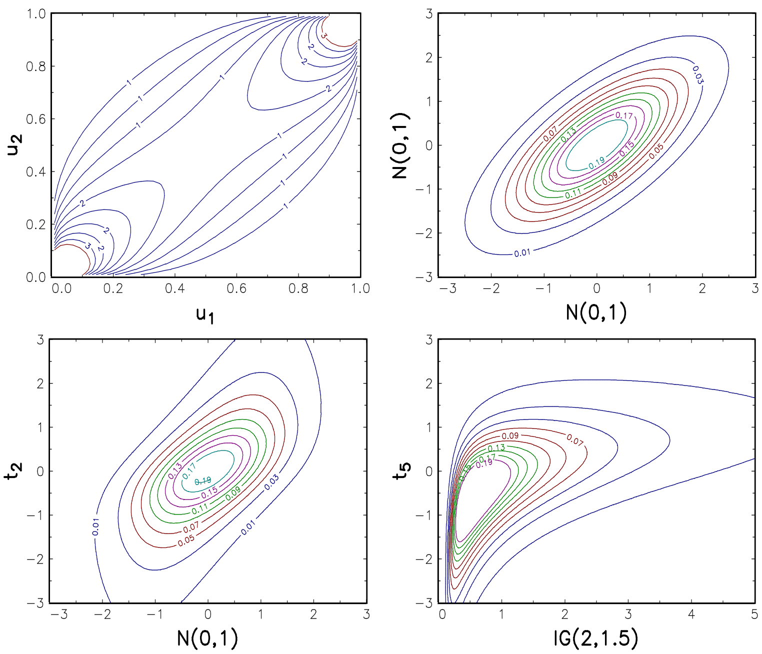

FIGURE 15.5: Contour lines of bivariate densities (normal copula)

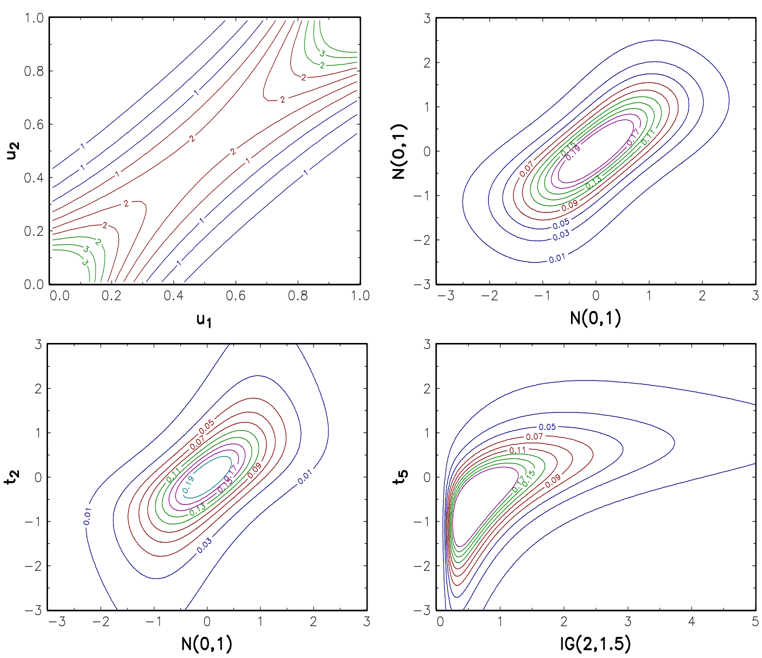

FIGURE 15.6: Contour lines of bivariate densities (Frank copula)

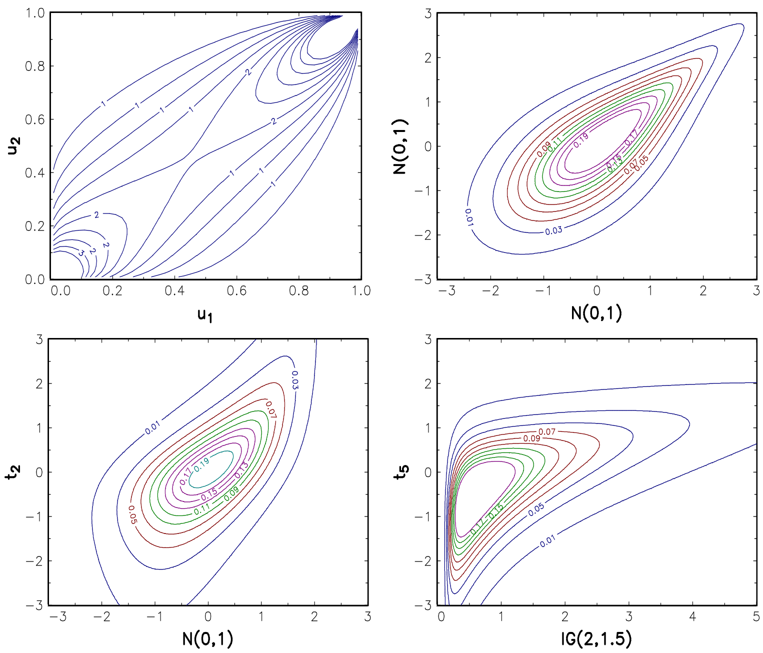

FIGURE 15.7: Contour lines of bivariate densities (Gumbel copula)

function is not the same. This is why Kendall's tau is not an exhaustive statistic of the dependence between two random variables.

We could build bivariate probability distributions, which are even less comparable. Indeed, the set of these three copula families (normal, Frank and Gumbel) is very small compared to the set $\mathcal{C}$ of copulas. However, there exists other dependence functions that are very far from the previous copulas. For instance, we consider the region $\mathcal{B}(\tau, \varrho)$ defined by:

$$(\tau, \varrho) \in \mathcal{B}(\tau, \varrho) \Leftrightarrow \begin{cases} (3\tau - 1)/2 \leq \varrho \leq \left(1 + 2\tau - \tau^2\right)/2 & \text{if} \quad \tau \geq 0 \\ \left(\tau^2 + 2\tau - 1\right)/2 \leq \varrho \leq (1 + 3\tau)/2 & \text{if} \quad \tau \leq 0 \end{cases}$$

Nelsen (2006) shows that these bounds can not be improved and there is always a copula function that corresponds to a point of the boundary $\mathcal{B}(\tau, \varrho)$. In Figure 15.8 we report the bounds $\mathcal{B}(\tau, \varrho)$ and the area reached by 8 copula families (normal, Plackett, Frank, Clayton, Gumbel, Galambos, Husler-Reiss, FGM). These copulas covered a small surface of the $\tau - \varrho$ region. These copula families are then relatively similar if we consider these concordance measures. Obtaining copulas that have a different behavior requires that the dependence is not monotone[4] on the whole domain $[0, 1]^2$.

[4]For instance, the dependence can be positive in one region and negative in another region.

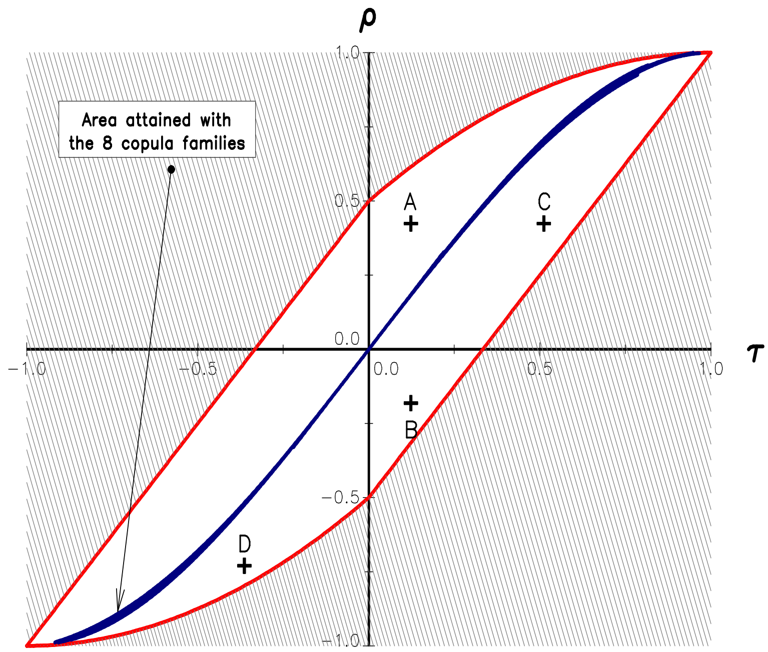

FIGURE 15.8: Bounds of (τ, ϱ) statistics

15.2.2.2 Linear correlation

We remind that the linear correlation (or Pearson's correlation) is defined as follows:

$$\rho \langle X_1, X_2 \rangle = \frac{\mathbb{E}[X_1 X_2] - \mathbb{E}[X_1]\,\mathbb{E}[X_2]}{\sigma(X_1)\,\sigma(X_2)}$$

Tchen (1980) showed the following properties of this measure:

- if the dependence of the random vector (X_1, X_2) is the product copula $\mathbf{C}^{\perp}$, then $\rho \langle X_1, X_2 \rangle = 0$;
- ρ is an increasing function with respect to the concordance measure:

$$\mathbf{C}_1 \succ \mathbf{C}_2 \Rightarrow \rho_1 \langle X_1, X_2 \rangle \geq \rho_2 \langle X_1, X_2 \rangle$$

- $\rho \langle X_1, X_2 \rangle$ is bounded:

$$\rho^{-} \langle X_1, X_2 \rangle \leq \rho \langle X_1, X_2 \rangle \leq \rho^{+} \langle X_1, X_2 \rangle$$

  and the bounds are reached for the Fréchet copulas $\mathbf{C}^{-}$ and $\mathbf{C}^{+}$.

However, a concordance measure must satisfy $m \langle \mathbf{C}^{-} \rangle = -1$ and $m \langle \mathbf{C}^{+} \rangle =$

+1. If we use the stochastic representation of Fréchet bounds, we have:

$$\rho^{-}\langle X_1, X_2\rangle = \rho^{+}\langle X_1, X_2\rangle = \frac{\mathbb{E}\left[f_1(X)\, f_2(X)\right] - \mathbb{E}\left[f_1(X)\right]\mathbb{E}\left[f_2(X)\right]}{\sigma\left(f_1(X)\right)\sigma\left(f_2(X)\right)}$$

The solution of the equation $\rho^{-}\langle X_1, X_2\rangle = -1$ is $f_1(x) = a_1 x + b_1$ and $f_2(x) = a_2 x + b_2$ where $a_1 a_2 < 0$. For the equation $\rho^{+}\langle X_1, X_2\rangle = +1$, the condition becomes $a_1 a_2 > 0$. Except for Gaussian random variables, there are few probability distributions that can satisfy these conditions. Moreover, if the linear correlation is a concordance measure, it is an invariant measure by increasing transformations:

$$\rho\langle X_1, X_2\rangle = \rho\langle f_1(X_1), f_2(X_2)\rangle$$

Again, the solution of this equation is $f_1(x) = a_1 x + b_1$ and $f_2(x) = a_2 x + b_2$ where $a_1 a_2 > 0$. We now have a better understanding why we say that this dependence measure is linear. In summary, the copula function generalizes the concept of linear correlation in a non-gaussian non-linear world.

Exercise 60 *We consider the bivariate log-normal random vector* (X_1, X_2) *where* $X_1 \sim \mathcal{LN}\left(\mu_1, \sigma_1^2\right)$, $X_2 \sim \mathcal{LN}\left(\mu_2, \sigma_2^2\right)$ *and* $\rho = \rho\langle \ln X_1, \ln X_2\rangle$.

We can show that:

$$\mathbb{E}\left[X_1^{p_1} X_2^{p_2}\right] = \exp\left(p_1\mu_1 + p_2\mu_2 + \frac{p_1^2\sigma_1^2 + p_2^2\sigma_2^2}{2} + p_1 p_2 \rho\sigma_1\sigma_2\right)$$

It follows that:

$$\rho\langle X_1, X_2\rangle = \frac{\exp(\rho\sigma_1\sigma_2) - 1}{\sqrt{\exp(\sigma_1^2) - 1} \times \sqrt{\exp(\sigma_2^2) - 1}}$$

We deduce that $\rho\langle X_1, X_2\rangle \in [\rho^{-}, \rho^{+}]$, but the bounds are not necessarily -1 and $+1$. For instance, when we use the parameters $\sigma_1 = 1$ and $\sigma_2 = 3$, we obtain the following results:

Copula	$\rho\langle X_1, X_2\rangle$	$\tau\langle X_1, X_2\rangle$	$\varrho\langle X_1, X_2\rangle$
$\mathbf{C}^{-}$	-0.008	-1.000	-1.000
$\rho = -0.7$	-0.007	-0.494	-0.683
$\mathbf{C}^{\perp}$	0.000	0.000	0.000
$\rho = 0.7$	0.061	0.494	0.683
$\mathbf{C}^{+}$	0.162	1.000	1.000

When the copula function is $\mathbf{C}^{-}$, the linear correlation takes a value close to zero! In Figure 15.9, we show that the bounds ρ^{-} and ρ^{+} of $\rho\langle X_1, X_2\rangle$ are not necessarily -1 and $+1$. When the marginals are log-normal, the upper bound $\rho^{+} = +1$ is reached only when $\sigma_1 = \sigma_2$ and the lower bound $\rho^{-} = -1$ is never reached. This poses a problem to interpret the value of a correlation. Let us consider two random vectors (X_1, X_2) and (Y_1, Y_2). What could we say about the dependence function when $\rho \langle X_1, X_2 \rangle \geq \rho \langle Y_1, Y_2 \rangle$? The answer is nothing if the marginals are not Gaussian. Indeed, we have seen previously that a 70% linear correlation between two Gaussian random vectors becomes a 6% linear correlation if we apply an exponential transformation. However, the two copulas of (X_1, X_2) and (Y_1, Y_2) are exactly the same. In fact, the drawback of the linear correlation is that this measure depends on the marginals and not only on the copula function.

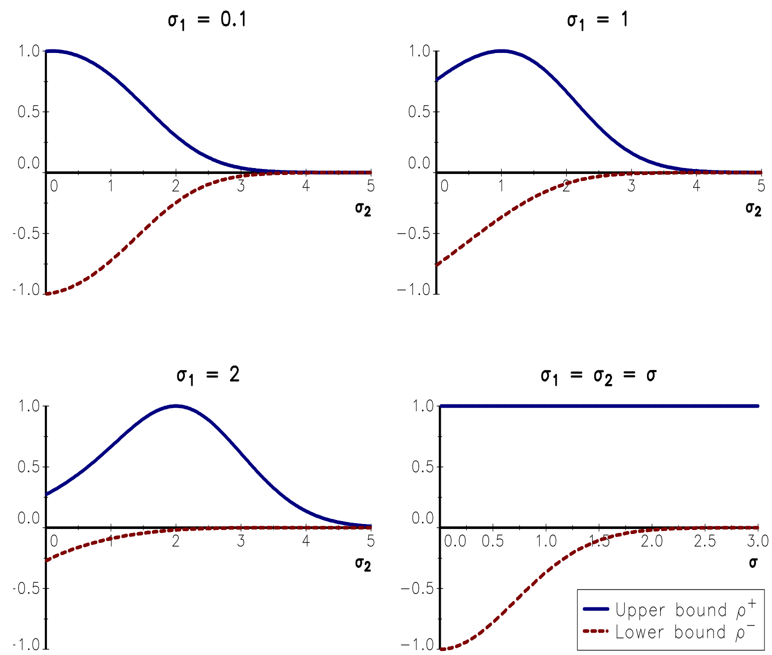

FIGURE 15.9: Bounds of the linear correlation between two log-normal random variables

15.2.2.3 Tail dependence

Contrary to concordance measures, tail dependence is a local measure that characterizes the joint behavior of the random variables X_1 and X_2 at the extreme points $x^- = \inf \{x : \mathbf{F}(x) > 0\}$ and $x^+ = \sup \{x : \mathbf{F}(x) < 1\}$. Let $\mathbf{C}$ be a copula function such that the following limit exists:

$$\lambda^+ = \lim_{u \to 1^-} \frac{1 - 2u + \mathbf{C}(u, u)}{1 - u}$$

We say that $\mathbf{C}$ has an upper tail dependence when $\lambda^+ \in (0, 1]$ and $\mathbf{C}$ has no upper tail dependence when $\lambda^+ = 0$ (Joe, 1997). For the lower tail dependence

λ^-, the limit becomes:

$$\lambda^- = \lim_{u \to 0^+} \frac{\mathbf{C}(u,u)}{u}$$

We notice that λ^+ and λ^- can also be defined as follows:

$$\lambda^+ = \lim_{u \to 1^-} \Pr\{U_2 > u \mid U_1 > u\}$$

and:

$$\lambda^- = \lim_{u \to 0^+} \Pr\{U_2 < u \mid U_1 < u\}$$

To compute the upper tail dependence, we consider the joint survival function $\bar{\mathbf{C}}$ defined by:

$$\begin{aligned}
\bar{\mathbf{C}}(u_1,u_2) &= \Pr\{U_1 > u_1, U_2 > u_2\} \\
&= 1 - u_1 - u_2 + \mathbf{C}(u_1,u_2)
\end{aligned}$$

The expression of the upper tail dependence is then:

$$\begin{aligned}
\lambda^+ &= \lim_{u \to 1^-} \frac{\bar{\mathbf{C}}(u,u)}{1-u} \\
&= -\lim_{u \to 1^-} \frac{\mathrm{d}\bar{\mathbf{C}}(u,u)}{\mathrm{d}u} \\
&= -\lim_{u \to 1^-} \left(-2 + \partial_1 \mathbf{C}(u,u) + \partial_2 \mathbf{C}(u,u)\right) \\
&= \lim_{u \to 1^-} \left(\Pr\{U_2 > u \mid U_1 = u\} + \Pr\{U_1 > u \mid U_2 = u\}\right)
\end{aligned}$$

By assuming that the copula is symmetric, we finally obtain:

$$\begin{aligned}
\lambda^+ &= 2 \lim_{u \to 1^-} \Pr\{U_2 > u \mid U_1 = u\} \\
&= 2 - 2 \lim_{u \to 1^-} \Pr\{U_2 < u \mid U_1 = u\} \\
&= 2 - 2 \lim_{u \to 1^-} \mathbf{C}_{2|1}(u,u) \qquad (15.4)
\end{aligned}$$

In a similar way, we find that the lower tail dependence of a symmetric copula is:

$$\lambda^- = 2 \lim_{u \to 0^+} \mathbf{C}_{2|1}(u,u) \qquad (15.5)$$

For the copula functions $\mathbf{C}^{\perp}$ and $\mathbf{C}^-$, we have $\lambda^+ = \lambda^- = 0$. For the copula $\mathbf{C}^+$, we obtain $\lambda^+ = \lambda^- = 1$. However, there exists copulas such that $\lambda^+ \neq \lambda^-$. This is the case of the Gumbel copula $\mathbf{C}(u_1,u_2;\theta) = \exp\left(-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right)$, because we have $\lambda^+ = 2 - 2^{1/\theta}$ and $\lambda^- = 0$. The Gumbel copula has then a upper tail dependence, but no lower tail dependence. If we consider the Clayton copula $\mathbf{C}(u_1, u_2; \theta) = \left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$, we obtain $\lambda^+ = 0$ and $\lambda^- = 2^{-1/\theta}$.

Coles *et al.* (1999) define the quantile-quantile dependence function as follows:

$$\lambda^+(\alpha) = \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha) \mid X_1 > \mathbf{F}_1^{-1}(\alpha)\right\}$$

It is the conditional probability that X_2 is larger than the quantile $\mathbf{F}_2^{-1}(\alpha)$ given that X_1 is larger than the quantile $\mathbf{F}_1^{-1}(\alpha)$. We have:

$$\begin{aligned}
\lambda^+(\alpha) &= \Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha) \mid X_1 > \mathbf{F}_1^{-1}(\alpha)\right\} \\
&= \frac{\Pr\left\{X_2 > \mathbf{F}_2^{-1}(\alpha), X_1 > \mathbf{F}_1^{-1}(\alpha)\right\}}{\Pr\left\{X_1 > \mathbf{F}_1^{-1}(\alpha)\right\}} \\
&= \frac{1 - \Pr\left\{X_1 \leq \mathbf{F}_1^{-1}(\alpha)\right\} - \Pr\left\{X_2 \leq \mathbf{F}_2^{-1}(\alpha)\right\}}{1 - \Pr\left\{X_1 \leq \mathbf{F}_1^{-1}(\alpha)\right\}} + \\
&\quad \frac{\Pr\left\{X_2 \leq \mathbf{F}_2^{-1}(\alpha), X_1 \leq \mathbf{F}_1^{-1}(\alpha)\right\}}{1 - \Pr\left\{\mathbf{F}_1(X_1) \leq \alpha\right\}} \\
&= \frac{1 - 2\alpha + \mathbf{C}(\alpha, \alpha)}{1 - \alpha}
\end{aligned}$$

The tail dependence λ^+ is then the limit of the conditional probability when the confidence level α tends to 1. It is also the probability of one variable being extreme given that the other is extreme. Because $\lambda^+(\alpha)$ is a probability, we verify that $\lambda^+ \in [0, 1]$. If the probability is zero, the extremes are independent. If λ^+ is equal to 1, the extremes are perfectly dependent. To illustrate the measures[5] $\lambda^+(\alpha)$ and $\lambda^-(\alpha)$, we represent their values for the Gumbel and Clayton copulas in Figure 15.10. The parameters are calibrated with respect to Kendall's tau.

Remark 55 *We consider two portfolios, whose losses correspond to the random variables L_1 and L_2 with probability distributions $\mathbf{F}_1$ and $\mathbf{F}_2$. The probability that the loss of the second portfolio is larger than its value-at-risk knowing that the value-at-risk of the first portfolio is exceeded is exactly equal to the quantile-quantile dependence measure $\lambda^+(\alpha)$:*

$$\begin{aligned}
\lambda(\alpha) &= \Pr\left\{L_2 > \mathbf{F}_2^{-1}(\alpha) \mid L_1 > \mathbf{F}_1^{-1}(\alpha)\right\} \\
&= \Pr\left\{L_2 > \mathrm{VaR}_\alpha(L_2) \mid L_1 > \mathrm{VaR}_\alpha(L_1)\right\}
\end{aligned}$$

[5] We have $\lambda^-(\alpha) = \Pr\left\{X_2 < \mathbf{F}_2^{-1}(\alpha) \mid X_1 < \mathbf{F}_1^{-1}(\alpha)\right\}$ and $\lim_{\alpha \to 0} \lambda^-(\alpha) = \lambda^-$.

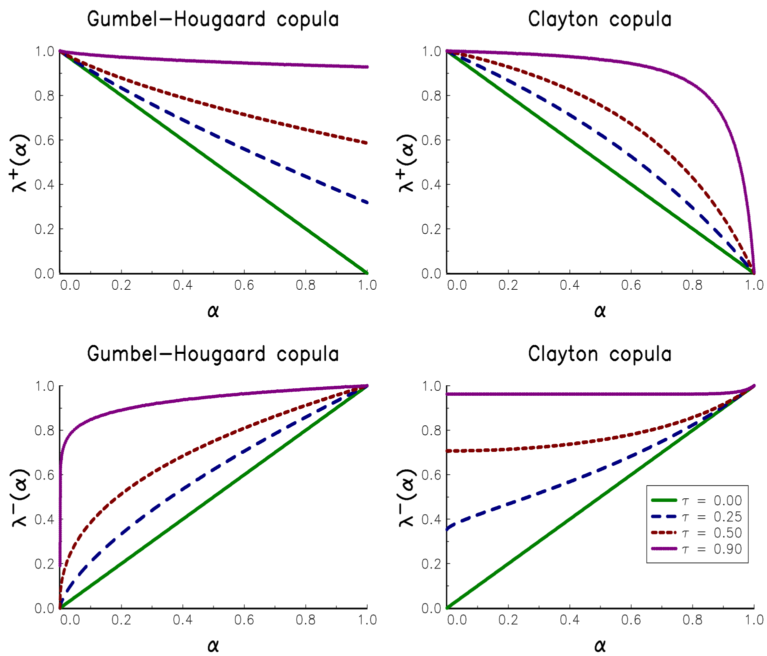

FIGURE 15.10: Quantile-quantile dependence measures $\lambda^+(\alpha)$ and $\lambda^-(\alpha)$

15.3 Parametric copula functions

In this section, we study the copula families, which are commonly used in risk management. They are parametric copulas, which depends on a set of parameters. Statistical inference, in particular parameter estimation, is developed in the next section.

15.3.1 Archimedean copulas

15.3.1.1 Definition

Genest and MacKay (1986b) define Archimedean copulas as follows:

$$\mathbf{C}(u_1, u_2) = \begin{cases} \varphi^{-1}(\varphi(u_1) + \varphi(u_2)) & \text{if } \varphi(u_1) + \varphi(u_2) \le \varphi(0) \\ 0 & \text{otherwise} \end{cases}$$

with φ a C^2 function which satisfies $\varphi(1) = 0$, $\varphi'(u) < 0$ and $\varphi''(u) > 0$ for all $u \in [0, 1]$. $\varphi(u)$ is called the generator of the copula function. If $\varphi(0) = \infty$, the generator is said to be strict. Genest and MacKay (1986a) link the construction of Archimedean copulas to the independence of random variables. Indeed, by considering the multiplicative generator $\lambda(u) = \exp(-\varphi(u))$, the authors shows that:

$$\mathbf{C}(u_1, u_2) = \lambda^{-1}(\lambda(u_1)\lambda(u_2))$$

This means that:

$$\lambda(\Pr\{U_1 \le u_1, U_2 \le u_2\}) = \lambda(\Pr\{U_1 \le u_1\}) \times \lambda(\Pr\{U_2 \le u_2\})$$

In this case, the random variables (U_1, U_2) become independent when the scale of probabilities has been transformed.

Example 61 *If $\varphi(u) = u^{-1} - 1$, we have $\varphi^{-1}(u) = (1+u)^{-1}$ and:*

$$\mathbf{C}(u_1, u_2) = \left(1 + \left(u_1^{-1} - 1 + u_2^{-1} - 1\right)\right)^{-1} = \frac{u_1 u_2}{u_1 + u_2 - u_1 u_2}$$

The Gumbel logistic copula is then an Archimedean copula.

Example 62 *The product copula $\mathbf{C}^{\perp}$ is Archimedean and the associated generator is $\varphi(u) = -\ln u$. Concerning Fréchet copulas, only $\mathbf{C}^{-}$ is Archimedean with $\varphi(u) = 1 - u$.*

In Table 15.1, we provide another examples of Archimedean copulas[6].

TABLE 15.1: Examples of Archimedean copula functions

Copula	$\varphi(u)$	$\mathbf{C}(u_1, u_2)$
$\mathbf{C}^{\perp}$	$-\ln u$	$u_1 u_2$
Clayton	$u^{-\theta} - 1$	$\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-1/\theta}$
Frank	$-\ln \frac{e^{-\theta u} - 1}{e^{-\theta} - 1}$	$-\frac{1}{\theta} \ln \left(1 + \frac{\left(e^{-\theta u_1} - 1\right)\left(e^{-\theta u_2} - 1\right)}{e^{-\theta} - 1}\right)$
Gumbel	$(-\ln u)^{\theta}$	$\exp\left(-\left(\tilde{u}_1^{\theta} + \tilde{u}_2^{\theta}\right)^{1/\theta}\right)$
Joe	$-\ln\left(1 - (1-u)^{\theta}\right)$	$1 - \left(\bar{u}_1^{\theta} + \bar{u}_2^{\theta} - \bar{u}_1^{\theta}\bar{u}_2^{\theta}\right)^{1/\theta}$

15.3.1.2 Properties

Archimedean copulas plays an important role in statistics, because they present many interesting properties, for example:

- $\mathbf{C}$ is symmetric, meaning that $\mathbf{C}(u_1, u_2) = \mathbf{C}(u_2, u_1)$;
- $\mathbf{C}$ is associative, implying that $\mathbf{C}(u_1, \mathbf{C}(u_1, u_3)) = \mathbf{C}(\mathbf{C}(u_1, u_2), u_3)$;
- the diagonal section $\delta(u) = \mathbf{C}(u, u)$ satisfies $\delta(u) < u$ for all $u \in (0, 1)$;

[6] We use the notations $\bar{u} = 1 - u$ and $\tilde{u} = -\ln u$.

- if a copula $\mathbf{C}$ is associative and $\delta(u) < u$ for all $u \in (0,1)$, then $\mathbf{C}$ is Archimedean.

Genest and MacKay (1986a) also showed that the expression of Kendall's tau is:

$$\tau\langle\mathbf{C}\rangle = 1 + 4\int_0^1 \frac{\varphi(u)}{\varphi'(u)}\,\mathrm{d}u$$

whereas the copula density is:

$$c(u_1,u_2) = -\frac{\varphi''(\mathbf{C}(u_1,u_2))\,\varphi'(u_1)\,\varphi'(u_2)}{[\varphi'(\mathbf{C}(u_1,u_2))]^3}$$

Example 63 *With the Clayton copula, we have $\varphi(u) = u^{-\theta} - 1$ and $\varphi'(u) = -\theta u^{-\theta-1}$. We deduce that:*

$$\begin{aligned}\tau &= 1 + 4\int_0^1 \frac{1-u^{-\theta}}{\theta u^{-\theta-1}}\,\mathrm{d}u \\ &= \frac{\theta}{\theta+2}\end{aligned}$$

15.3.1.3 Two-parameter Archimedean copulas

Nelsen (2006) showed that if $\varphi(t)$ is a strict generator, then we can build two-parameter Archimedean copulas by considering the following generator:

$$\varphi_{\alpha,\beta}(t) = (\varphi(t^\alpha))^\beta$$

where $\alpha > 0$ and $\beta > 1$. For instance, if $\varphi(t) = t^{-1}-1$, the two-parameter generator is $\varphi_{\alpha,\beta}(t) = (t^{-\alpha}-1)^\beta$. Therefore, the corresponding copula function is defined by:

$$\mathbf{C}(u_1,u_2) = \left(\left[\left(u_1^{-\alpha}-1\right)^\beta + \left(u_2^{-\alpha}-1\right)^\beta\right]^{1/\beta} + 1\right)^{-1/\alpha}$$

This is a generalization of the Clayton copula, which is obtained when the parameter β is equal to 1.

15.3.1.4 Extension to the multivariate case

We can build multivariate Archimedean copulas in the following way:

$$\mathbf{C}(u_1,\ldots,u_n) = \varphi^{-1}(\varphi(u_1) + \ldots + \varphi(u_n))$$

However, $\mathbf{C}$ is a copula function if and only if the function $\varphi^{-1}(u)$ is completely monotone (Nelsen, 2006):

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}u^k}\varphi^{-1}(u) \geq 0 \qquad \forall\, k \geq 1$$

For instance, the multivariate Gumbel copula is:

$$\mathbf{C}(u_1, \ldots, u_n) = \exp\left(-\left((-\ln u_1)^\theta + \ldots + (-\ln u_n)^\theta\right)^{1/\theta}\right)$$

The previous construction is related to an important class of multivariate distributions, which are called frailty models (Oakes, 1989). Let $\mathbf{F}_1, \ldots, \mathbf{F}_n$ be univariate distribution functions, and let $\mathbf{G}$ be an n-variate distribution function with univariate marginals $\mathbf{G}_i$, such that $\bar{\mathbf{G}}(0, \ldots, 0) = 1$. We denote by ψ_i the Laplace transform of $\mathbf{G}_i$. Marshall and Olkin (1988) showed that the function defined by:

$$\mathbf{F}(x_1, \ldots, x_n) = \int \cdots \int \mathbf{C}\left(\mathbf{H}_1^{t_1}(x_1), \ldots, \mathbf{H}_n^{t_n}(x_n)\right) \mathrm{d}\mathbf{G}(t_1, \ldots, t_n)$$

is a multivariate probability distribution with marginals $\mathbf{F}_1, \ldots, \mathbf{F}_n$ if $\mathbf{H}_i(x) = \exp\left(-\psi_i^{-1}(\mathbf{F}_i(x))\right)$. If we assume that the univariate distributions $\mathbf{G}_i$ are the same and equal to $\mathbf{G}_1$, $\mathbf{G}$ is the upper Fréchet bound and $\mathbf{C}$ is the product copula $\mathbf{C}^\perp$, the previous expression becomes:

$$\begin{aligned}
\mathbf{F}(x_1, \ldots, x_n) &= \int \prod_{i=1}^n \mathbf{H}_i^{t_1}(x_i)\ \mathrm{d}\mathbf{G}_1(t_1) \\
&= \int \exp\left(-t_1 \sum_{i=1}^n \psi^{-1}(\mathbf{F}_i(x_i))\right) \mathrm{d}\mathbf{G}_1(t_1) \\
&= \psi\left(\psi^{-1}(\mathbf{F}_1(x_1)) + \ldots + \psi^{-1}(\mathbf{F}_n(x_n))\right)
\end{aligned}$$

The corresponding copula is then given by:

$$\mathbf{C}(u_1, \ldots, u_n) = \psi\left(\psi^{-1}(u_1) + \ldots + \psi^{-1}(u_n)\right)$$

This is a special case of Archimedean copulas where the generator φ is the inverse of a Laplace transform. For instance, the Clayton copula is a frailty copula where $\psi(x) = (1 + \theta x)^{-1/\theta}$ is the Laplace transform of a Gamma random variate. The Gumbel-Hougaard copula is frailty too and we have $\psi(x) = \exp\left(-x^{1/\theta}\right)$. This is the Laplace transform of a positive stable distribution.

For frailty copulas, Joe (1997) showed that upper and lower tail dependence measures are given by:

$$\lambda^+ = 2 - 2 \lim_{x \to 0} \frac{\psi'(2x)}{\psi'(x)}$$

and:

$$\lambda^- = 2 \lim_{x \to \infty} \frac{\psi'(2x)}{\psi'(x)}$$

Exercise 64 *In the case of the Clayton copula, the Laplace transform is* $\psi(x) = (1+\theta x)^{-1/\theta}$. *We have:*

$$\frac{\psi'(2x)}{\psi'(x)} = \frac{(1+2\theta x)^{-1/\theta-1}}{(1+\theta x)^{-1/\theta-1}}$$

We deduce that:

$$\begin{aligned}\lambda^+ &= 2 - 2\lim_{x\to 0}\frac{(1+2\theta x)^{-1/\theta-1}}{(1+\theta x)^{-1/\theta-1}} \\ &= 2-2 \\ &= 0\end{aligned}$$

and:

$$\begin{aligned}\lambda^- &= 2\lim_{x\to\infty}\frac{(1+2\theta x)^{-1/\theta-1}}{(1+\theta x)^{-1/\theta-1}} \\ &= 2\times 2^{-1/\theta-1} \\ &= 2^{-1/\theta}\end{aligned}$$

15.3.2 Normal copula

The normal copula is the dependency function of the multivariate normal distribution with a correlation matrix ρ:

$$\mathbf{C}(u_1,\ldots,u_n;\rho) = \Phi_n\left(\Phi^{-1}(u_1),\ldots,\Phi^{-1}(u_n);\rho\right)$$

By using the canonical decomposition of the multivariate density function:

$$f(x_1,\ldots,x_n) = c(\mathbf{F}_1(x_1),\ldots,\mathbf{F}_n(x_n))\prod_{i=1}^{n} f_i(x_i)$$

We deduce that the density of the normal copula is:

$$c(u_1,\ldots,u_n,;\rho) = \frac{1}{|\rho|^{\frac{1}{2}}}\exp\left(-\frac{1}{2}x^\top\left(\rho^{-1}-I_n\right)x\right)$$

where $x_i = \mathbf{\Phi}^{-1}(u_i)$. In the bivariate case, we obtain[7]:

$$c(u_1,u_2;\rho) = \frac{1}{\sqrt{1-\rho^2}}\exp\left(-\frac{x_1^2+x_2^2-2\rho x_1x_2}{2(1-\rho^2)}+\frac{x_1^2+x_2^2}{2}\right)$$

[7] In the bivariate case, the parameter ρ is the cross-correlation between X_1 and X_2, that is the element $(1,2)$ of the correlation matrix.

It follows that the expression of the bivariate normal copula function is also equal to:

$$\mathbf{C}(u_1, u_2; \rho) = \int_{-\infty}^{\mathbf{\Phi}^{-1}(u_1)} \int_{-\infty}^{\mathbf{\Phi}^{-1}(u_2)} \phi_2(x_1, x_2; \rho)\, \mathrm{d}x_1\, \mathrm{d}x_2 \tag{15.6}$$

where $\phi_2(x_1, x_2; \rho)$ is the bivariate normal density:

$$\phi_2(x_1, x_2; \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{2(1-\rho^2)}\right)$$

Exercise 65 *Let (X_1, X_2) be a standardized Gaussian random vector, whose cross-correlation is ρ. Using the Cholesky decomposition, we write X_2 as follows:*

$$X_2 = \rho X_1 + \sqrt{1-\rho^2} X_3 \leq x_2$$

where $X_3 \sim \mathcal{N}(0,1)$ is independent from X_1 and X_2. We have:

$$\begin{aligned}
\Phi_2(x_1, x_2; \rho) &= \Pr\{X_1 \leq x_1, X_2 \leq x_2\} \\
&= \mathbb{E}\left[\Pr\left\{X_1 \leq x_1, \rho X_1 + \sqrt{1-\rho^2} X_3 \leq x_2 \mid X_1\right\}\right] \\
&= \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\, \mathrm{d}x
\end{aligned}$$

It follows that:

$$\mathbf{C}(u_1, u_2; \rho) = \int_{-\infty}^{\Phi^{-1}(u_1)} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x)\, \mathrm{d}x$$

We finally obtain that the bivariate normal copula function is equal to:

$$\mathbf{C}(u_1, u_2; \rho) = \int_0^{u_1} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u \tag{15.7}$$

This expression is more convenient to use than Equation (15.6).

Like the normal distribution, the normal copula is easy to manipulate for computational purposes. For instance, Kendall's tau and Spearman's rho are equal to:

$$\tau = \frac{2}{\pi} \arcsin \rho$$

and:

$$\varrho = \frac{6}{\pi} \arcsin \frac{\rho}{2}$$

The conditional distribution $\mathbf{C}_{2|1}(u_1, u_2)$ has the following expression:

$$\begin{aligned}
\mathbf{C}_{2|1}(u_1, u_2) &= \partial_1 \mathbf{C}(u_1, u_2) \\
&= \Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)
\end{aligned}$$

To compute the tail dependence, we apply Equation (15.4) and obtain:

$$\begin{aligned}\lambda^{+} &= 2-2\lim_{u\to 1^{-}}\Phi\left(\frac{\Phi^{-1}(u)-\rho\Phi^{-1}(u)}{\sqrt{1-\rho^{2}}}\right)\\ &= 2-2\lim_{u\to 1^{-}}\Phi\left(\frac{\sqrt{1-\rho}}{\sqrt{1+\rho}}\Phi^{-1}(u)\right)\end{aligned}$$

We finally deduce that:

$$\lambda^{+}=\lambda^{-}=\begin{cases}0 & \text{if } \rho<1\\ 1 & \text{if } \rho=1\end{cases}$$

In Figure 15.11, we have represented the quantile-quantile dependence measure $\lambda^{+}(\alpha)$ for several values of the parameter ρ. When ρ is equal to 90% and α is close to one, we notice that $\lambda^{+}(\alpha)$ dramatically decreases. This means that even if the correlation is high, the extremes are independent.

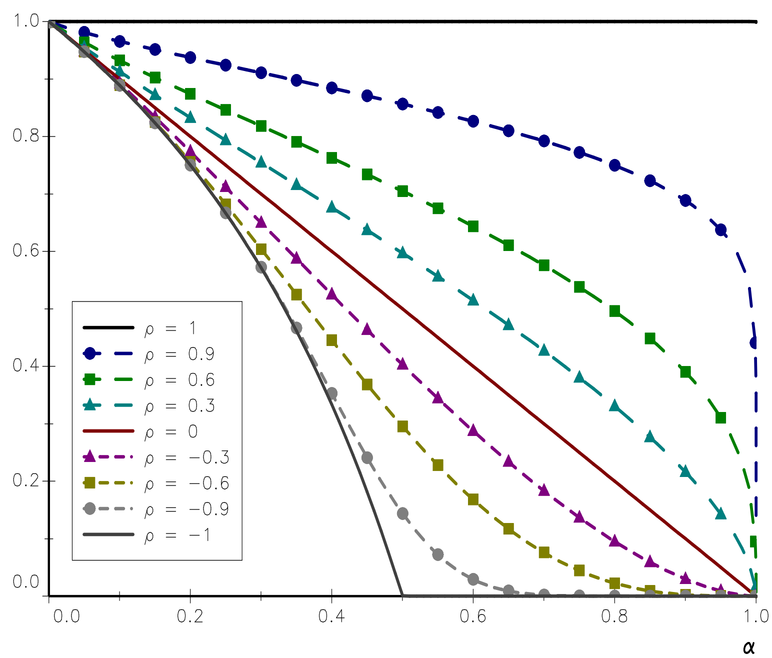

FIGURE 15.11: Tail dependence $\lambda^{+}(\alpha)$ for the normal copula

15.3.3 Student's t copula

In a similar way, the Student's t copula is the dependency function associated with the multivariate Student's t probability distribution:

$$\mathbf{C}(u_1,\ldots,u_n;\rho,\nu)=\mathbf{T}_n\left(\mathbf{T}_\nu^{-1}(u_1),\ldots,\mathbf{T}_\nu^{-1}(u_n);\rho,\nu\right)$$

By using the definition of the cumulative distribution function:

$$\mathbf{T}_n(x_1,\ldots,x_n;\rho,\nu)=\int_{-\infty}^{x_1}\cdots\int_{-\infty}^{x_n}\frac{\Gamma\left(\frac{\nu+n}{2}\right)|\rho|^{-\frac{1}{2}}}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{\frac{n}{2}}}\left(1+\frac{1}{\nu}x^\top\rho^{-1}x\right)^{-\frac{\nu+n}{2}}\,\mathrm{d}x$$

we can show that the copula density is then:

$$c(u_1,\ldots,u_n,;\rho,\nu)=|\rho|^{-\frac{1}{2}}\frac{\Gamma\left(\frac{\nu+n}{2}\right)\left[\Gamma\left(\frac{\nu}{2}\right)\right]^n}{\left[\Gamma\left(\frac{\nu+1}{2}\right)\right]^n\Gamma\left(\frac{\nu}{2}\right)}\frac{\left(1+\frac{1}{\nu}x^\top\rho^{-1}x\right)^{-\frac{\nu+n}{2}}}{\prod_{i=1}^n\left(1+\frac{x_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}}$$

where $x_i=\mathbf{T}_\nu^{-1}(u_i)$. In the bivariate case, we deduce that the t copula has the following expression:

$$\begin{aligned}\mathbf{C}(u_1,u_2;\rho,\nu) &= \int_{-\infty}^{\mathbf{T}_\nu^{-1}(u_1)}\int_{-\infty}^{\mathbf{T}_\nu^{-1}(u_2)}\frac{1}{2\pi\sqrt{1-\rho^2}}\times\\ &\quad\left(1+\frac{x_1^2+x_2^2-2\rho x_1x_2}{\nu(1-\rho^2)}\right)^{-\frac{\nu+2}{2}}\,\mathrm{d}x_1\,\mathrm{d}x_2\end{aligned}$$

Like the normal copula, we can obtain another expression, which is easier to manipulate. Let (X_1,X_2) be a random vector whose probability distribution is $\mathbf{T}_2(x_1,x_2;\rho,\nu)$. Conditional to $X_1=x_1$, we have:

$$\left(\frac{\nu+1}{\nu+x_1^2}\right)^{1/2}\frac{X_2-\rho x_1}{\sqrt{1-\rho^2}}\sim\mathbf{T}_{\nu+1}$$

The conditional distribution $\mathbf{C}_{2|1}(u_1,u_2)$ is then:

$$\mathbf{C}_{2|1}(u_1,u_2;\rho,\nu)=\mathbf{T}_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[\mathbf{T}_\nu^{-1}(u_1)\right]^2}\right)^{1/2}\frac{\mathbf{T}_\nu^{-1}(u_2)-\rho\mathbf{T}_\nu^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$$

We deduce that:

$$\mathbf{C}(u_1,u_2;\rho,\nu)=\int_0^{u_1}\mathbf{C}_{2|1}(u,u_2;\rho,\nu)\,\mathrm{d}u$$

We can show that the expression of Kendall's tau for the t copula is the one obtained for the normal copula. In the case of Spearman's rho, there is no analytical expression. We denote by $\varrho_t(\rho,\nu)$ and $\varrho_n(\rho)$ the values of Spearman's rho for Student's t and normal copulas with same parameter ρ. We can show that $\varrho_t(\rho,\nu)>\varrho_n(\rho)$ for negative values of ρ and $\varrho_t(\rho,\nu)<\varrho_n(\rho)$ for positive values of ρ. In Figure 15.12, we report the relationship between τ and ϱ for different degrees of freedom ν.

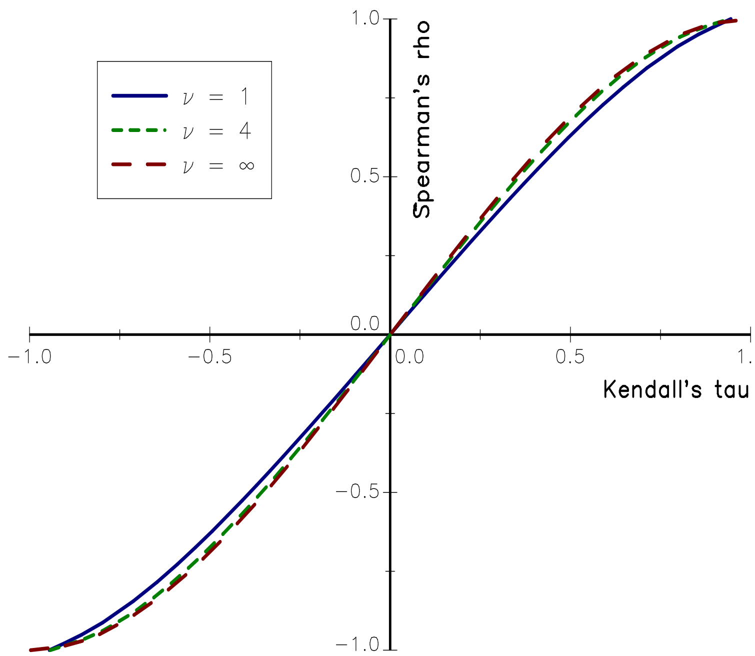

FIGURE 15.12: Relationship between τ and ϱ of the Student's t copula

Because the t copula is symmetric, we can apply Equation (15.4) and obtain:

$$\begin{aligned}\lambda^{+} &= 2-2\lim_{u\to 1^{-}}\mathbf{T}_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[\mathbf{T}_{\nu}^{-1}(u)\right]^{2}}\right)^{1/2}\frac{\mathbf{T}_{\nu}^{-1}(u)-\rho\mathbf{T}_{\nu}^{-1}(u)}{\sqrt{1-\rho^{2}}}\right)\\ &= 2-2\mathbf{T}_{\nu+1}\left(\left(\frac{(\nu+1)(1-\rho)}{(1+\rho)}\right)^{1/2}\right)\end{aligned}$$

We finally deduce that:

$$\lambda^{+}=\begin{cases}0 & \text{if } \rho=-1\\ >0 & \text{if } \rho>-1\end{cases}$$

Contrary to the normal copula, the t copula has an upper tail dependence. In Figures 15.13 and 15.14, we represent the quantile-quantile dependence measure $\lambda^{+}(\alpha)$ for two degrees of freedom ν. We observe that the behavior of $\lambda^{+}(\alpha)$ is different than the one obtained in Figure 15.11 with the normal copula. In Table 15.2, we give the numerical values of the coefficient λ^{+} for various values of ρ and ν. We notice that it is strictly positive for small degrees of freedom even if the parameter ρ is negative. For instance, λ^{+} is equal to 13.40% when ν and ρ are equal to 1 and −50%. We also observe that the convergence to the Gaussian case is low when the parameter ρ is positive.

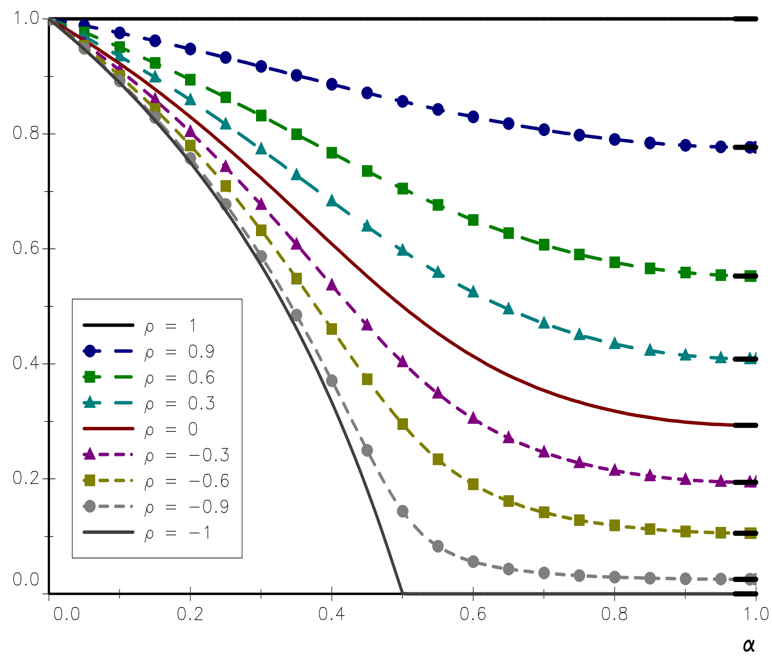

FIGURE 15.13: Tail dependence $\lambda^{+}(\alpha)$ for the Student's t copula ($\nu = 1$)

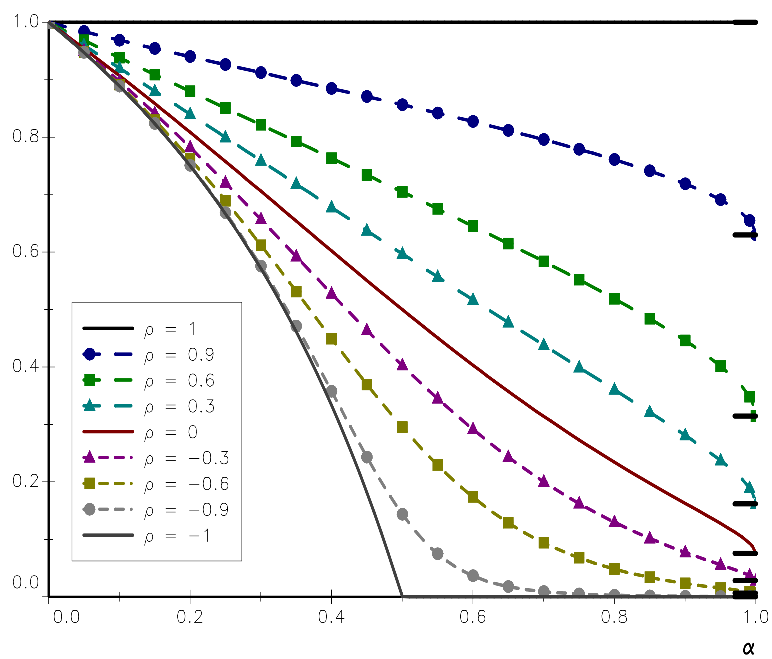

FIGURE 15.14: Tail dependence $\lambda^{+}(\alpha)$ for the Student's t copula ($\nu = 4$)

TABLE 15.2: Values in % of the upper tail dependence λ^+ for the t copula

ν	Parameter ρ (in %)					
	-70.00	-50.00	0.00	50.00	70.00	90.00
1	7.80	13.40	29.29	50.00	61.27	77.64
2	2.59	5.77	18.17	39.10	51.95	71.77
3	0.89	2.57	11.61	31.25	44.81	67.02
4	0.31	1.17	7.56	25.32	39.07	62.98
6	0.04	0.25	3.31	17.05	30.31	56.30
10	0.00	0.01	0.69	8.19	19.11	46.27
∞	0.00	0.00	0.00	0.00	0.00	0.00

Remark 56 *The normal copula is a particular case of the Student's t copula when ν tends to ∞. This is why these two copulas are often compared for a given value of ρ. However, we must be careful because the previous analysis of the tail dependence has shown that these two copulas are very different. Let us consider the bivariate case. We can write the Student's t random vector (T_1, T_2) as follows:*

$$\begin{aligned}(T_1, T_2) &= \frac{(N_1, N_2)}{\sqrt{X/\nu}} \\ &= \left(\frac{N_1}{\sqrt{X/\nu}}, \rho\frac{N_1}{\sqrt{X/\nu}} + \sqrt{1-\rho^2}\frac{N_3}{\sqrt{X/\nu}}\right)\end{aligned}$$

where N_1 and N_3 are two independent Gaussian random variables and X is a random variable, whose probability distribution is χ^2_ν. This is the introduction of the random variable X that produces a strong dependence between T_1 and T_2, and correlates the extremes. Even if the parameter ρ is equal to zero, we obtain:

$$(T_1, T_2) = \left(\frac{N_1}{\sqrt{X/\nu}}, \frac{N_3}{\sqrt{X/\nu}}\right)$$

This implies that the product copula $\mathbf{C}^{\perp}$ can never be attained by the t copula.

15.4 Statistical inference and estimation of copula functions

We now consider the estimation problem of copula functions. We first introduce the empirical copula, which may viewed as a non-parametric estimator of the copula function. Then, we discuss the method of moments to estimate the parameters of copula functions. Finally, we apply the method of maximum likelihood and show the different forms of implementation.

15.4.1 The empirical copula

Let $\hat{\mathbf{F}}$ be the empirical distribution associated to a sample of T observations of the random vector $(X_1, \ldots, X_n)$. Following Deheuvels (1979), any copula $\hat{\mathbf{C}} \in \mathcal{C}$ defined on the lattice $\mathfrak{L}$:

$$\mathfrak{L} = \left\{ \left(\frac{t_1}{T}, \ldots, \frac{t_n}{T} \right) : 1 \leq j \leq n, t_j = 0, \ldots, T \right\}$$

by the function:

$$\hat{\mathbf{C}} \left(\frac{t_1}{T}, \ldots, \frac{t_n}{T} \right) = \frac{1}{T} \sum_{t=1}^{T} \prod_{i=1}^{n} \mathbf{1} \left\{ \mathfrak{R}_{t,i} \leq t_i \right\}$$

is an empirical copula. Here $\mathfrak{R}_{t,i}$ is the rank statistic of the random variable X_i meaning that $X_{\mathfrak{R}_{t,i}:T,i} = X_{t,i}$. We notice that $\hat{\mathbf{C}}$ is the copula function associated to the empirical distribution $\hat{\mathbf{F}}$. However, $\hat{\mathbf{C}}$ is not unique because $\hat{\mathbf{F}}$ is not continuous. In the bivariate case, we obtain:

$$\begin{aligned} \hat{\mathbf{C}} \left(\frac{t_1}{T}, \frac{t_2}{T} \right) &= \frac{1}{T} \sum_{t=1}^{T} \mathbf{1} \left\{ \mathfrak{R}_{t,1} \leq t_1, \mathfrak{R}_{t,2} \leq t_2 \right\} \\ &= \frac{1}{T} \sum_{t=1}^{T} \mathbf{1} \left\{ x_{t,1} \leq x_{t_1:T,1}, x_{t,2} \leq x_{t_2:T,2} \right\} \end{aligned}$$

where $\{(x_{t,1}, x_{t,2}), t = 1, \ldots, T\}$ denotes the sample of (X_1, X_2). Nelsen (2006) defines the empirical copula frequency function as follows:

$$\begin{aligned} \hat{c} \left(\frac{t_1}{T}, \frac{t_2}{T} \right) &= \hat{\mathbf{C}} \left(\frac{t_1}{T}, \frac{t_2}{T} \right) - \hat{\mathbf{C}} \left(\frac{t_1 - 1}{T}, \frac{t_2}{T} \right) - \\ &\quad \hat{\mathbf{C}} \left(\frac{t_1}{T}, \frac{t_2 - 1}{T} \right) + \hat{\mathbf{C}} \left(\frac{t_1 - 1}{T}, \frac{t_2 - 1}{T} \right) \\ &= \frac{1}{T} \sum_{t=1}^{T} \mathbf{1} \left\{ x_{t,1} = x_{t_1:T,1}, x_{t,2} = x_{t_2:T,2} \right\} \end{aligned}$$

We have then:

$$\hat{\mathbf{C}} \left(\frac{t_1}{T}, \frac{t_2}{T} \right) = \sum_{j_1=1}^{t_1} \sum_{j_2=1}^{t_2} \hat{c} \left(\frac{j_1}{T}, \frac{j_2}{T} \right)$$

We can interpret $\hat{c}$ as the probability density function of the sample.

Example 66 *We consider the daily returns of European (EU) and American (US) MSCI equity indices from January 2006 and December 2015. In Figure 15.15, we represent the level lines of the empirical copula and compare them with the level lines of the normal copula. For this copula function, the parameter ρ is estimated by the linear correlation between the daily returns of the two MSCI equity indices. We notice that the normal copula does not exactly fit the empirical copula.*

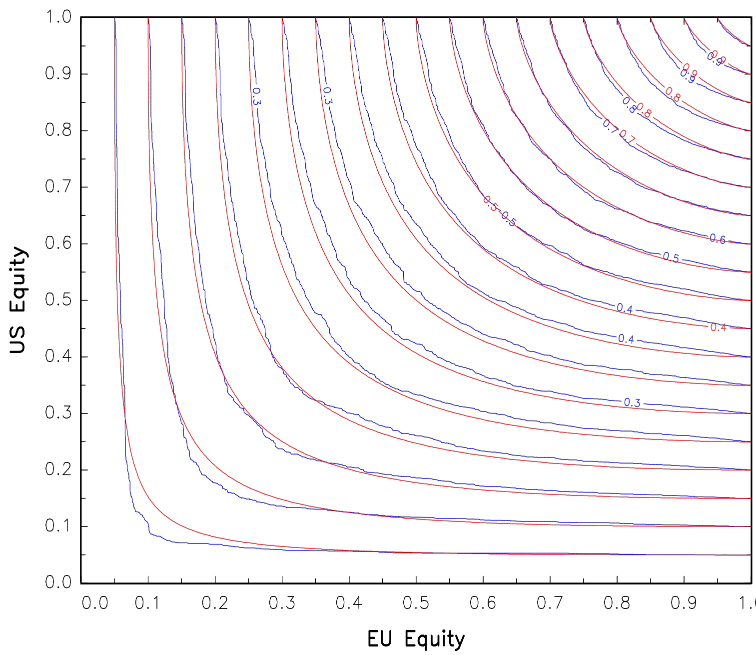

FIGURE 15.15: Comparison of the empirical copula (blue line) and the normal copula (red line)

Like the histogram of the empirical distribution function $\hat{\mathbf{F}}$, it is difficult to extract information from $\hat{\mathbf{C}}$ or $\hat{c}$, because these functions are not smooth[8]. It is better to use a dependogram. This representation has been introduced by Deheuvels (1981), and consists in transforming the sample $\{(x_{t,1}, x_{t,2}), t = 1, \ldots, T\}$ of the random vector (X_1, X_2) into a sample $\{(u_{t,1}, u_{t,2}), t = 1, \ldots, T\}$ of uniform random variables (U_1, U_2) by considering the rank statistics:

$$u_{t,i} = \frac{1}{T}\mathfrak{R}_{t,i}$$

The dependogram is then the scatter plot between $u_{t,1}$ and $u_{t,2}$. For instance,

[8] This is why they are generally coupled with approximation methods based on Bernstein polynomials (Sancetta and Satchell, 2004).

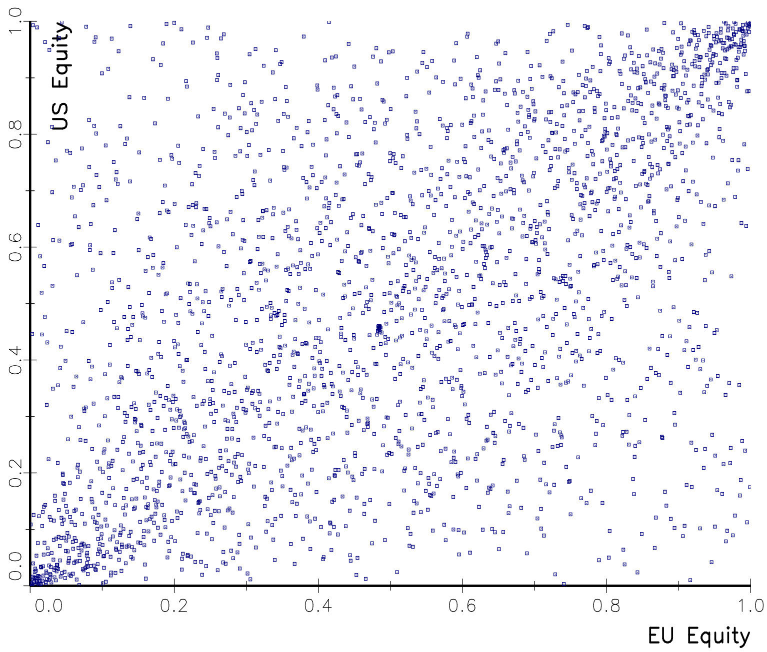

FIGURE 15.16: Dependogram of EU and US equity returns

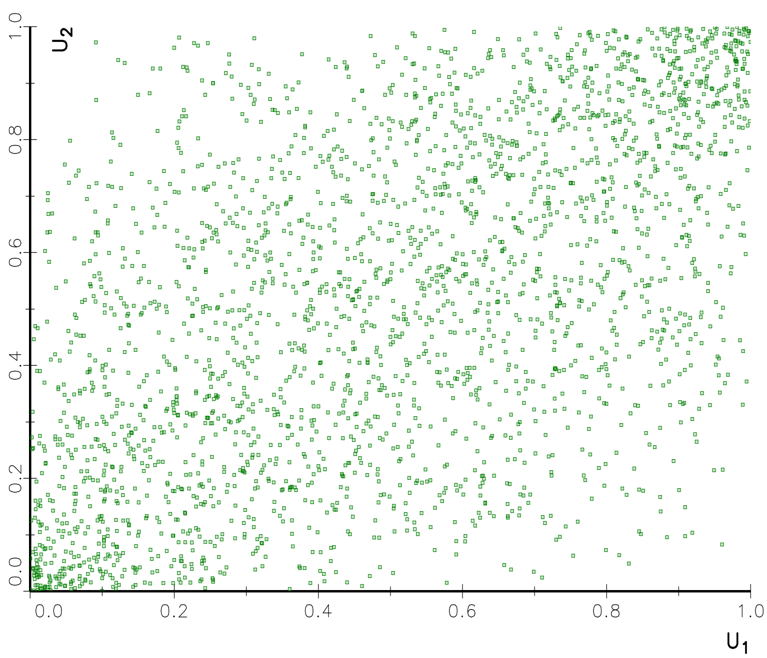

FIGURE 15.17: Dependogram of simulated Gaussian returns

Figure 15.16 shows the dependogram of EU and US equity returns. We can compare this figure with the one obtained by assuming that equity returns are Gaussian. Indeed, Figure 15.17 shows the dependogram of a simulated bivariate Gaussian random vector when the correlation is equal to 57.8%, which is the estimated value between EU and US equity returns during the study period.

15.4.2 The method of moments

When it is applied to copulas, this method is different than the one presented in Chapter 14. Indeed, it consists in estimating the parameters θ of the copula function from the population version of concordance measures. For instance, if $\tau = f_\tau(\theta)$ is the relationship between θ and Kendall's tau, the MM estimator is simply the inverse of this relationship:

$$\hat{\theta} = f_\tau^{-1}(\hat{\tau})$$

where $\hat{\tau}$ is the estimate of Kendall's tau based on the sample[9]. For instance, in the case of the Gumbel copula, we obtain:

$$\hat{\theta} = \frac{1}{1-\hat{\tau}}$$

Remark 57 *This approach is also valid for other concordance measures like Spearman's rho. We have then:*

$$\hat{\theta} = f_\varrho^{-1}(\hat{\varrho})$$

where $\hat{\varrho}$ is the estimate[10] *of Spearman's rho and f_ϱ is the theoretical relationship between θ and Spearman's rho.*

Example 67 *We consider the daily returns of 5 asset classes from January 2006 and December 2015. These asset classes are represented by the European MSCI equity index, the American MSCI equity index, the Barclays sovereign bond index, the Barclays corporate investment grade bond index and the Bloomberg commodity index. In Table 15.3, we report the correlation matrix. In Tables 15.4 and 15.5, we assume that the dependence function is a normal copula and give the matrix $\hat{\rho}$ of estimated parameters using the method of moments based on Kendall's tau and Spearman's rho. We notice that these two matrices are very close, but we also also observe some important differences with the correlation matrix reported in Table 15.3.*

[9] We have:

$$\hat{\tau} = \frac{c-d}{c+d}$$

where c and d are respectively the number of concordant and discordant pairs.

[10] It is equal to the the linear correlation between the rank statistics.

TABLE 15.3: Matrix of linear correlations $\hat{\rho}_{i,j}$

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	57.8	100.0			
Sovereign	−34.0	−32.6	100.0		
Credit	−15.1	−28.6	69.3	100.0	
Commodity	51.8	34.3	−22.3	−14.4	100.0

TABLE 15.4: Matrix of parameters $\hat{\rho}_{i,j}$ estimated using Kendall's tau

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	57.7	100.0			
Sovereign	−31.8	−32.1	100.0		
Credit	−17.6	−33.8	73.9	100.0	
Commodity	43.4	30.3	−19.6	−15.2	100.0

15.4.3 The method of maximum likelihood

Let us denote by $\{(x_{t,1}, \ldots, x_{t,n}), t = 1 \ldots, T\}$ the sample of the random vector $(X_1, \ldots, X_n)$, whose multivariate distribution function has the following canonical decomposition:

$$\mathbf{F}(x_1, \ldots, x_n) = \mathbf{C}(\mathbf{F}_1(x_1; \theta_1), \ldots, \mathbf{F}_n(x_n; \theta_n); \theta_c)$$

This means that this statistical model depends on two types of parameters:

- the parameters $(\theta_1, \ldots, \theta_n)$ of univariate distribution functions;
- the parameters θ_c of the copula function.

TABLE 15.5: Matrix of parameters $\hat{\rho}_{i,j}$ estimated using Spearman's rho

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	55.4	100.0			
Sovereign	−31.0	−31.3	100.0		
Credit	−17.1	−32.7	73.0	100.0	
Commodity	42.4	29.4	−19.2	−14.9	100.0

The expression of the log-likelihood function is:

$$\begin{aligned}\ell\left(\theta_1,\ldots,\theta_n,\theta_c\right) &= \sum_{t=1}^{T}\ln c\left(\mathbf{F}_1\left(x_{t,1};\theta_1\right),\ldots,\mathbf{F}_n\left(x_{t,n};\theta_n\right);\theta_c\right)+ \\ &\quad \sum_{t=1}^{T}\sum_{i=1}^{n}\ln f_i\left(x_{t,i};\theta_i\right)\end{aligned}$$

where c is the copula density and f_i is the probability density function associated to $\mathbf{F}_i$. The ML estimator is then defined as follows:

$$\left(\hat{\theta}_1,\ldots,\hat{\theta}_n,\hat{\theta}_c\right)=\arg\max\ell\left(\theta_1,\ldots,\theta_n,\theta_c\right)$$

The estimation by maximum likelihood method can be time-consuming when the number of parameters is large. However, the copula approach suggests a two-stage parametric method (Shih and Louis, 1995):

1. the first stage involves maximum likelihood from univariate marginals, meaning that we estimate the parameters $\theta_1,\ldots,\theta_n$ separately for each marginal:

$$\hat{\theta}_i=\arg\max\sum_{t=1}^{T}\ln f_i\left(x_{t,i};\theta_i\right)$$

2. the second stage involves maximum likelihood of the copula parameters θ_c with the univariate parameters $\hat{\theta}_1,\ldots,\hat{\theta}_n$ held fixed from the first stage:

$$\hat{\theta}_c=\arg\max\sum_{t=1}^{T}\ln c\left(\mathbf{F}_1\left(x_{t,1};\hat{\theta}_1\right),\ldots,\mathbf{F}_n\left(x_{t,n};\hat{\theta}_n\right);\theta_c\right)$$

This approach is known as the method of inference functions for marginals or IFM (Joe, 1997). Let $\hat{\theta}_{\mathrm{IFM}}$ be the IFM estimator obtained with this two-stage procedure. We have:

$$T^{1/2}\left(\hat{\theta}_{\mathrm{IFM}}-\theta_0\right)\rightarrow\mathcal{N}\left(\mathbf{0},\mathcal{V}^{-1}\left(\theta_0\right)\right)$$

where $\mathcal{V}\left(\theta_0\right)$ is the Godambe matrix (Joe, 1997).

Genest *et al.* (1995) propose a third estimation method, which consists in estimating the copula parameters θ_c by considering the nonparametric estimates of the marginals $\mathbf{F}_1,\ldots,\mathbf{F}_n$:

$$\hat{\theta}_c=\arg\max\sum_{t=1}^{T}\ln c\left(\hat{\mathbf{F}}_1\left(x_{t,1}\right),\ldots,\hat{\mathbf{F}}_n\left(x_{t,n}\right);\theta_c\right)$$

In this case, $\hat{\mathbf{F}}_i\left(x_{t,i}\right)$ is the normalized rank $\mathfrak{R}_{t,i}/T$. This estimator called omnibus or OM is then the ML estimate applied to the dependogram.

Example 68 *Let us assume that the dependence function of asset returns (X_1, X_2) is the Frank copula whereas the marginals are Gaussian. The log-likelihood function for observation t is then equal to:*

$$\begin{aligned}
\ell_t &= \ln\left(\theta_c\left(1-e^{-\theta_c}\right)e^{-\theta_c(\Phi(y_{t,1})+\Phi(y_{t,2}))}\right) - \\
&\quad \ln\left(\left(1-e^{-\theta_c}\right)-\left(1-e^{-\theta_c\Phi(y_{t,1})}\right)\left(1-e^{-\theta_c\Phi(y_{t,2})}\right)\right)^2 - \\
&\quad \left(\frac{1}{2}\ln 2\pi+\frac{1}{2}\ln\sigma_1^2+\frac{1}{2}y_{t,1}^2\right) - \\
&\quad \left(\frac{1}{2}\ln 2\pi+\frac{1}{2}\ln\sigma_2^2+\frac{1}{2}y_{t,2}^2\right)
\end{aligned}$$

where $y_{t,i} = \sigma_i^{-1}(x_{t,i} - \mu_i)$ is the standardized return of Asset i for the observation t. The vector of parameters to estimate is $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \theta_c)$. In the case of the IFM approach, the parameters $(\mu_1, \sigma_1, \mu_2, \sigma_2)$ are estimated in a first step. Then, we estimate the copula parameter θ_c by considering the following log-likelihood:

$$\begin{aligned}
\ell_t &= \ln\left(\theta_c\left(1-e^{-\theta_c}\right)e^{-\theta_c(\Phi(\hat{y}_{t,1})+\Phi(\hat{y}_{t,2}))}\right) - \\
&\quad \ln\left(\left(1-e^{-\theta_c}\right)-\left(1-e^{-\theta_c\Phi(\hat{y}_{t,1})}\right)\left(1-e^{-\theta_c\Phi(\hat{y}_{t,2})}\right)\right)^2
\end{aligned}$$

where $\hat{y}_{t,i}$ is equal to $\hat{\sigma}_i^{-1}(x_{t,i} - \hat{\mu}_i)$. Finally, the OM approach uses the uniform variates $u_{t,i} = \Re_{t,i}/T$ in the expression of the log-likelihood function:

$$\begin{aligned}
\ell_t &= \ln\left(\theta_c\left(1-e^{-\theta_c}\right)e^{-\theta_c(u_{t,1}+u_{t,2})}\right) - \\
&\quad \ln\left(\left(1-e^{-\theta_c}\right)-\left(1-e^{-\theta_c u_{t,1}}\right)\left(1-e^{-\theta_c u_{t,2}}\right)\right)^2
\end{aligned}$$

Using the returns of MSCI Europe and US indices for the last 10 years, we obtain the following results for the parameter θ_c of the Frank copula:

	ML	IFM	OM	Method of Moments Kendall	Method of Moments Spearman
$\hat{\theta}_c$	6.809	6.184	4.149	3.982	3.721
$\hat{\tau}$	0.554	0.524	0.399	0.387	0.367
$\hat{\varrho}$	0.754	0.721	0.571	0.555	0.529

We obtain $\hat{\theta}_c = 6.809$ for the method of maximum likelihood and $\hat{\theta}_c = 6.184$ for the IFM approach. These results are very close, that is not the case with the omnibus approach where we obtain $\hat{\theta}_c = 4.149$. This means that the assumption of Gaussian marginals is far to be verified. The specification of wrong marginals in ML and IFM approaches induces then a bias in the estimation of the copula parameter. With the omnibus approach, we do not face this issue because we consider non-parametric marginals. This explains that we obtain a value, which is close to the MM estimates (Kendall's tau and Spearman's rho).

For IFM and OM approaches, we can obtain a semi-analytical expression of $\hat{\theta}_c$ for some specific copula functions. In the case of the normal copula, the matrix ρ of the parameters is estimated with the following algorithm:

1. we first transform the uniform variates $u_{t,i}$ into Gaussian variates:
$$n_{t,i} = \Phi^{-1}(u_{t,i})$$
2. we then calculate the correlation matrix of the Gaussian variates $n_{t,i}$.

For the Student's t copula, Bouyé *et al.* (2000) suggest the following algorithm:

1. Let $\hat{\rho}_0$ be the estimated value of ρ for the Gaussian copula;
2. $\hat{\rho}_{k+1}$ is obtained using the following equation:
$$\hat{\rho}_{k+1} = \frac{1}{T}\sum_{t=1}^{T} \frac{(\nu + n)\,\varsigma_t \varsigma_t^\top}{\nu + \varsigma_t^\top \hat{\rho}_k^{-1} \varsigma_t}$$
where:
$$\varsigma_t = \begin{pmatrix} \mathbf{t}_\nu^{-1}(u_{t,1}) \\ \vdots \\ \mathbf{t}_\nu^{-1}(u_{t,n}) \end{pmatrix}$$
3. Repeat the second step until convergence: $\hat{\rho}_{k+1} = \hat{\rho}_k := \hat{\rho}_\infty$.

Let us consider Example 67. We have estimated the parameter matrix ρ of Gaussian and Student's t copula using the omnibus approach. Results are given in Tables 15.6, 15.7 and 15.8. We notice that these matrices are different than the correlation matrix calculated in Table 15.3. The reason is that we have previously assumed that the marginals were Gaussian. In this case, the ML estimate introduced a bias in the copula parameter in order to compensate the bias induced by the wrong specification of the marginals.

TABLE 15.6: Omnibus estimate $\hat{\rho}$ (Gaussian copula)

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	56.4	100.0			
Sovereign	−32.5	−32.1	100.0		
Credit	−16.3	−30.3	70.2	100.0	
Commodity	46.5	30.7	−21.1	−14.7	100.0

TABLE 15.7: Omnibus estimate $\hat{\rho}$ (Student's t copula with $\nu = 1$)

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	47.1	100.0			
Sovereign	−20.3	−18.9	100.0		
Credit	−9.3	−22.1	57.6	100.0	
Commodity	28.0	17.1	−7.4	−6.2	100.0

TABLE 15.8: Omnibus estimate $\hat{\rho}$ (Student's t copula with $\nu = 4$)

	EU Equity	US Equity	Sovereign	Credit	Commodity
EU Equity	100.0				
US Equity	59.6	100.0			
Sovereign	−31.5	−31.9	100.0		
Credit	−18.3	−32.9	71.3	100.0	
Commodity	43.0	30.5	−17.2	−13.4	100.0

Remark 58 *The discrepancy between the ML or IFM estimate and the OM estimate is an interesting information for knowing if the specification of the marginals are right or not. In particular, a large discrepancy indicates that the estimated marginals are far from the empirical marginals.*

15.5 Exercises

15.5.1 Gumbel logistic copula

1. Calculate the density of the Gumbel logistic copula.
2. Show that it has a lower tail dependence, but no upper tail dependence.

15.5.2 Farlie-Gumbel-Morgenstern copula

We consider the following function:

$$\mathbf{C}(u_1, u_2) = u_1 u_2 (1 + \theta (1 - u_1)(1 - u_2)) \tag{15.8}$$

1. Show that $\mathbf{C}$ is a copula function for $\theta \in [-1, 1]$.
2. Calculate the tail dependence coefficient λ, the Kendall's τ statistic and the Spearman's ϱ statistic.
3. Let $X = (X_1, X_2)$ be a bivariate random vector. We assume that $X_1 \sim$

$\mathcal{N}\left(\mu, \sigma^2\right)$ and $X_2 \sim \mathcal{E}(\lambda)$. Propose an algorithm to simulate (X_1, X_2) when the copula is the function (15.8).

4. Calculate the log-likelihood function of the sample $\left\{(x_{1,i}, x_{2,i})_{i=1}^{i=n}\right\}$.

15.5.3 Survival copula

Let $\mathbf{S}$ be the bivariate function defined by:

$$\mathbf{S}(x_1, x_2) = \exp\left(-\left(x_1 + x_2 - \theta \frac{x_1 x_2}{x_1 + x_2}\right)\right)$$

with $\theta \in [0, 1]$, $x_1 \geq 0$ et $x_2 \geq 0$.

1. Verify that $\mathbf{S}$ is a survival distribution.
2. Define the survival copula associated to $\mathbf{S}$.

15.5.4 Method of moments

Let (X_1, X_2) be a bivariate random vector such that $X_1 \sim \mathcal{N}\left(\mu_1, \sigma_1^2\right)$ and $X_2 \sim \mathcal{N}\left(\mu_2, \sigma_2^2\right)$. We consider that the dependence function is given by the following copula:

$$\mathbf{C}(u_1, u_2) = \theta\, \mathbf{C}^-(u_1, u_2) + (1 - \theta)\, \mathbf{C}^+(u_1, u_2)$$

where $\theta \in [0, 1]$ is the copula parameter.

1. We assume that $\mu_1 = \mu_2 = 0$ and $\sigma_1 = \sigma_2 = 1$. Find the parameter θ such that the linear correlation of X_1 and X_2 is equal to zero. Show that there exists a function f such that $X_1 = f(X_2)$. Comment on this result.
2. Calculate the linear correlation of X_1 and X_2 as a function of the parameters μ_1, μ_2, σ_1, σ_2 and θ.
3. Propose a method of moments to estimate θ.

15.5.5 Correlated loss given default rates

We assume that the probability distribution of the (annual) loss given default rate associated to a risk class $\mathcal{C}$ is given by:

$$\begin{aligned} \mathbf{F}(x) &= \Pr\{\mathrm{LGD} \leq x\} \\ &= x^\gamma \end{aligned}$$

1. Find the conditions on the parameter γ that are necessary for $\mathbf{F}$ to be a probability distribution.

2. Let $\{x_1, \ldots, x_n\}$ be a sample of loss given default rates. Calculate the log-likelihood function and deduce the ML estimator $\hat{\gamma}_{\mathrm{ML}}$.

3. Calculate the first moment $\mathbb{E}[\mathrm{LGD}]$. Then find the method of moments estimator $\hat{\gamma}_{\mathrm{MM}}$.

4. We assume that $x_i = 50\%$ for all i. Calculate the numerical values taken by $\hat{\gamma}_{\mathrm{ML}}$ and $\hat{\gamma}_{\mathrm{MM}}$. Comment on these results.

5. We now consider two risk classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and note LGD_1 and LGD_2 the corresponding LGD rates. We assume that the dependence function between LGD_1 and LGD_2 is given by the Gumbel-Barnett copula:

$$\mathbf{C}(u_1, u_2) = u_1 u_2 e^{-\theta \ln u_1 \ln u_2}$$

where θ is the copula parameter. Show that the density function of the copula is equal to:

$$c(u_1, u_2; \theta) = \left(1 - \theta - \theta \ln(u_1 u_2) + \theta^2 \ln u_1 \ln u_2\right) e^{-\theta \ln u_1 \ln u_2}$$

6. Deduce the log-likelihood function of the historical sample $\left\{(x_i, y_i)_{i=1}^{i=n}\right\}$.

7. We note $\hat{\gamma}_1$, $\hat{\gamma}_2$ and $\hat{\theta}$ the ML estimators of the parameters γ_1 (risk class $\mathcal{C}_1$), γ_2 (risk class $\mathcal{C}_2$) and θ (copula parameter). Why the ML estimator $\hat{\gamma}_1$ does not correspond to the ML estimator $\hat{\gamma}_{\mathrm{ML}}$ except in the case $\hat{\theta} = 0$. Illustrate with an example.

15.5.6 Calculation of correlation bounds

1. Give the mathematical definition of the copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$. What is the probabilistic interpretation of these copulas?

2. We note $\boldsymbol{\tau}$ and LGD the default time and the loss given default of a counterparty. We assume that $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$ and $\mathrm{LGD} \sim \mathcal{U}_{[0,1]}$.

   (a) Show that the dependence between $\boldsymbol{\tau}$ and LGD is maximum when the following equality holds:

   $$\mathrm{LGD} + e^{-\lambda \boldsymbol{\tau}} - 1 = 0$$

   (b) Show that the linear correlation $\rho(\boldsymbol{\tau}, \mathrm{LGD})$ verifies the following inequality:

   $$|\rho\langle \boldsymbol{\tau}, \mathrm{LGD}\rangle| \leq \frac{\sqrt{3}}{2}$$

   (c) Comment on these results.

3. We consider two exponential default times $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ with parameters λ_1 and λ_2.

(a) We assume that the dependence function between $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ is $\mathbf{C}^+$. Demonstrate that the following relation is true:

$$\boldsymbol{\tau}_1 = \frac{\lambda_2}{\lambda_1}\boldsymbol{\tau}_2$$

(b) Show that there exists a function f such that $\boldsymbol{\tau}_2 = f(\boldsymbol{\tau}_2)$ when the dependence function is $\mathbf{C}^-$.

(c) Show that the lower and upper bounds of the linear correlation satisfy the following relationship:

$$-1 < \rho\langle\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle \leq 1$$

(d) In the more general case, show that the linear correlation of a random vector (X_1, X_2) can not be equal to -1 if the support of the random variables X_1 and X_2 is $[0, +\infty]$.

4. We assume that (X_1, X_2) is a Gaussian random vector where $X_1 \sim \mathcal{N}(\mu_1, \sigma_1^2)$, $X_2 \sim \mathcal{N}(\mu_2, \sigma_2^2)$ and ρ is the linear correlation between X_1 and X_2. We note $\theta = (\mu_1, \sigma_1, \mu_2, \sigma_2, \rho)$ the set of parameters.

(a) Find the probability distribution of $X_1 + X_2$.

(b) Then show that the covariance between $Y_1 = e^{X_1}$ and $Y_2 = e^{X_2}$ is equal to:

$$\operatorname{cov}(Y_1, Y_2) = e^{\mu_1 + \frac{1}{2}\sigma_1^2} e^{\mu_2 + \frac{1}{2}\sigma_2^2}\left(e^{\rho\sigma_1\sigma_2} - 1\right)$$

(c) Deduce the correlation between Y_1 and Y_2.

(d) For which values of θ does the equality $\rho\langle Y_1, Y_2\rangle = +1$ hold? Same question when $\rho\langle Y_1, Y_2\rangle = -1$.

(e) We consider the bivariate Black-Scholes model:

$$\begin{cases} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{cases}$$

with $\mathbb{E}[W_1(t)\,W_2(t)] = \rho t$. Deduce the linear correlation between $S_1(t)$ and $S_2(t)$. Find the limit case $\lim_{t\to\infty}\rho\langle S_1(t), S_2(t)\rangle$.

(f) Comment on these results.

Chapter 16

Extreme Value Theory

This chapter is dedicated to tail (or extreme) risk modeling. Tail risk recovers two notions. The first one is related to rare events, meaning that a severe loss may occur with a very small probability. The second one concerns the magnitude of a loss that is difficult to reconciliate with the observed volatility of the portfolio. Of course, the two notions are connected, but the second is more frequent. For instance, stock market crashes are numerous since the end of the eighties. The study of these rare or abnormal events needs an appropriate framework to analyze their risk. This is the subject of this chapter. In a first section, we consider order statistics, which are very useful to understand the underlying concept of tail risk. Then, we present the extreme value theory (EVT) in the unidimensional case. Finally, the last section deals with the correlation issue between extreme risks.

16.1 Order statistics

16.1.1 Main Properties

Let $X_1, \ldots, X_n$ be *iid* random variables, whose probability distribution is denoted by $\mathbf{F}$. We rank these random variables by increasing order:

$$X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n-1:n} \leq X_{n:n}$$

$X_{i:n}$ is called the i^{th} order statistic in the sample of size n. We note $x_{i:n}$ the corresponding random variate or the value taken by $X_{i:n}$. We have:

$$\begin{aligned}
\mathbf{F}_{i:n}(x) &= \Pr\{X_{i:n} \leq x\} \\
&= \Pr\{\text{at least } i \text{ variables among } X_1, \ldots, X_n \text{ are less or equal to } x\} \\
&= \sum_{k=i}^{n} \Pr\{k \text{ variables among } X_1, \ldots, X_n \text{ are less or equal to } x\} \\
&= \sum_{k=i}^{n} \binom{n}{k} \mathbf{F}(x)^k (1 - \mathbf{F}(x))^{n-k} \qquad (16.1)
\end{aligned}$$

We note f the density function of $\mathbf{F}$. We deduce that the density function of $X_{i:n}$ has the following expression[1]:

$$\begin{aligned}
f_{i:n}(x) &= \sum_{k=i}^{n}\binom{n}{k}k\mathbf{F}(x)^{k-1}(1-\mathbf{F}(x))^{n-k}f(x)- \\
&\quad \sum_{k=i}^{n-1}\binom{n}{k}\mathbf{F}(x)^{k}(n-k)(1-\mathbf{F}(x))^{n-k-1}f(x) \\
&= \sum_{k=i}^{n}\frac{n!}{(k-1)!\,(n-k)!}\mathbf{F}(x)^{k-1}(1-\mathbf{F}(x))^{n-k}f(x)- \\
&\quad \sum_{k=i}^{n-1}\frac{n!}{k!\,(n-k-1)!}\mathbf{F}(x)^{k}(1-\mathbf{F}(x))^{n-k-1}f(x) \\
&= \sum_{k=i}^{n}\frac{n!}{(k-1)!\,(n-k)!}\mathbf{F}(x)^{k-1}(1-\mathbf{F}(x))^{n-k}f(x)- \\
&\quad \sum_{k=i+1}^{n}\frac{n!}{(k-1)!\,(n-k)!}\mathbf{F}(x)^{k-1}(1-\mathbf{F}(x))^{n-k}f(x) \\
&= \frac{n!}{(i-1)!\,(n-i)!}\mathbf{F}(x)^{i-1}(1-\mathbf{F}(x))^{n-i}f(x) \qquad (16.2)
\end{aligned}$$

Example 69 *If $X_1,\ldots,X_n$ follow a uniform distribution $\mathcal{U}_{[0,1]}$, we obtain:*

$$\begin{aligned}
\mathbf{F}_{i:n}(x) &= \sum_{k=i}^{n}\binom{n}{k}x^{k}(1-x)^{n-k} \\
&= B(x;i,n-i+1)
\end{aligned}$$

where $B(x;a,b)$ is the regularized incomplete beta function[2]:

$$B(x;a,b)=\frac{1}{B(a,b)}\int_0^x t^{a-1}(1-t)^{b-1}\,\mathrm{d}t$$

We deduce that $X_{i:n}\sim\mathcal{B}(i,n-i+1)$. It follows that the expected value of the order statistic $X_{i:n}$ is equal to:

$$\begin{aligned}
\mathbb{E}[X_{i:n}] &= \mathbb{E}[\mathcal{B}(i,n-i+1)] \\
&= \frac{i}{n+1}
\end{aligned}$$

[1]When k is equal to n, the derivative of $(1-\mathbf{F}(x))^{n-k}$ is equal to zero. This explains that the second summation does not include the case $k=n$.

[2]It is also the Beta probability distribution: $B(x;a,b)=\Pr\{X\leq x\}$ where $X\sim\mathcal{B}(a,b)$.

We verify the stochastic ordering:

$$j > i \Rightarrow \mathbf{F}_{i:n} \succ \mathbf{F}_{j:n}$$

Indeed, we have:

$$\begin{aligned}
\mathbf{F}_{i:n}(x) &= \sum_{k=i}^{n} \binom{n}{k} \mathbf{F}(x)^k (1 - \mathbf{F}(x))^{n-k} \\
&= \sum_{k=i}^{j-1} \binom{n}{k} \mathbf{F}(x)^k (1 - \mathbf{F}(x))^{n-k} + \sum_{k=j}^{n} \binom{n}{k} \mathbf{F}(x)^k (1 - \mathbf{F}(x))^{n-k} \\
&= \mathbf{F}_{j:n}(x) + \sum_{k=i}^{j-1} \binom{n}{k} \mathbf{F}(x)^k (1 - \mathbf{F}(x))^{n-k}
\end{aligned}$$

meaning that $\mathbf{F}_{i:n}(x) \geq \mathbf{F}_{j:n}(x)$. In Figure 16.1, we illustrate this property when the random variables $X_1, \ldots, X_n$ follow the normal distribution $\mathcal{N}(0, 1)$. We verify that $\mathbf{F}_{i:n}(x)$ increases with the ordering value i.

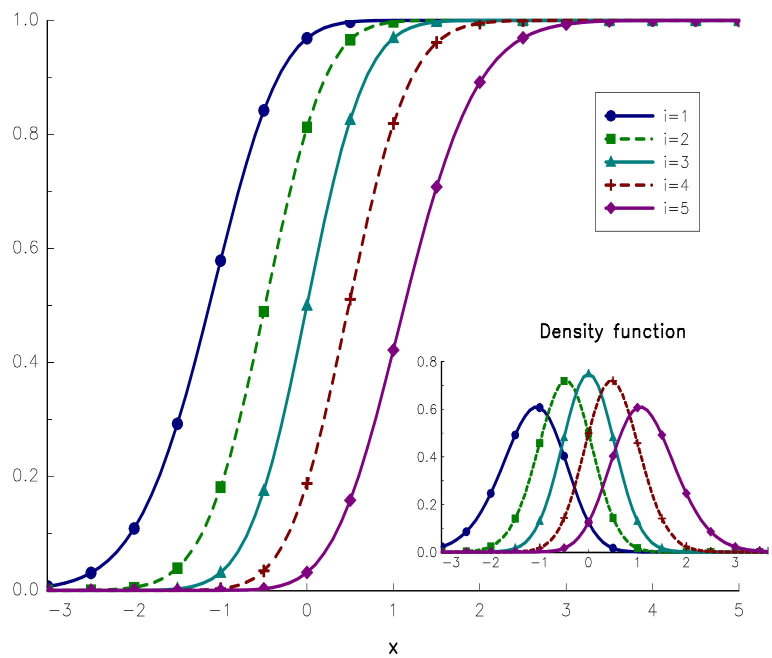

FIGURE 16.1: Distribution function $\mathbf{F}_{i:n}$ when the random variables $X_1, \ldots, X_n$ are Gaussian

16.1.2 Extreme order statistics

Two order statistics are particularly interesting for the study of rare events. They are the lower and higher order statistics:

$$M_n^- = X_{1:n} = \min\left(X_1, \ldots, X_n\right)$$

and:

$$M_n^+ = X_{n:n} = \max\left(X_1, \ldots, X_n\right)$$

We can find their probability distributions by setting $i = 1$ and $i = n$ in Formula (16.1). We can also retrieve their expression by noting that:

$$\begin{aligned}
\mathbf{F}_{1:n}(x) &= \Pr\left\{\min\left(X_1, \ldots, X_n\right) \leq x\right\} \\
&= 1 - \Pr\left\{\min\left(X_1, \ldots, X_n\right) \geq x\right\} \\
&= 1 - \Pr\left\{X_1 \geq x, X_2 \geq x, \ldots, X_n \geq x\right\} \\
&= 1 - \prod_{i=1}^{n} \Pr\left\{X_i \geq x\right\} \\
&= 1 - \prod_{i=1}^{n} \left(1 - \Pr\left\{X_i \leq x\right\}\right) \\
&= 1 - \left(1 - \mathbf{F}(x)\right)^n
\end{aligned}$$

and:

$$\begin{aligned}
\mathbf{F}_{n:n}(x) &= \Pr\left\{\max\left(X_1, \ldots, X_n\right) \leq x\right\} \\
&= \Pr\left\{X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x\right\} \\
&= \prod_{i=1}^{n} \Pr\left\{X_i \leq x\right\} \\
&= \mathbf{F}(x)^n
\end{aligned}$$

We deduce that the density functions are equal to:

$$f_{1:n}(x) = n\left(1 - \mathbf{F}(x)\right)^{n-1} f(x)$$

and

$$f_{n:n}(x) = n\mathbf{F}(x)^{n-1} f(x)$$

Let us consider an example with the Gaussian distribution $\mathcal{N}(0,1)$. Figure 16.2 shows the evolution of the density function $f_{n:n}$ with respect to the sample size n. We verify the stochastic ordering: $n > m \Rightarrow \mathbf{F}_{n:n} \succ \mathbf{F}_{m:m}$.

Let us now illustrate the impact of distribution tails on order statistics. We consider the daily returns of the MSCI USA index from 1995 to 2015. We consider three hypotheses:

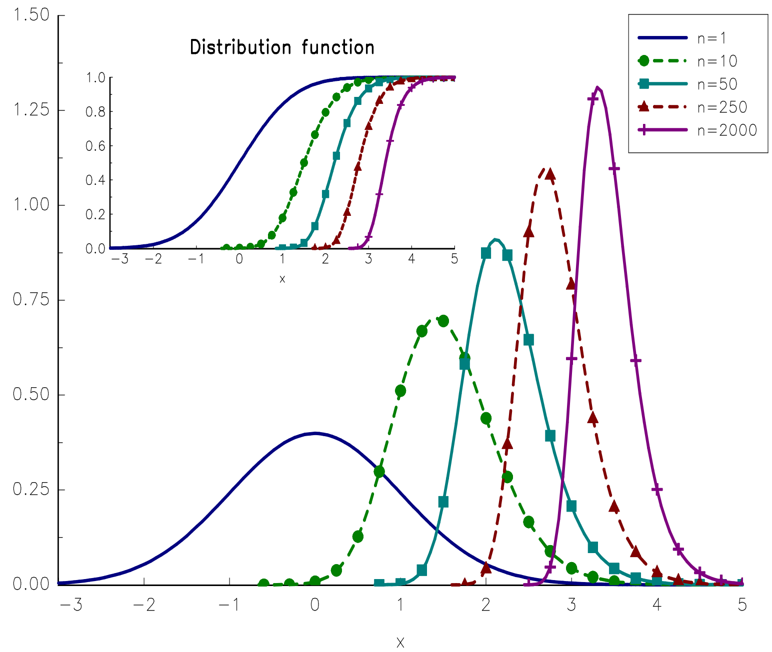

FIGURE 16.2: Density function $f_{n:n}$ of the Gaussian random variable $\mathcal{N}(0,1)$

$\mathcal{H}_1$ Daily returns are Gaussian, meaning that:

$$R_t = \hat{\mu} + \hat{\sigma} X_t$$

where $X_t \sim N(0,1)$, $\hat{\mu}$ is the empirical mean of daily returns and $\hat{\sigma}$ is the daily standard deviation.

$\mathcal{H}_2$ Daily returns follow a Student's t distribution[3]:

$$R_t = \hat{\mu} + \hat{\sigma}\sqrt{\frac{\nu - 2}{\nu}} X_t$$

where $X_t \sim \mathbf{t}_\nu$. We consider two alternative assumptions: $\mathcal{H}_{2a} : \nu = 3$ and $\mathcal{H}_{2b} : \nu = 6$.

We represent the probability density function of $R_{n:n}$ for several values of n in Figure 16.3. When n is equal to one trading day, $R_{n:n}$ is exactly the daily return. We notice that it is difficult to measure the impact of the distribution tail. However, when n increases, the impact becomes more and more important. Order statistics allow to amplify local phenomena of probability

[3]We add the factor $\sqrt{\frac{\nu-2}{\nu}}$ in order to verify that $\operatorname{var}(R_t) = \hat{\sigma}^2$.

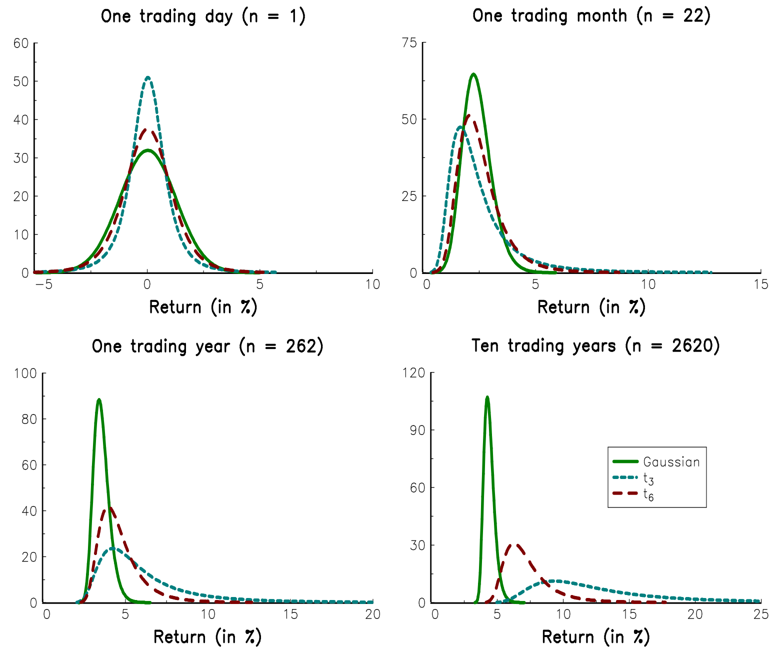

FIGURE 16.3: Density function of the maximum order statistic (daily return of the MSCI USA index, 1995-2015)

distributions. In particular, extreme order statistics are a very useful tool to analyze left and right tails.

Remark 59 *The limit distributions of minima and maxima are given by the following results:*

$$\begin{aligned}\lim_{n\to\infty} \mathbf{F}_{1:n}(x) &= \lim_{n\to\infty} 1-(1-\mathbf{F}(x))^n \\ &= \begin{cases} 0 & \text{if } \mathbf{F}(x)=0 \\ 1 & \text{if } \mathbf{F}(x)>0 \end{cases}\end{aligned}$$

and:

$$\begin{aligned}\lim_{n\to\infty} \mathbf{F}_{n:n}(x) &= \lim_{n\to\infty} \mathbf{F}(x)^n \\ &= \begin{cases} 0 & \text{if } \mathbf{F}(x)<1 \\ 1 & \text{if } \mathbf{F}(x)=1 \end{cases}\end{aligned}$$

We deduce that the limit distributions are degenerate as they only take values of 0 *and* 1*. This property is very important, because it means that we cannot study extreme events by considering these limit distributions. This is why the extreme value theory is based on another convergence approach of extreme order statistics.*

16.1.3 Inference statistics

The common approach to estimate the parameters θ of the probability density function $f(x;\theta)$ is to maximize the log-likelihood function of a given sample $\{x_1,\ldots,x_T\}$:

$$\hat{\theta} = \arg\max \sum_{t=1}^{T} \ln f(x_t;\theta)$$

In a similar way, we can consider the sample[4] $\left\{x'_1,\ldots,x'_{n_S}\right\}$ of the order statistic $X_{i:n}$ and estimate the parameters θ by the method of maximum likelihood:

$$\hat{\theta}_{i:n} = \arg\max \boldsymbol{\ell}_{i:n}(\theta)$$

where:

$$\begin{aligned}
\boldsymbol{\ell}_{i:n}(\theta) &= \sum_{s=1}^{n_S} \ln f_{i:n}(x'_s;\theta) \\
&= \sum_{s=1}^{n_S} \ln \frac{n!}{(i-1)!\,(n-i)!} \mathbf{F}(x'_s;\theta)^{i-1} \left(1-\mathbf{F}(x'_s;\theta)\right)^{n-i} f(x'_s;\theta)
\end{aligned}$$

The computation of the log-likelihood function gives:

$$\begin{aligned}
\boldsymbol{\ell}_{i:n}(\theta) &= n_S \ln n! - n_S \ln(i-1)! - n_S \ln(n-i)! + \\
&\quad (i-1)\sum_{s=1}^{n_S} \ln \mathbf{F}(x'_s;\theta) + (n-i)\sum_{s=1}^{n_S} \ln\left(1-\mathbf{F}(x'_s;\theta)\right) + \\
&\quad \sum_{s=1}^{n_S} \ln f(x'_s;\theta)
\end{aligned}$$

By definition, the traditional ML estimator is equal to new ML estimator when $n=1$ and $i=1$:

$$\hat{\theta} = \hat{\theta}_{1:1}$$

In the other cases ($n>1$), there is no reason that the two estimators coincide exactly:

$$\hat{\theta}_{i:n} \neq \hat{\theta}$$

However, if the random variates are drawn from the distribution function $X \sim \mathbf{F}(x;\theta)$, we can test the hypothesis $\mathcal{H}: \hat{\theta}_{i:n} = \theta$ for all n and $i \leq n$. If two estimates $\hat{\theta}_{i:n}$ and $\hat{\theta}_{i':n'}$ are very different, this indicates that the distribution function is certainly not appropriate for modeling the random variable X.

Let us consider the previous example with the returns of the MSCI USA index. We assume that the daily returns can be modeled with the Student's t distribution:

$$\frac{R_t-\mu}{\sigma} \sim \mathbf{t}_\nu$$

[4]The size of the sample n_S is equal to the size of the original sample T divided by n.

TABLE 16.1: ML estimate of σ (in bps) for the distribution $\mathbf{t}_1$

Size n	Order i									
	1	2	3	4	5	6	7	8	9	10
1	50									
2	48	49								
3	44	54	44							
4	41	53	53	41						
5	38	52	55	51	37					
6	35	51	56	56	48	33				
7	32	49	55	56	55	45	29			
8	31	48	53	55	54	50	43	26		
9	29	46	55	56	57	55	49	40	25	
10	28	43	53	58	57	56	53	48	37	20

TABLE 16.2: ML estimate of σ (in bps) for the distribution $\mathbf{t}_6$

Size n	Order i									
	1	2	3	4	5	6	7	8	9	10
1	88									
2	89	87								
3	91	91	85							
4	95	92	89	87						
5	98	99	87	90	88					
6	101	104	95	88	92	89				
7	101	112	100	88	94	95	89			
8	102	116	103	89	85	89	98	89		
9	105	121	117	97	85	86	94	101	88	
10	105	123	120	108	91	87	92	99	104	88

TABLE 16.3: ML estimate of σ (in bps) for the distribution $\mathbf{t}_\infty$

Size n	Order i									
	1	2	3	4	5	6	7	8	9	10
1	125									
2	125	124								
3	136	116	129							
4	147	116	112	140						
5	155	133	103	114	150					
6	163	142	118	107	122	157				
7	171	152	125	105	117	134	162			
8	175	165	130	106	99	111	139	170		
9	180	174	155	122	95	99	128	152	171	
10	183	182	162	136	110	100	111	127	155	181

The vector of parameters to estimate is then $\theta = (\mu, \sigma)$. In Tables 16.1, 16.2 and 16.3, we report the values taken by the ML estimator $\hat{\sigma}_{i:n}$ obtained by considering several order statistics and three values of ν. For instance, the ML estimate $\hat{\sigma}_{1:1}$ in the case of the $\mathbf{t}_1$ distribution is equal to 50 bps. We notice that the values taken by $\hat{\sigma}_{i:n}$ are not very stable with respect to i and n. This indicates that the three distributions functions ($\mathbf{t}_1$, $\mathbf{t}_6$ and $\mathbf{t}_\infty$) are not well appropriate to represent the index returns. In Figure 16.4, we have reported the corresponding annualized volatility[5] calculated from the order statistics $R_{i:10}$. In the case of the $\mathbf{t}_1$ distribution, we notice that it is lower for median order statistics than extreme order statistics. The $\mathbf{t}_1$ distribution has then the property to over-estimate extreme events. In the case of the Gaussian (or $\mathbf{t}_\infty$) distribution, we obtain contrary results. The Gaussian distribution has the property to under-estimate extreme events. In order to compensate this bias, the method of maximum likelihood applied to extreme order statistics will over-estimate the volatility.

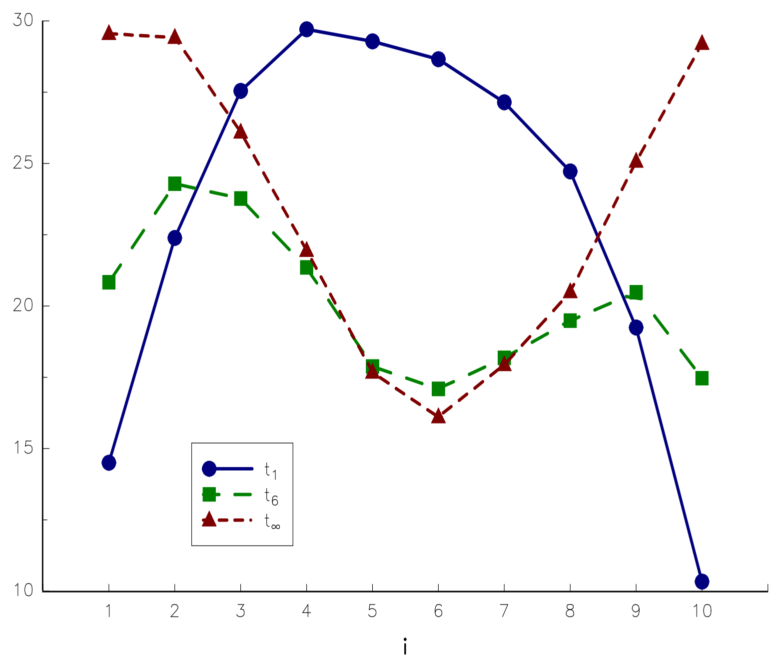

FIGURE 16.4: Annualized volatility (in %) calculated from the order statistics $R_{i:10}$

Remark 60 *The approach based on extreme order statistics to calculate the volatility is then a convenient way to reduce the under-estimation of the Gaussian value-at-risk.*

[5] The annualized volatility takes the value $\sqrt{260} \times c \times \hat{\sigma}_{i:n}$ where the constant c is equal to $\sqrt{\nu / (\nu - 2)}$. In the case of the $\mathbf{t}_1$ distribution, c is equal to 3.2.

16.1.4 Extension to dependent random variables

Let us now assume that $X_1, \ldots, X_n$ are not *iid*. We note $\mathbf{C}$ the copula of the corresponding random vector. It follows that[6]:

$$\begin{aligned}\mathbf{F}_{n:n}(x) &= \Pr\{X_{n:n} \leq x\} \\ &= \Pr\{X_1 \leq x, \ldots, X_n \leq x\} \\ &= \mathbf{C}(\mathbf{F}_1(x), \ldots, \mathbf{F}_n(x))\end{aligned}$$

and:

$$\begin{aligned}\mathbf{F}_{1:n}(x) &= \Pr\{X_{1:n} \leq x\} \\ &= 1 - \Pr\{X_{1:n} \geq x\} \\ &= 1 - \Pr\{X_1 \geq x, \ldots, X_n \geq x\} \\ &= 1 - \breve{\mathbf{C}}(1 - \mathbf{F}_1(x), \ldots, 1 - \mathbf{F}_n(x))\end{aligned}$$

where $\breve{\mathbf{C}}$ is the survival copula associated to $\mathbf{C}$. If we are interested in other order statistics, we use the following formula given in Georges *et al.* (2001):

$$\mathbf{F}_{i:n}(x) = \sum_{k=i}^{n}\left[\sum_{l=i}^{k}(-1)^{k-l}\binom{k}{l}\sum_{\mathbf{v}(\mathbf{F}_1(x),\ldots,\mathbf{F}_n(x))\in\mathcal{Z}(n-k,n)}\mathbf{C}(u_1, \ldots, u_n)\right]$$

where:

$$\mathcal{Z}(m,n) = \left\{\mathbf{v} \in [0,1]^n \mid v_i \in \{u_i, 1\}, \sum_{i=1}^{n}\mathbf{1}\{v_i = 1\} = m\right\}$$

In order to understand this formula, we consider the case $n = 3$. We have[7]:

$$\begin{aligned}\mathbf{F}_{1:3}(x) &= \mathbf{F}_1(x) + \mathbf{F}_2(x) + \mathbf{F}_3(x) - \\ &\quad \mathbf{C}(\mathbf{F}_1(x), \mathbf{F}_2(x), 1) - \mathbf{C}(\mathbf{F}_1(x), 1, \mathbf{F}_3(x)) - \mathbf{C}(1, \mathbf{F}_2(x), \mathbf{F}_3(x)) + \\ &\quad \mathbf{C}(\mathbf{F}_1(x), \mathbf{F}_2(x), \mathbf{F}_3(x)) \\ \mathbf{F}_{2:3}(x) &= \mathbf{C}(\mathbf{F}_1(x), \mathbf{F}_2(x), 1) + \mathbf{C}(\mathbf{F}_1(x), 1, \mathbf{F}_3(x)) + \mathbf{C}(1, \mathbf{F}_2(x), \mathbf{F}_3(x)) - \\ &\quad 2\mathbf{C}(\mathbf{F}_1(x), \mathbf{F}_2(x), \mathbf{F}_3(x)) \\ \mathbf{F}_{3:3}(x) &= \mathbf{C}(\mathbf{F}_1(x), \mathbf{F}_2(x), \mathbf{F}_3(x))\end{aligned}$$

---

[6] In the case of the product copula and identical distributions, we retrieve the previous results:

$$\begin{aligned}\mathbf{F}_{n:n}(x) &= \mathbf{C}^{\perp}(\mathbf{F}(x), \ldots, \mathbf{F}(x)) \\ &= \mathbf{F}(x)^n\end{aligned}$$

and:

$$\begin{aligned}\mathbf{F}_{1:n}(x) &= 1 - \mathbf{C}^{\perp}(1 - \mathbf{F}(x), \ldots, 1 - \mathbf{F}(x)) \\ &= 1 - (1 - \mathbf{F}(x))^n\end{aligned}$$

[7] Because $\mathbf{C}(\mathbf{F}_1(x), 1, 1) = \mathbf{F}_1(x)$.

We verify that:

$$\mathbf{F}_{1:3}(x) + \mathbf{F}_{2:3}(x) + \mathbf{F}_{3:3}(x) = \mathbf{F}_1(x) + \mathbf{F}_2(x) + \mathbf{F}_3(x)$$

The dependence structure has a big impact on the distribution of order statistics. For instance, if we assume that $X_1, \ldots, X_n$ are *iid*, we obtain:

$$\mathbf{F}_{n:n}(x) = \mathbf{F}(x)^n$$

If the copula function is the Upper Fréchet copula, this result becomes:

$$\begin{aligned}\mathbf{F}_{n:n}(x) &= \mathbf{C}^+(\mathbf{F}(x), \ldots, \mathbf{F}(x)) \\ &= \min(\mathbf{F}(x), \ldots, \mathbf{F}(x)) \\ &= \mathbf{F}(x)\end{aligned}$$

This implies that the occurrence probability of extreme events is lower in this second case.

We consider n Weibull default times $\boldsymbol{\tau}_i \sim \mathcal{W}(\lambda_i, \gamma_i)$. The survival function is equal to $\mathbf{S}_i(t) = \exp(-\lambda_i t^{\gamma_i})$. The hazard rate $\lambda_i(t)$ is then $\lambda_i \gamma_i t^{\gamma_i - 1}$ and the expression of the density is $f_i(t) = \lambda_i(t)\mathbf{S}_i(t)$. If we assume that the survival copula is the Gumbel-Hougaard copula with parameter $\theta \geq 1$, the survival function of the first-to-default is equal to:

$$\begin{aligned}\mathbf{S}_{1:n}(t) &= \exp\left(-\left((-\ln \mathbf{S}_1(t))^\theta + \ldots + (-\ln \mathbf{S}_n(t))^\theta\right)^{1/\theta}\right) \\ &= \exp\left(-\left(\sum_{i=1}^n \lambda_i^\theta t^{\theta\gamma_i}\right)^{1/\theta}\right)\end{aligned}$$

We deduce the expression of the density function:

$$f_{1:n}(t) = \left(\sum_{i=1}^n \lambda_i^\theta t^{\theta\gamma_i}\right)^{1/\theta - 1} \left(\sum_{i=1}^n \gamma_i \lambda_i^\theta t^{\theta\gamma_i - 1}\right) \exp\left(-\left(\sum_{i=1}^n \lambda_i^\theta t^{\theta\gamma_i}\right)^{1/\theta}\right)$$

In the case where the default times are identically distributed, the first-to-default time is a Weibull default time: $\boldsymbol{\tau}_{1:n} \sim \mathcal{W}\left(n^{1/\theta}\lambda, \gamma\right)$. In Figure 16.5, we report the density function $f_{1:10}(t)$ for the parameters $\lambda = 3\%$ and $\gamma = 2$. We notice that the parameter θ has a big influence on the first-to-default. The case $\theta = 1$ corresponds to the product copula and we retrieve the previous result:

$$\mathbf{S}_{1:n}(t) = \mathbf{S}(t)^n$$

When the Gumbel-Hougaard is the upper Fréchet copula ($\theta \to \infty$), we verify that the density function of $\boldsymbol{\tau}_{1:n}$ is this of any default time $\boldsymbol{\tau}_i$.

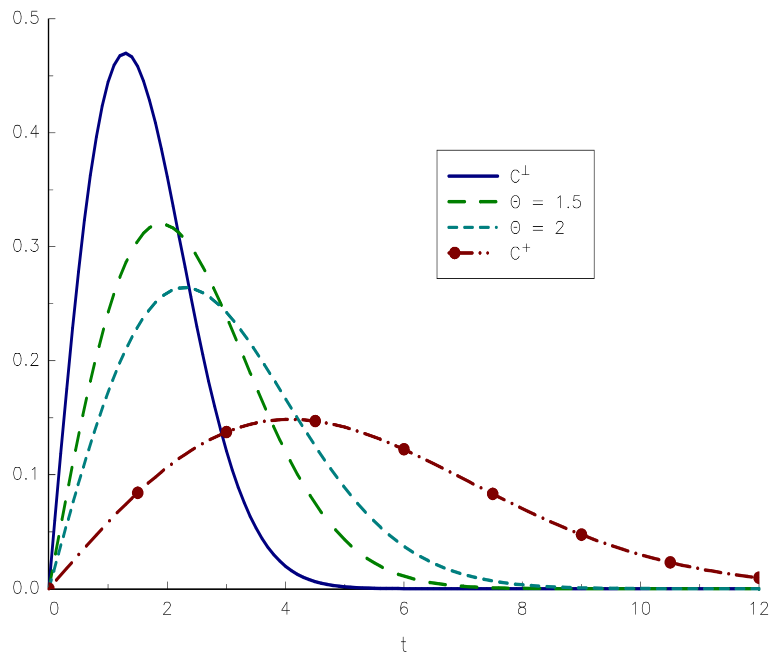

FIGURE 16.5: Density function of the first-to-default $\boldsymbol{\tau}_{1:10}$

16.2 Univariate extreme value theory

The extreme value theory consists in studying the limit distribution of extreme order statistics $X_{1:n}$ and $X_{n:n}$ when the sample size tends to infinity. We will see that the limit distribution converges to three probability distributions. This result will help to evaluate stress scenarios and to build a stress testing framework.

Remark 61 *In what follows, we only consider the largest order statistic $X_{n:n}$. Indeed, the minimum order statistic $X_{1:n}$ can be defined with respect to the maximum order statistic $Y_{n:n}$ by setting $Y_i = -X_i$:*

$$\begin{aligned} X_{1:n} &= \min(X_1, \ldots, X_n) \\ &= \min(-Y_1, \ldots, -Y_n) \\ &= -\max(Y_1, \ldots, Y_n) \\ &= -Y_{n:n} \end{aligned}$$

16.2.1 Fisher-Tippet theorem

Theorem 2 (Embrechts *et al.*, 1997) *Let $X_1, \ldots, X_n$ be a sequence of iid random variables, whose distribution function is $\mathbf{F}$. If there exist two constants a_n and b_n and a non-degenerate distribution function $\mathbf{G}$ such that:*

$$\lim_{n\to\infty} \Pr\left\{\frac{X_{n:n} - b_n}{a_n} \leq x\right\} = \mathbf{G}(x) \tag{16.3}$$

then $\mathbf{G}$ can be classified as one of the following three types[8]*:*

Type I	*(Gumbel)*	$\mathbf{\Lambda}(x) = \exp(-e^{-x})$
Type II	*(Fréchet)*	$\mathbf{\Phi}_\alpha(x) = \mathbb{1}(x \geq 0) \cdot \exp(-x^{-\alpha})$
Type III	*(Weibull)*	$\mathbf{\Psi}_\alpha(x) = \mathbb{1}(x \leq 0) \cdot \exp(-(-x)^\alpha)$

The distribution functions $\mathbf{\Lambda}$, $\mathbf{\Phi}_\alpha$ et $\mathbf{\Psi}_\alpha$ are called extreme value distributions. The Fisher-Tippet theorem is very important, because the set of extreme value distributions is very small although the set of distribution functions is very large. We can draw a parallel with the normal distribution and the sum of random variables. In some sense, the Fisher-Tippet theorem provides an extreme value analog of the central limit theorem.

Let us consider the case of exponential random variables, whose probability distribution is $\mathbf{F}(x) = 1 - \exp(-\lambda x)$. We have[9]:

$$\begin{aligned}
\lim_{n\to\infty} \mathbf{F}_{n:n}(x) &= \lim_{n\to\infty} \left(1 - e^{-\lambda x}\right)^n \\
&= \lim_{n\to\infty} \left(1 - \frac{ne^{-\lambda x}}{n}\right)^n \\
&= \lim_{n\to\infty} \exp\left(-ne^{-\lambda x}\right) \\
&= 0
\end{aligned}$$

We verify that the limit distribution is degenerate. If we consider the affine

[8] In terms of probability density function, we have:

$$\begin{aligned}
g(x) &= \exp\left(-x - e^{-x}\right) && \text{(Gumbel)} \\
&= \mathbb{1}(x \geq 0) \cdot \alpha x^{-(1+\alpha)} \exp\left(-x^{-\alpha}\right) && \text{(Fréchet)} \\
&= \mathbb{1}(x \leq 0) \cdot \alpha(-x)^{\alpha-1} \exp(-(-x)^\alpha) && \text{(Weibull)}
\end{aligned}$$

[9] Because we have:

$$\begin{aligned}
\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\
&= \exp(x)
\end{aligned}$$

transformation with $a_n = 1/\lambda$ et $b_n = (\ln n)/\lambda$, we obtain:

$$\begin{aligned}
\Pr\left\{\frac{X_{n:n} - b_n}{a_n} \leq x\right\} &= \Pr\left\{X_{n:n} \leq a_n x + b_n\right\} \\
&= \left(1 - e^{-\lambda(a_n x + b_n)}\right)^n \\
&= \left(1 - e^{-x - \ln n}\right)^n \\
&= \left(1 - \frac{e^{-x}}{n}\right)^n
\end{aligned}$$

We deduce that:

$$\begin{aligned}
\mathbf{G}(x) &= \lim_{n \to \infty} \left(1 - \frac{e^{-x}}{n}\right)^n \\
&= \exp\left(-e^{-x}\right)
\end{aligned}$$

It follows that the limit distribution of the affine transformation is not degenerate. In Figure 16.6, we illustrate the convergence of $\mathbf{F}^n(a_n x + b_n)$ to the Gumbel distribution $\mathbf{\Lambda}(x)$.

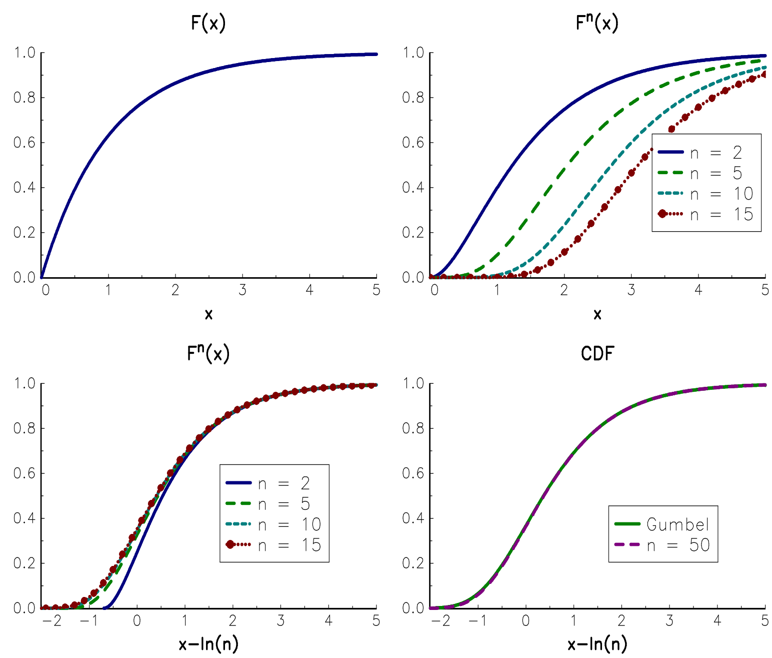

FIGURE 16.6: Max-convergence of the exponential distribution $\mathcal{E}(1)$ to the Gumbel distribution

Example 70 *If we consider the Pareto distribution, we have:*

$$\mathbf{F}(x) = 1 - \left(\frac{x}{x_-}\right)^{-\alpha}$$

The normalizing constants are $a_n = x_- n^{1/\alpha}$ and $b_n = 0$. We obtain:

$$\begin{aligned}
\Pr\left\{\frac{X_{n:n} - b_n}{a_n} \leq x\right\} &= \left(1 - \left(\frac{x_- n^{1/\alpha} x}{x_-}\right)^{-\alpha}\right)^n \\
&= \left(1 - \frac{x^{-\alpha}}{n}\right)^n
\end{aligned}$$

We deduce that the law of the maximum tends to the Fréchet distribution:

$$\lim_{n\to\infty}\left(1 - \frac{x^{-\alpha}}{n}\right)^n = \exp\left(-x^{-\alpha}\right)$$

Example 71 *For the uniform distribution, the normalizing constants become $a_n = n^{-1}$ and $b_n = 1$ and we obtain the Weibull distribution with $\alpha = 1$:*

$$\begin{aligned}
\lim_{n\to\infty}\Pr\left\{\frac{X_{n:n} - b_n}{a_n} \leq x\right\} &= \left(1 + \frac{x}{n}\right)^n \\
&= \exp(x)
\end{aligned}$$

16.2.2 Maximum domain of attraction

The application of the Fisher-Tippet theorem is limited because it can be extremely difficult to find the normalizing constants and the extreme value distribution for a given probability distribution $\mathbf{F}$. However, the graphical representation of $\mathbf{\Lambda}$, $\mathbf{\Phi}_\alpha$ and $\mathbf{\Psi}_\alpha$ given in Figure 16.7 already provides some information. For instance, the Weibull probability distribution concerns random variables that are right bounded. This is why it has less interest in finance than the Fréchet or Gumbel distribution functions[10]. We also notice some difference in the shape of the curves. In particular, the Gumbel distribution is more 'normal' than the Fréchet distribution, whose shape and tail depend to the parameter α (see Figure 16.8).

We say that the distribution function $\mathbf{F}$ belongs to the max-domain of attraction of the distribution function $\mathbf{G}$ and we write $\mathbf{F} \in \mathrm{MDA}(\mathbf{G})$ if the distribution function of the normalized maximum converges to $\mathbf{G}$. For instance, we have already seen that $\mathcal{E}(\lambda) \in \mathrm{MDA}(\mathbf{\Lambda})$. In what follows, we indicate how to characterize the set $\mathrm{MDA}(\mathbf{G})$ and which normalizing constants are[11].

[10]However, the Weibull probability distribution is related to the Fréchet probability distribution thanks to the relationship $\mathbf{\Psi}_\alpha(x) = \mathbf{\Phi}_\alpha\left(-x^{-1}\right)$.

[11]Most of the following results come from Resnick (1987).

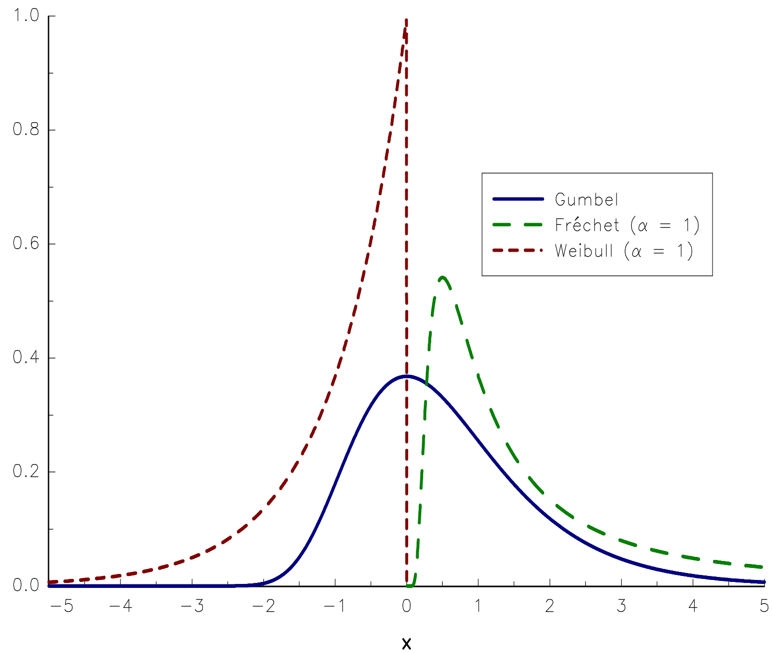

FIGURE 16.7: Density function of $\mathbf{\Lambda}$, $\mathbf{\Phi}_1$ and $\mathbf{\Psi}_1$

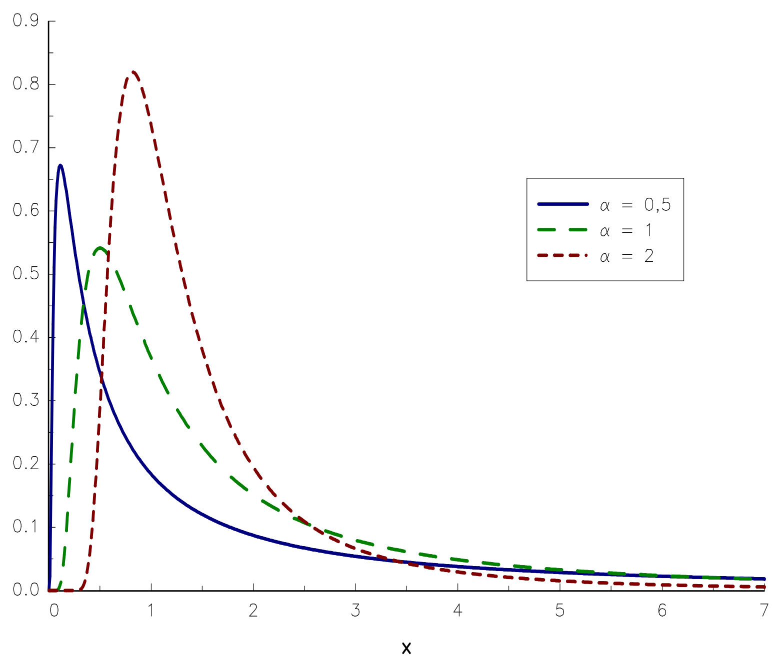

FIGURE 16.8: Density function of the Fréchet probability distribution

16.2.2.1 MDA of the Gumbel distribution

Theorem 3 $\mathbf{F} \in \text{MDA}(\mathbf{\Lambda})$ *if and only if there exists a function* $h(t)$ *such that:*

$$\lim_{t \to x_0} \frac{1 - \mathbf{F}(t + xh(t))}{1 - \mathbf{F}(t)} = \exp(-x)$$

where $x_0 \leq \infty$. *The normalizing constants are then* $a_n = h\left(\mathbf{F}^{-1}\left(1 - n^{-1}\right)\right)$ *and* $b_n = \mathbf{F}^{-1}\left(1 - n^{-1}\right)$.

The previous characterization of $\text{MDA}(\mathbf{\Lambda})$ is difficult to use because we have to define the function $h(t)$. However, we can show that if the distribution function $\mathbf{F}$ is C^2, a sufficient condition is:

$$\lim_{x \to \infty} \frac{(1 - \mathbf{F}(x)) \times \partial_x^2 \mathbf{F}(x)}{(\partial_x \mathbf{F}(x))^2} = -1$$

For instance, in the case of the Exponential distribution, we have $\mathbf{F}(x) = 1 - \exp(-\lambda x)$, $\partial_x \mathbf{F}(x) = \lambda \exp(-\lambda x)$ and $\partial_x^2 \mathbf{F}(x) = -\lambda^2 \exp(-\lambda x)$. We verify that:

$$\lim_{x \to \infty} \frac{(1 - \mathbf{F}(x)) \times \partial_x^2 \mathbf{F}(x)}{(\partial_x \mathbf{F}(x))^2} = \lim_{x \to \infty} \frac{\exp(-\lambda x) \times \left(-\lambda^2 \exp(-\lambda x)\right)}{(\lambda \exp(-\lambda x))^2} = -1$$

If we consider the Gaussian distribution $\mathcal{N}(0,1)$, we have $\mathbf{F}(x) = \Phi(x)$, $\partial_x \mathbf{F}(x) = \phi(x)$ and $\partial_x^2 \mathbf{F}(x) = -x\phi(x)$. Using L'Hospital's rule, we deduce that:

$$\lim_{x \to \infty} \frac{(1 - \mathbf{F}(x)) \times \partial_x^2 \mathbf{F}(x)}{(\partial_x \mathbf{F}(x))^2} = \lim_{x \to \infty} -\frac{x\Phi(-x)}{\phi(x)} = -1$$

16.2.2.2 MDA of the Fréchet distribution

Definition 1 *A function* f *is regularly varying with index* α *and we write* $f \in \text{RV}_\alpha$ *if we have:*

$$\lim_{t \to \infty} \frac{f(tx)}{f(t)} = x^\alpha$$

for every $x > 0$.

Theorem 4 $\mathbf{F} \in \text{MDA}(\mathbf{\Phi}_\alpha)$ *if and only if* $1 - \mathbf{F} \in \text{RV}_{-\alpha}$. *The normalizing constants are then* $a_n = \mathbf{F}^{-1}\left(1 - n^{-1}\right)$ *and* $b_n = 0$.

Using this theorem, we deduce that the distribution function $\mathbf{F} \in \text{MDA}(\mathbf{\Phi}_\alpha)$ if it satisfies the following condition:

$$\lim_{t \to \infty} \frac{1 - \mathbf{F}(tx)}{1 - \mathbf{F}(t)} = x^{-\alpha}$$

If we apply this result to the Pareto distribution, we obtain:

$$\begin{aligned}\lim_{t\to\infty}\frac{1-\mathbf{F}(tx)}{1-\mathbf{F}(t)} &= \lim_{t\to\infty}\frac{(tx/x_-)^{-\alpha}}{(t/x_-)^{-\alpha}} \\ &= x^{-\alpha}\end{aligned}$$

We deduce that $1-\mathbf{F}\in\mathrm{RV}_{-\alpha}$, $\mathbf{F}\in\mathrm{MDA}(\mathbf{\Phi}_\alpha)$, $a_n=\mathbf{F}^{-1}\left(1-n^{-1}\right)=x_-n^{1/\alpha}$ and $b_n=0$.

Remark 62 *The previous theorem suggests that:*

$$\frac{1-\mathbf{F}(tx)}{1-\mathbf{F}(t)}\approx x^{-\alpha}$$

when t is sufficiently large. This means that we must observe a linear relationship between $\ln(x)$ *and* $\ln(1-\mathbf{F}(tx))$*:*

$$\ln(1-\mathbf{F}(tx))\approx\ln(1-\mathbf{F}(t))-\alpha\ln(x)$$

This property can be used to check graphically if a given distribution function belongs or not to the maximum domain of attraction of the Fréchet distribution. For instance, we observe that $\mathcal{N}(0,1)\notin$ MDA $(\mathbf{\Phi}_\alpha)$ in Figure 16.9, because the curve is not a straight line.

16.2.2.3 MDA of the Weibull distribution

Theorem 5 $\mathbf{F}\in\mathrm{MDA}(\mathbf{\Psi}_\alpha)$ *if and only if* $1-\mathbf{F}\left(x_0-x^{-1}\right)\in\mathrm{RV}_{-\alpha}$ *and* $x_0<\infty$*. The normalizing constants are then* $a_n=x_0-\mathbf{F}^{-1}\left(1-n^{-1}\right)$ *and* $b_n=x_0$*.*

If we consider the uniform distribution with $x_0=1$, we have:

$$\mathbf{F}\left(x_0-x^{-1}\right)=1-\frac{1}{x}$$

and:

$$\begin{aligned}\lim_{t\to\infty}\frac{1-\mathbf{F}\left(1-t^{-1}x^{-1}\right)}{1-\mathbf{F}\left(1-t^{-1}\right)} &= \lim_{t\to\infty}\frac{t^{-1}x^{-1}}{t^{-1}} \\ &= x^{-1}\end{aligned}$$

We deduce that $\mathbf{F}\in\mathrm{MDA}(\mathbf{\Psi}_1)$, $a_n=1-\mathbf{F}^{-1}\left(1-n^{-1}\right)=n^{-1}$ and $b_n=1$.

16.2.2.4 Main results

In Table 16.4, we report the maximum domain of attraction and normalizing constants of some well-known distribution functions.

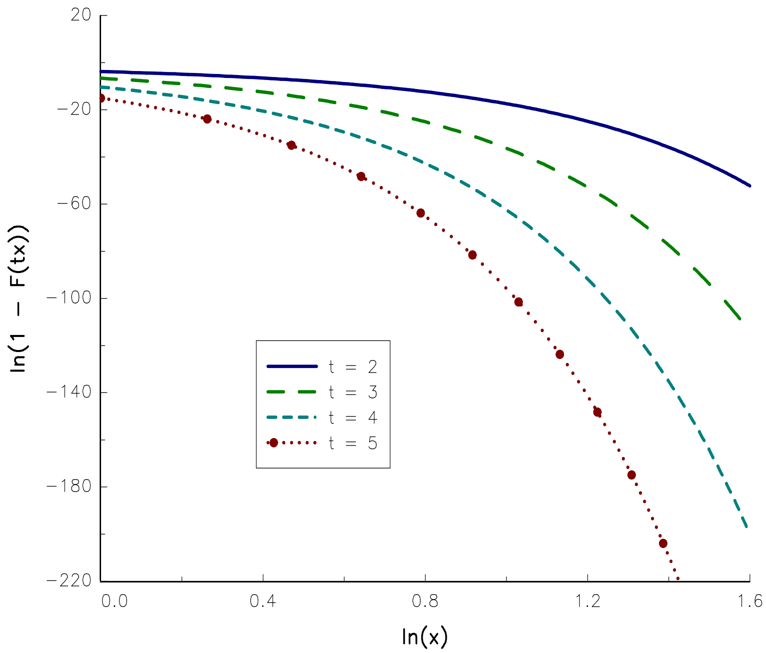

FIGURE 16.9: The graphical verification of the regular variation property for the $\mathcal{N}(0,1)$ distribution function

Remark 63 *Let* $\mathbf{G}(x)$ *be the non-degenerate distribution of* $X_{n:n}$*. We note* a_n *and* b_n *the normalizing constants. We consider the linear transformation* $Y = cX + d$ *with* $c > 0$*. Because we have* $Y_{n:n} = cX_{n:n} + d$*, we deduce that:*

$$\begin{aligned}
\mathbf{G}(x) &= \lim_{n\to\infty} \Pr\{X_{n:n} \leq a_n x + b_n\} \\
&= \lim_{n\to\infty} \Pr\left\{\frac{Y_{n:n} - d}{c} \leq a_n x + b_n\right\} \\
&= \lim_{n\to\infty} \Pr\{Y_{n:n} \leq a_n c x + b_n c + d\} \\
&= \lim_{n\to\infty} \Pr\{Y_{n:n} \leq a'_n x + b'_n\}
\end{aligned}$$

where $a'_n = a_n c$ *and* $b'_n = b_n c + d$*. This means that* $\mathbf{G}(x)$ *is also the non-degenerate distribution of* $Y_{n:n}$*, and* a'_n *and* b'_n *are the normalizing constants. For instance, if we consider the distribution function* $\mathcal{N}(\mu, \sigma^2)$*, we deduce that the normalizing constants are:*

$$a_n = \sigma (2 \ln n)^{-1/2}$$

and:

$$b_n = \mu + \sigma \left(\frac{4 \ln n - \ln(4\pi) + \ln \ln n}{2\sqrt{2 \ln n}} \right)$$

TABLE 16.4: Maximum domain of attraction and normalizing constants of some distribution functions

Distribution	$\mathbf{G}(x)$	a_n	b_n
$\mathcal{E}(\lambda)$	$\boldsymbol{\Lambda}$	λ^{-1}	$\lambda^{-1}\ln n$
$\mathcal{G}(\alpha,\beta)$	$\boldsymbol{\Lambda}$	β^{-1}	$\beta^{-1}\left(\ln n+(\alpha-1)\ln(\ln n)-\ln\Gamma(\alpha)\right)$
$\mathcal{N}(0,1)$	$\boldsymbol{\Lambda}$	$(2\ln n)^{-1/2}$	$\dfrac{4\ln n-\ln(4\pi)-\ln(\ln n)}{2\sqrt{2\ln n}}$
$\mathcal{LN}\left(\mu,\sigma^2\right)$	$\boldsymbol{\Lambda}$	$\sigma(2\ln n)^{-1/2}\,b_n$	$\exp\left(\mu+\sigma\left(\dfrac{4\ln n-\ln(4\pi)+\ln\ln n}{2\sqrt{2\ln n}}\right)\right)$
$\mathcal{P}a(\alpha,x_-)$	$\boldsymbol{\Phi}_\alpha$	$x_-n^{1/\alpha}$	0
$\mathcal{LG}(\alpha,\beta)$	$\boldsymbol{\Phi}_\beta$	$\dfrac{\left(n(\ln n)^{\alpha-1}\right)^{1/\beta}}{\Gamma(\alpha)}$	0
$\mathbf{T}_\nu$	$\boldsymbol{\Phi}_\nu$	$\mathbf{T}_\nu^{-1}\left(1-n^{-1}\right)$	0
$\mathcal{U}_{[0,1]}$	$\boldsymbol{\Psi}_1$	n^{-1}	1
$\mathcal{B}(\alpha,\beta)$	$\boldsymbol{\Psi}_\alpha$	$\left(\dfrac{n\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta+1)}\right)^{-1/\beta}$	1

Remark 64 *The normalizing constants are uniquely defined. In the case of the Gaussian distribution $\mathcal{N}(0,1)$, they are equal to $a_n = h(b_n) = b_n / \left(1 + b_n^2\right)$ and $b_n = \Phi^{-1}\left(1 - n^{-1}\right)$. In Table ??, we report an approximation, which is not necessarily unique. For instance, Gasull et al. (2015) propose the following alternative value of b_n:*

$$b_n \approx \sqrt{\ln\left(\frac{n^2}{2\pi}\right) - \ln\left(\ln\left(\frac{n^2}{2\pi}\right)\right) + \frac{\ln\left(0.5 + \ln n^2\right) - 2}{\ln\left(n^2\right) - \ln\left(2\pi\right)}}$$

and show that this solution is more accurate than the classical approximation.

16.2.3 Generalized extreme value distribution

16.2.3.1 Definition

From a statistical point of view, the previous results of the extreme value theory are difficult to use. Indeed, they are many issues concerning the choice of the distribution function, the normalizing constants or the convergence rate as explained by Coles (2001):

> "*The three types of limits that arise in Theorem 2 have distinct forms of behavior, corresponding to the different forms of tail behaviour for the distribution function* **F** *of the X_i. This can be made precise by considering the behavior of the limit distribution* **G** *at x_+, its upper end-point. For the Weibull distribution x_+ is finite, while for both the Fréchet and Gumbel distributions $x_+ = \infty$. However, the density of* **G** *decays exponentially for the Gumbel distribution and polynomially for the Fréchet distribution, corresponding to relatively different rates of decay in the tail of* **F**. *It follows that in applications the three different families give quite different representations of extreme value behavior. In early applications of extreme value theory, it was usual to adopt one of the three families, and then to estimate the relevant parameters of that distribution. But there are two weakness: first, a technique is required to choose which of the three families is most appropriate for the data at hand; second, once such a decision is made, subsequent inferences presume this choice to be correct, and do not allow for the uncertainty such a selection involves, even though this uncertainty may be substantial*".

In practice, the statistical inference on extreme values takes another route. Indeed, the three types can be combined into a single distribution function:

$$\mathbf{G}(x) = \exp\left\{-\left(1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right\}$$

defined on the support $\Delta = \left\{x : 1 + \xi\sigma^{-1}(x - \mu) > 0\right\}$. It is known as the *generalized extreme value* distribution and we denote it by $\mathcal{GEV}(\mu, \sigma, \xi)$. We obtain the following cases:

- the limit case $\xi \to 0$ corresponds to the Gumbel distribution;
- $\xi = -\alpha^{-1} > 0$ defines the Fréchet distribution;
- the Weibull distribution is obtained by considering $\xi = -\alpha^{-1} < 0$.

We also notice that the parameters μ and σ are the limits of the normalizing constants b_n and a_n. The corresponding density function is equal to:

$$g(x) = \frac{1}{\sigma}\left(1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-(1+\xi)/\xi} \exp\left\{-\left(1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right\}$$

It is represented in Figure 16.10 for various values of parameters. We notice that μ is a parameter of localization, σ controls the standard deviation and ξ is related to the tail of the distribution. The parameters can be estimated using the method of maximum likelihood and we obtain;

$$\boldsymbol{\ell}_t = -\ln\sigma - \left(\frac{1+\xi}{\xi}\right)\ln\left(1 + \xi\left(\frac{x_t-\mu}{\sigma}\right)\right) - \left(1 + \xi\left(\frac{x_t-\mu}{\sigma}\right)\right)^{-1/\xi}$$

where x_t is the observed maximum for the t^{th} period.

We consider again the example of the MSCI USA index. Using daily returns, we calculate the block maximum for each period of 22 trading days. We then estimate the GEV distribution using the method of maximum likelihood. For the period 1995-2015, we obtain $\hat{\mu} = 0.0149$, $\hat{\sigma} = 0.0062$ and $\hat{\xi} = 0.3736$. In Figure 16.11, we compared the estimated GEV distribution with the distribution function $\mathbf{F}_{22:22}(x)$ when we assume that daily returns are Gaussian. We notice that the Gaussian hypothesis largely underestimates extreme events as illustrated by the quantile function in the table below:

α	90%	95%	96%	97%	98%	99%
Gaussian	3.26%	3.56%	3.65%	3.76%	3.92%	4.17%
GEV	3.66%	4.84%	5.28%	5.91%	6.92%	9.03%

For instance, the probability is 1% to observe a maximum daily return during a period of one month larger than 4.17% in the case of the Gaussian distribution and 9.03% in the case of the GEV distribution.

16.2.3.2 Estimating the value-at-risk

Let us consider a portfolio w, whose mark-to-market value is $P_t(w)$ at time t. We remind that the P&L between t and $t+1$ is equal to:

$$\begin{aligned} \Pi(w) &= P_{t+1}(w) - P_t(w) \\ &= P_t(w) \times R(w) \end{aligned}$$

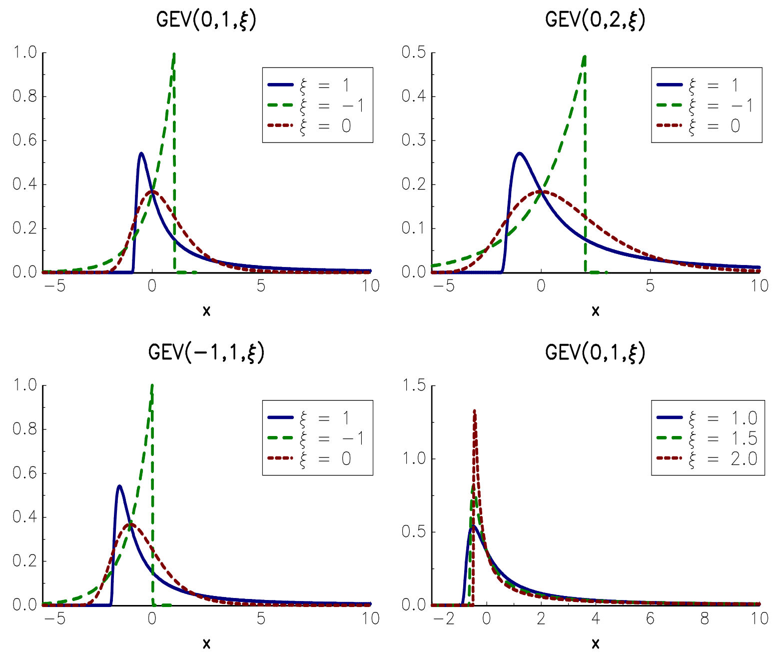

FIGURE 16.10: Probability density function of the GEV distribution

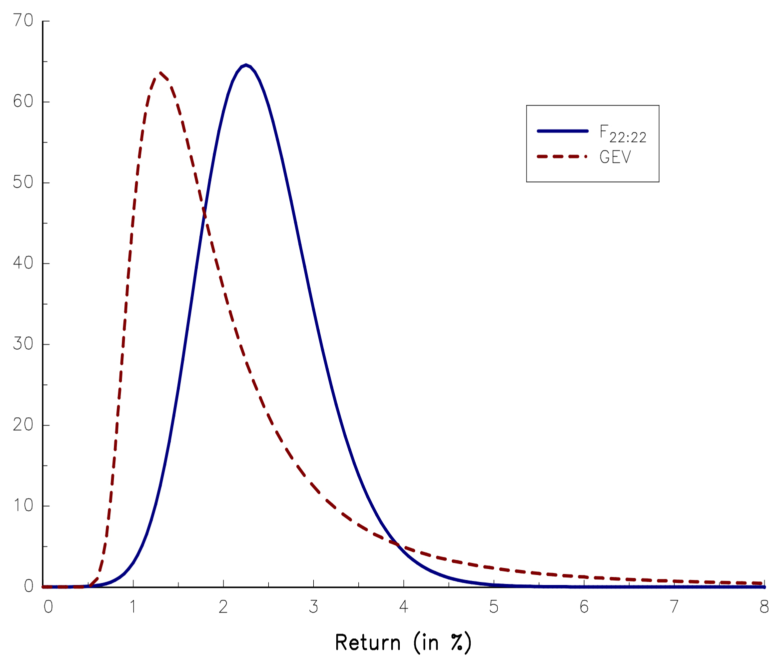

FIGURE 16.11: Probability density function of the maximum return $R_{22:22}$

where $R(w)$ is the daily return of the portfolio. If we note $\hat{\mathbf{F}}$ the estimated probability distribution of $R(w)$, the expression of the value-at-risk at the confidence level α is equal to:

$$\mathrm{VaR}_\alpha(w) = -P_t(w) \times \hat{\mathbf{F}}^{-1}(1-\alpha)$$

We now estimate the GEV distribution $\hat{\mathbf{G}}$ of the maximum of $-R(w)$ for a period of n trading days[12]. We have to define the confidence level α_{GEV} when we consider block minima of daily returns that corresponds to the same confidence level α when we consider daily returns. For that, we assume that the two exception events have the same return period, implying that:

$$\frac{1}{1-\alpha} \times 1 \text{ day} = \frac{1}{1-\alpha_{\mathrm{GEV}}} \times n \text{ days}$$

We deduce that:

$$\alpha_{\mathrm{GEV}} = 1 - (1-\alpha) \times n$$

It follows that the value-at-risk calculated with the GEV distribution is equal to[13]:

$$\mathrm{VaR}_\alpha(w) = P(t) \times \hat{\mathbf{G}}^{-1}(\alpha_{\mathrm{GEV}})$$

We consider four portfolios invested in the MSCI USA index and the MSCI EM index: (1) long on the MSCI USA, (2) long on the MSCI EM index, (3) long on the MSCI USA and short on the MSCI EM index and (4) long on the MSCI EM index and short on the MSCI USA index. Using daily returns from January 1995 to December 2015, we estimate the daily value-at-risk of these portfolios for different confidence level α. We report the results in Table 16.5 for Gaussian and historical value-at-risk and compare them with those calculated with the GEV approach. In this case, we estimate the parameters of the extreme value distribution using block maxima of 22 trading days. When we consider a 99% confidence level, the lowest value is obtained by the GEV method followed by Gaussian and historical methods. For higher quantile, the GEV VaR is between the Gaussian VaR and the historical VaR. The value-at-risk calculated with the GEV approach can therefore be interpreted as a parametric value-at-risk, which is estimated using only tail events.

[12]We model the maximum of the opposite of daily returns, because we are interested in extreme losses, and not in extreme profits.

[13]The inverse function of the probability distribution $\mathcal{GEV}(\mu,\sigma,\xi)$ is equal to:

$$\mathbf{G}^{-1}(\alpha) = \mu - \frac{\sigma}{\xi}\left(1 - (-\ln\alpha)^{-\xi}\right)$$

TABLE 16.5: Comparing Gaussian, historical and GEV value-at-risk

VaR	α	Long US	Long EM	Long US Short EM	Long EM Short US
Gaussian	99.0%	2.88%	2.83%	3.06%	3.03%
	99.5%	3.19%	3.14%	3.39%	3.36%
	99.9%	3.83%	3.77%	4.06%	4.03%
Historical	99.0%	3.46%	3.61%	3.37%	3.81%
	99.5%	4.66%	4.73%	3.99%	4.74%
	99.9%	7.74%	7.87%	6.45%	7.27%
GEV	99.0%	2.64%	2.61%	2.72%	2.93%
	99.5%	3.48%	3.46%	3.41%	3.82%
	99.9%	5.91%	6.05%	5.35%	6.60%

16.2.4 Peak over threshold

16.2.4.1 Definition

The estimation of the GEV distribution is a "*block component-wise*" approach. This means that from a sample of random variates, we build a sample of maxima by considering blocks with the same length. This implies a loss of information, because some blocks may contain several extreme events whereas some other blocks may not be impacted by extremes. Another approach consists in using the "*peak over threshold*" (POT) method. In this case, we are interested in estimating the distribution of exceedance over a certain threshold u:

$$\mathbf{F}_u(x) = \Pr\{X - u \leq x \mid X > u\}$$

where $0 \leq x < x_0 - u$ and $x_0 = \sup\{x \in \mathbb{R} : \mathbf{F}(x) < 1\}$. $\mathbf{F}_u(x)$ is also called the conditional excess distribution function. It is also equal to:

$$\begin{aligned}
\mathbf{F}_u(x) &= 1 - \Pr\{X - u \leq x \mid X \leq u\} \\
&= 1 - \left(\frac{1 - \mathbf{F}(u + x)}{1 - \mathbf{F}(u)}\right) \\
&= \frac{\mathbf{F}(u + x) - \mathbf{F}(u)}{1 - \mathbf{F}(u)}
\end{aligned}$$

Pickands (1975) showed that, for very large u, $\mathbf{F}_u(x)$ follows a generalized Pareto distribution (GPD):

$$\mathbf{F}_u(x) \approx \mathbf{H}(x)$$

where[14]:

$$\mathbf{H}(x) = 1 - \left(1 + \frac{\xi x}{\sigma}\right)^{-1/\xi}$$

[14] If $\xi \to 0$, we have $\mathbf{H}(x) = 1 - \exp(-x/\sigma)$.

The distribution function $\mathcal{GPD}(\sigma, \xi)$ depends on two parameters: σ is the scale parameter and ξ is the shape parameter.

Exercise 72 *If* $\mathbf{F}$ *is an exponential distribution* $\mathcal{E}(\lambda)$*, we have:*

$$\frac{1 - \mathbf{F}(u + x)}{1 - \mathbf{F}(u)} = \exp(-\lambda x)$$

It is the GPD where $\sigma = 1/\lambda$ *and* $\xi = 0$.

Exercise 73 *If* $\mathbf{F}$ *is a uniform distribution, we have:*

$$\frac{1 - \mathbf{F}(u + x)}{1 - \mathbf{F}(u)} = 1 - \frac{x}{1 - u}$$

It corresponds to the generalized Pareto distribution with the following parameters: $\sigma = 1 - u$ *and* $\xi = -1$.

In fact, there is a strong link between the block maxima approach and the peak over threshold method. Suppose that $X_{n:n} \sim \mathcal{GEV}(\mu, \sigma, \xi)$. It follows that:

$$\mathbf{F}^n(x) \approx \exp\left\{-\left(1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right)^{-1/\xi}\right\}$$

We deduce that:

$$n \ln \mathbf{F}(x) \approx -\left(1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right)^{-1/\xi}$$

Using the approximation $\ln \mathbf{F}(x) \approx -(1 - \mathbf{F}(x))$ for large x, we obtain:

$$1 - \mathbf{F}(x) \approx \frac{1}{n}\left(1 + \xi\left(\frac{x - \mu}{\sigma}\right)\right)^{-1/\xi}$$

We find that $\mathbf{F}_u(x)$ is a generalized Pareto distribution $\mathcal{GPD}(\tilde{\sigma}, \xi)$:

$$\begin{aligned}\Pr\{X > u + x \mid X > u\} &= \frac{1 - \mathbf{F}(u + x)}{1 - \mathbf{F}(u)} \\ &= \left(1 + \frac{\xi x}{\tilde{\sigma}}\right)^{-1/\xi}\end{aligned}$$

where:

$$\tilde{\sigma} = \sigma + \xi(u - \mu)$$

Therefore, we have a duality between GEV and GPD distribution functions:

> "*[...] if block maxima have approximating distribution* $\mathbf{G}$*, then threshold excesses have a corresponding approximate distribution within the generalized Pareto family. Moreover, the parameters of the generalized Pareto distribution of threshold excesses are uniquely determined by those of the associated GEV distribution of block maxima. In particular, the parameter* ξ *is equal to that of the corresponding GEV distribution. Choosing a different, but still large, block size size* n *would affect the values of the GEV parameters, but not those of the corresponding generalized Pareto distribution of threshold excesses:* ξ *is invariant to block size, while the calculation of* $\tilde{\sigma}$ *is unperturbed by the changes in* μ *and* σ *which are self-compensating*" (Coles, 2001, page 75).

The estimation of the parameters (σ, ξ) is not obvious because it depends on the value taken by the threshold u. It must be sufficiently large to apply the previous theorem, but we also need enough data to obtain good estimates. We notice that the mean residual life $e(u)$ is a linear function of u:

$$\begin{aligned} e(u) &= \mathbb{E}[X - u \mid X > u] \\ &= \frac{\sigma + \xi u}{1 - \xi} \end{aligned}$$

when $\xi < 1$. If the GPD approximation is valid for a value u_0, it is therefore valid for any value $u > u_0$. To determine u_0, we can use a mean residual life plot, which consists in plotting u against the empirical mean excess $\hat{e}(u)$:

$$\hat{e}(u) = \frac{\sum_{i=1}^{n} (x_i - u)^+}{\sum_{i=1}^{n} \mathbf{1}\{x_i > u\}}$$

Once u_0 is found, we estimate the parameters (σ, ξ) by the method of maximum likelihood or linear regression[15].

Let us consider our previous example. In Figure 16.12, we have reported the mean residual life plot for the left tail of the four portfolios[16]. The determination of u_0 consists in finding linear relationships. We have a first linear relationship between $u = -3\%$ and $u = -1\%$, but it is not valid because it is followed by a change in slope. We prefer to consider that the linear relationship is valid for $u \geq 2\%$. By assuming that $u_0 = 2\%$ for all the four portfolios, we obtain the estimates given in Table 16.6.

[15] In this case, we estimate the linear model $\hat{e}(u) = a + bu + \varepsilon$ for $u \geq u_0$ and deduce that $\hat{\sigma} = \hat{a} / \left(1 + \hat{b}\right)$ and $\hat{\xi} = \hat{b} / \left(1 + \hat{b}\right)$.

[16] This means that $\hat{e}(u)$ is calculated using the portfolio's loss, that is the opposite of the portfolio's return.

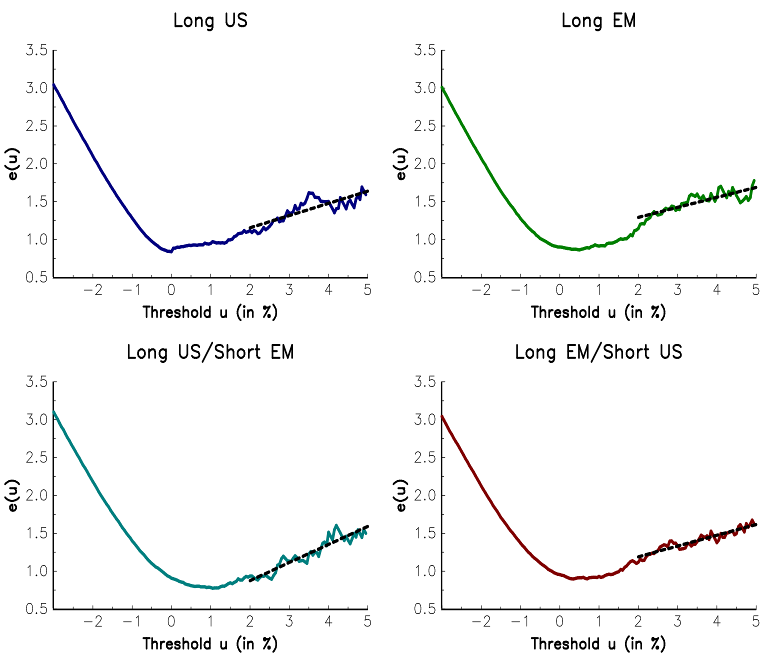

FIGURE 16.12: Mean residual life plot

TABLE 16.6: Estimation of the generalized Pareto distribution

Parameter	Long US	Long EM	Long US Short EM	Long EM Short US
$\hat{a}$	0.834	1.029	0.394	0.904
$\hat{b}$	0.160	0.132	0.239	0.142
$\hat{\sigma}$	0.719	0.909	0.318	0.792
$\hat{\xi}$	0.138	0.117	0.193	0.124

16.2.4.2 Estimating the expected shortfall

We remind that:

$$\mathbf{F}_u(x) = \frac{\mathbf{F}(u+x) - \mathbf{F}(u)}{1 - \mathbf{F}(u)} \approx \mathbf{H}(x)$$

where $\mathbf{H} \sim \mathcal{GPD}(\sigma, \xi)$. We deduce that:

$$\begin{aligned}
\mathbf{F}(x) &= \mathbf{F}(u) + (1 - \mathbf{F}(u)) \times \mathbf{F}_u(x - u) \\
&\approx \mathbf{F}(u) + (1 - \mathbf{F}(u)) \times \mathbf{H}(x - u)
\end{aligned}$$

We consider a sample of size n. We note n_u the number of observations whose the value x_i is larger than the threshold u. The non-parametric estimate of $\mathbf{F}(u)$ is then equal to:

$$\hat{\mathbf{F}}(u) = 1 - \frac{n_u}{n}$$

Therefore, we obtain the following semi-parametric estimate of $\mathbf{F}(x)$ for x larger than u:

$$\begin{aligned}
\hat{\mathbf{F}}(x) &= \hat{\mathbf{F}}(u) + \left(1 - \hat{\mathbf{F}}(u)\right) \times \hat{\mathbf{H}}(x - u) \\
&= \left(1 - \frac{n_u}{n}\right) + \frac{n_u}{n}\left(1 - \left(1 + \frac{\hat{\xi}(x-u)}{\hat{\sigma}}\right)^{-1/\hat{\xi}}\right) \\
&= 1 - \frac{n_u}{n}\left(1 + \frac{\hat{\xi}(x-u)}{\hat{\sigma}}\right)^{-1/\hat{\xi}}
\end{aligned}$$

We can interpret $\hat{\mathbf{F}}(x)$ as the historical estimate of the distribution tail that is improved by the extreme value theory. We deduce that:

$$\begin{aligned}
\mathrm{VaR}_\alpha &= \hat{\mathbf{F}}^{-1}(\alpha) \\
&= u + \frac{\hat{\sigma}}{\hat{\xi}}\left(\left(\frac{n}{n_u}(1-\alpha)\right)^{-\hat{\xi}} - 1\right)
\end{aligned}$$

and:

$$\begin{aligned}
\mathrm{ES}_\alpha &= \mathbb{E}\left[X \mid X > \mathrm{VaR}_\alpha\right] \\
&= \mathrm{VaR}_\alpha + \mathbb{E}\left[X - \mathrm{VaR}_\alpha \mid X > \mathrm{VaR}_\alpha\right] \\
&= \mathrm{VaR}_\alpha + \frac{\hat{\sigma} + \hat{\xi}(\mathrm{VaR}_\alpha - u)}{1 - \hat{\xi}} \\
&= \frac{\mathrm{VaR}_\alpha}{1 - \hat{\xi}} + \frac{\hat{\sigma} - \hat{\xi}u}{1 - \hat{\xi}} \\
&= \frac{u}{1 - \hat{\xi}} + \frac{\hat{\sigma}}{\left(1 - \hat{\xi}\right)\hat{\xi}}\left(\left(\frac{n}{n_u}(1-\alpha)\right)^{-\hat{\xi}} - 1\right) + \frac{\hat{\sigma} - \hat{\xi}u}{1 - \hat{\xi}}
\end{aligned}$$

TABLE 16.7: Estimating value-at-risk and expected shortfall using the generalized Pareto distribution

Risk measure	α	Long US	Long EM	Long US Short EM	Long EM Short US
	99.0%	3.20%	3.42%	2.56%	3.43%
VaR	99.5%	3.84%	4.20%	2.88%	4.13%
	99.9%	5.60%	6.26%	3.80%	6.02%
	99.0%	4.22%	4.64%	3.09%	4.54%
ES	99.5%	4.97%	5.52%	3.48%	5.34%
	99.9%	7.01%	7.86%	4.62%	7.49%

Finally, we obtain:

$$\mathrm{ES}_\alpha = u - \frac{\hat{\sigma}}{\hat{\xi}} + \frac{\hat{\sigma}}{\left(1-\hat{\xi}\right)\hat{\xi}}\left(\frac{n}{n_u}(1-\alpha)\right)^{-\hat{\xi}}$$

We consider again the example of the four portfolios with exposures on US and EM equities. In the sample, we have 3 815 observations, whereas the value taken by n_u when u is equal to 2% is 171, 161, 174 and 195 respectively. Using the estimates given in Table 16.6, we calculate the daily value-at-risk and expected shortfall of the four portfolios. The results are reported in Table 16.7. If we compare them with those obtained in Table 16.5, we notice that the GPD VaR is close to the GEV VaR.

16.3 Multivariate extreme value theory

The extreme value theory is generally formulated and used in the univariate case. It can be easily extended to the multivariate case, but its implementation is more difficult. This section is essentially based on the works of Deheuvels (1978), Galambos (1987) and Joe (1997).

16.3.1 Multivariate extreme value distributions

16.3.1.1 Extreme value copulas

An extreme value (EV) copula satisfies the following relationship:

$$\mathbf{C}\left(u_1^t,\ldots,u_n^t\right) = \mathbf{C}^t\left(u_1,\ldots,u_n\right)$$

for all $t > 0$. For instance, the Gumbel copula is an EV copula:

$$\begin{aligned}\mathbf{C}\left(u_1^t, u_2^t\right) &= \exp\left(-\left(\left(-\ln u_1^t\right)^\theta + \left(-\ln u_2^t\right)^\theta\right)^{1/\theta}\right) \\ &= \exp\left(-\left(t^\theta\left(\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right)\right)^{1/\theta}\right) \\ &= \left(\exp\left(-\left(\left(-\ln u_1\right)^\theta + \left(-\ln u_2\right)^\theta\right)^{1/\theta}\right)\right)^t \\ &= \mathbf{C}^t\left(u_1, u_2\right)\end{aligned}$$

but it is not the case of the Farlie-Gumbel-Morgenstern copula:

$$\begin{aligned}\mathbf{C}\left(u_1^t, u_2^t\right) &= u_1^t u_2^t + \theta u_1^t u_2^t\left(1 - u_1^t\right)\left(1 - u_2^t\right) \\ &= u_1^t u_2^t\left(1 + \theta - \theta u_1^t - \theta u_2^t + \theta u_1^t u_2^t\right) \\ &\neq u_1^t u_2^t\left(1 + \theta - \theta u_1 - \theta u_2 + \theta u_1 u_2\right)^t \\ &\neq \mathbf{C}^t\left(u_1, u_2\right)\end{aligned}$$

The term "*extreme value copula*" suggests a relationship between the extreme value theory and these copula functions. Let $X = (X_1, \ldots, X_n)$ be a random vector of dimension n. We note $X_{m:m}$ the random vector of maxima:

$$X_{m:m} = \begin{pmatrix} X_{m:m,1} \\ \vdots \\ X_{m:m,n} \end{pmatrix}$$

and $\mathbf{F}_{m:m}$ the corresponding distribution function:

$$\mathbf{F}_{m:m}\left(x_1, \ldots, x_n\right) = \Pr\left\{X_{m:m,1} \leq x_1, \ldots, X_{m:m,n} \leq x_n\right\}$$

The multivariate extreme value (MEV) theory considers the asymptotic behavior of the non-degenerate distribution function $\mathbf{G}$ such that:

$$\lim_{m\to\infty} \Pr\left(\frac{X_{m:m,1} - b_{m,1}}{a_{m,1}} \leq x_1, \ldots, \frac{X_{m:m,n} - b_{m,n}}{a_{m,n}} \leq x_n\right) = \mathbf{G}\left(x_1, \ldots, x_n\right)$$

Using Sklar's theorem, there exists a copula function $\mathbf{C}\langle\mathbf{G}\rangle$ such that:

$$\mathbf{G}\left(x_1, \ldots, x_n\right) = \mathbf{C}\langle\mathbf{G}\rangle\left(\mathbf{G}_1\left(x_1\right), \ldots, \mathbf{G}_n\left(x_n\right)\right)$$

It is obvious that the margins $\mathbf{G}_1, \ldots, \mathbf{G}_n$ satisfy the Fisher-Tippet theorem, meaning that the margins of a multivariate extreme value distribution can only be Gumbel, Fréchet or Weibull distribution functions. For the copula $\mathbf{C}\langle\mathbf{G}\rangle$, we have the following result:

Theorem 6 $\mathbf{C}\langle\mathbf{G}\rangle$ *is an extreme value copula.*

With the copula representation, we can then easily define MEV distributions. For instance, if we consider the random vector (X_1, X_2), whose joint distribution function is:

$$\mathbf{F}(x_1, x_2) = \exp\left(-\left((-\ln\Phi(x_1))^\theta + (-\ln x_2)^\theta\right)^{1/\theta}\right)$$

we notice that X_1 is a Gaussian random variable and X_2 is a uniform random variable. We conclude that the corresponding limit distribution function of maxima is:

$$\mathbf{G}(x_1, x_2) = \exp\left(-\left((-\ln\boldsymbol{\Lambda}(x_1))^\theta + (-\ln\boldsymbol{\Psi}_1(x_2))^\theta\right)^{1/\theta}\right)$$

In Figure 16.13, we have reported the contour plot of four MEV distribution functions, whose margins are $\mathcal{GEV}(0,1,1)$ and $\mathcal{GEV}(0,1,1.5)$. For the dependence function, we have considered the Gumbel-Hougaard copula and calibrated the parameter θ with respect to the Kendall's tau.

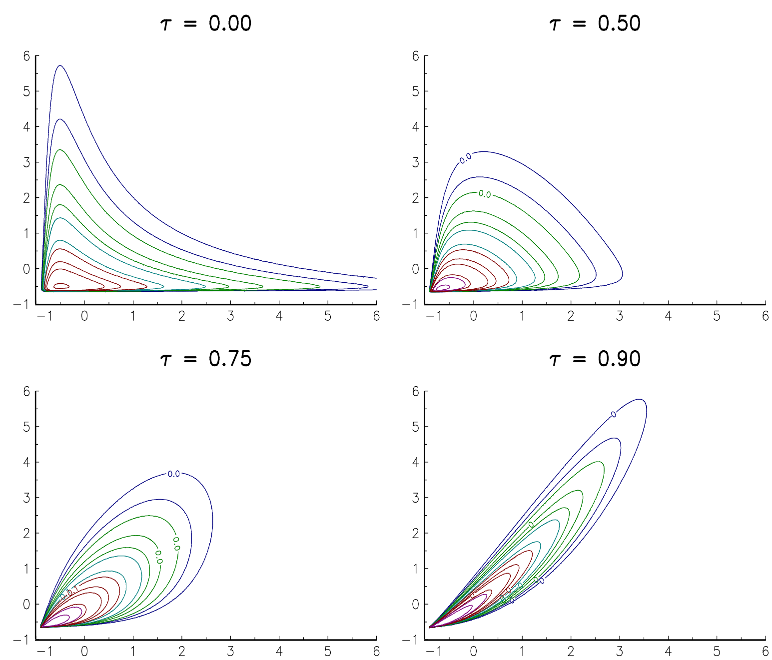

FIGURE 16.13: Multivariate extreme value distributions

16.3.1.2 Deheuvels-Pickands representation

Let $\mathbf{D}$ be a multivariate distribution function, whose survival margins are exponential and the dependence is an extreme value copula. By using the relationship[17] $\mathbf{C}(u_1,\ldots,u_n) = \mathbf{C}\left(e^{-\tilde{u}_1},\ldots,e^{-\tilde{u}_n}\right) = \mathbf{D}(\tilde{u}_1,\ldots,\tilde{u}_n)$, we have $\mathbf{D}^t(\tilde{\mathbf{u}}) = \mathbf{D}(t\tilde{\mathbf{u}})$. Therefore, $\mathbf{D}$ is a *min-stable multivariate exponential* (MSMVE) distribution.

Theorem 7 (Deheuvels/Pickands MSMVE representation) *Let* $\mathbf{D}(\tilde{\mathbf{u}})$ *be a survival function with exponential margins.* $\mathbf{D}$ *satisfies the relationship:*

$$-\ln \mathbf{D}(t\tilde{\mathbf{u}}) = -t \ln \mathbf{D}(\tilde{\mathbf{u}}) \qquad \forall\, t > 0$$

if and only if the representation of $\mathbf{D}$ *is:*

$$-\ln \mathbf{D}(\tilde{\mathbf{u}}) = \int \cdots \int_{\mathcal{S}_n} \max_{1 \le i \le n} (q_i \tilde{u}_i)\, \mathrm{d}S(\mathbf{q}) \qquad \forall\, \tilde{\mathbf{u}} \ge \mathbf{0}$$

where $\mathcal{S}_n$ *is the* n*-dimensional unit simplex and* S *is a finite measure on* $\mathcal{S}_n$.

This is the formulation[18] given by Joe (1997). Sometimes, the Deheuvels/Pickands representation is presented using a dependence function $B(\mathbf{w})$ defined by:

$$\begin{aligned}
\mathbf{D}(\tilde{\mathbf{u}}) &= \exp\left(-\left(\sum_{i=1}^n \tilde{u}_i\right) B(w_1,\ldots,w_n)\right) \\
B(\mathbf{w}) &= \int \cdots \int_{\mathcal{S}_n} \max_{1 \le i \le n} (q_i w_i)\, \mathrm{d}S(\mathbf{q})
\end{aligned}$$

where $w_i = \left(\sum_{i=1}^n \tilde{u}_i\right)^{-1} \tilde{u}_i$. Tawn (1990) showed that B is a convex function and satisfies the following condition:

$$\max(w_1,\ldots,w_n) \le B(w_1,\ldots,w_n) \le 1 \tag{16.4}$$

We deduce that an extreme value copula satisfies the PQD property:

$$\mathbf{C}^{\perp} \prec \mathbf{C} \prec \mathbf{C}^{+}$$

In the bivariate case, the formulation can be simplified because the convexity of B and the condition (16.4) are sufficient (Tawn, 1988). We have:

$$\begin{aligned}
\mathbf{C}(u_1,u_2) &= \mathbf{D}(\tilde{u}_1,\tilde{u}_2) \\
&= \exp\left(-(\tilde{u}_1+\tilde{u}_2)\, B\left(\frac{\tilde{u}_1}{\tilde{u}_1+\tilde{u}_2}, \frac{\tilde{u}_2}{\tilde{u}_1+\tilde{u}_2}\right)\right) \\
&= \exp\left(\ln(u_1 u_2)\, B\left(\frac{\ln u_1}{\ln(u_1 u_2)}, \frac{\ln u_2}{\ln(u_1 u_2)}\right)\right) \\
&= \exp\left(\ln(u_1 u_2)\, A\left(\frac{\ln u_1}{\ln(u_1 u_2)}\right)\right)
\end{aligned}$$

where $A(w) = B(w, 1-w)$. A is a convex function where $A(0) = A(1) = 1$ and satisfies $\max(w, 1-w) \le A(w) \le 1$.

[17] We recall that $\tilde{u} = -\ln u$.

[18] Note that it is similar to Proposition 5.11 of Resnick (1987), although the author does not use copulas.

TABLE 16.8: List of extreme value copulas

Copula	θ	$\mathbf{C}(u_1, u_2)$	$A(w)$
$\mathbf{C}^{\perp}$		$u_1 u_2$	1
Gumbel	$[1, \infty)$	$\exp\left(-\left(\tilde{u}_1^{\theta} + \tilde{u}_2^{\theta}\right)^{1/\theta}\right)$	$\left(w^{\theta} + (1-w)^{\theta}\right)^{1/\theta}$
Gumbel II	$[0, 1]$	$u_1 u_2 \exp\left(\theta \dfrac{\tilde{u}_1 \tilde{u}_2}{\tilde{u}_1 + \tilde{u}_2}\right)$	$\theta w^2 - \theta w + 1$
Galambos	$[0, \infty)$	$u_1 u_2 \exp\left(\left(\tilde{u}_1^{-\theta} + \tilde{u}_2^{-\theta}\right)^{-1/\theta}\right)$	$1 - \left(w^{-\theta} + (1-w)^{-\theta}\right)^{-1/\theta}$
Hüsler-Reiss	$[0, \infty)$	$\exp\left(-\tilde{u}_1 \vartheta(u_1, u_2; \theta) - \tilde{u}_2 \vartheta(u_2, u_1; \theta)\right)$	$w\kappa(w; \theta) + (1-w)\kappa(1-w; \theta)$
Marshall-Olkin	$[0, 1]^2$	$u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)$	$\max\left(1 - \theta_1 w, 1 - \theta_2(1-w)\right)$
$\mathbf{C}^{+}$		$\min(u_1, u_2)$	$\max(w, 1-w)$

$\vartheta(u_1, u_2; \theta) = \Phi\left(\frac{1}{\theta} + \frac{\theta}{2}\ln\left(\ln u_1 / \ln u_2\right)\right)$
$\kappa(w; \theta) = \vartheta(w, 1-w; \theta)$

Source: Ghoudi *et al.* (1998).

Example 74 *For the Gumbel copula, we have:*

$$\begin{aligned}
-\ln \mathbf{D}(\tilde{u}_1, \tilde{u}_2) &= \left(\tilde{u}_1^\theta + \tilde{u}_2^\theta\right)^{1/\theta} \\
B(w_1, w_2) &= \frac{\left(\tilde{u}_1^\theta + \tilde{u}_2^\theta\right)^{1/\theta}}{(\tilde{u}_1 + \tilde{u}_2)} = \left(w_1^\theta + w_2^\theta\right)^{1/\theta} \\
A(w) &= \left(w^\theta + (1-w)^\theta\right)^{1/\theta}
\end{aligned}$$

We verify that a bivariate EV copula satisfies the PQD property:

$$\begin{aligned}
& \max(w, 1-w) \leq A(w) \leq 1 \\
\Leftrightarrow\quad & \max\left(\frac{\ln u_1}{\ln(u_1 u_2)}, \frac{\ln u_2}{\ln(u_1 u_2)}\right) \leq A\left(\frac{\ln u_1}{\ln(u_1 u_2)}\right) \leq 1 \\
\Leftrightarrow\quad & \min(\ln u_1, \ln u_2) \geq \ln(u_1 u_2)\, A\left(\frac{\ln u_1}{\ln(u_1 u_2)}\right) \geq \ln(u_1 u_2) \\
\Leftrightarrow\quad & \min(u_1, u_2) \geq \exp\left[\ln(u_1 u_2)\, A\left(\frac{\ln u_1}{\ln(u_1 u_2)}\right)\right] \geq u_1 u_2 \\
\Leftrightarrow\quad & \mathbf{C}^+ \succ \mathbf{C} \succ \mathbf{C}^\perp
\end{aligned}$$

When the extreme values are independent, we have $A(w) = 1$ whereas the case of perfect dependence corresponds to $A(w) = \max(w, 1-w)$:

$$\begin{aligned}
\mathbf{C}(u_1, u_2) &= \exp\left[\ln(u_1 u_2) \max\left(\frac{\ln u_1}{\ln(u_1 u_2)}, \frac{\ln u_2}{\ln(u_1 u_2)}\right)\right] \\
&= \min(u_1, u_2) \\
&= \mathbf{C}^+(u_1, u_2)
\end{aligned}$$

In Table 16.8, we have reported the dependence function $A(w)$ of the most used EV copula functions.

16.3.2 Maximum domain of attraction

Let $\mathbf{F}$ be a multivariate distribution function whose margins are $\mathbf{F}_1, \ldots, \mathbf{F}_n$ and the copula is $\mathbf{C}\langle\mathbf{F}\rangle$. We note $\mathbf{G}$ the corresponding multivariate extreme value distribution, $\mathbf{G}_1, \ldots, \mathbf{G}_n$ the margins of $\mathbf{G}$ and $\mathbf{C}\langle\mathbf{G}\rangle$ the associated copula function. We can show that $\mathbf{F} \in \mathrm{MDA}(\mathbf{G})$ if and only if $\mathbf{F}_i \in \mathrm{MDA}(\mathbf{G}_i)$ for all $i = 1, \ldots, n$ and $\mathbf{C}\langle\mathbf{F}\rangle \in \mathrm{MDA}(\mathbf{C}\langle\mathbf{G}\rangle)$. Previously, we have seen how to characterize the max-domain of attraction in the univariate case and how to calculate the normalizing constants. These constants remains the same in the multivariate case, meaning that the only difficulty is to determine the EV copula $\mathbf{C}\langle\mathbf{G}\rangle$.

Theorem 8 $\mathbf{C}\langle\mathbf{F}\rangle \in \mathrm{MDA}(\mathbf{C}\langle\mathbf{G}\rangle)$ *if* $\mathbf{C}\langle\mathbf{F}\rangle$ *satisfies the following relationship:*

$$\lim_{t \to \infty} \mathbf{C}^t \langle\mathbf{F}\rangle \left(u_1^{1/t}, \ldots, u_n^{1/t}\right) = \mathbf{C}\langle\mathbf{G}\rangle(u_1, \ldots, u_n)$$

If $\mathbf{C}\langle\mathbf{F}\rangle$ *is an EV copula, then* $\mathbf{C}\langle\mathbf{F}\rangle \in \mathrm{MDA}(\mathbf{C}\langle\mathbf{F}\rangle)$.

We can show that an equivalent condition is:

$$\lim_{u\to 0}\frac{1-\mathbf{C}\langle\mathbf{F}\rangle\left((1-u)^{w_1},\ldots,(1-u)^{w_n}\right)}{u}=B(w_1,\ldots,w_n)$$

In the bivariate case, we obtain:

$$\lim_{u\to 0}\frac{1-\mathbf{C}\langle\mathbf{F}\rangle\left((1-u)^{1-t},(1-u)^{t}\right)}{u}=A(t)$$

for all $t\in[0,1]$.

Example 75 *We consider the random vector* (X_1, X_2) *defined by the following distribution function:*

$$\mathbf{F}(x_1,x_2)=\left(\left(1-e^{-x_1}\right)^{-\theta}+x_2^{-\theta}-1\right)^{-1/\theta}$$

on $[0,\infty]\times[0,1]$. *The margins of* $\mathbf{F}(x_1,x_2)$ *are* $\mathbf{F}_1(x_1)=\mathbf{F}(x_1,1)=1-e^{-x_1}$ *and* $\mathbf{F}_2(x_2)=\mathbf{F}(\infty,x_2)=x_2$. X_1 *is an exponential random variable and* X_2 *is a uniform random variable. We know that*

$$\lim_{n\to\infty}\Pr\left(\frac{X_{n:n,1}-\ln n}{1}\leq x_1\right)=\mathbf{\Lambda}(x_1)$$

and:

$$\lim_{n\to\infty}\Pr\left(\frac{X_{n:n,2}-1}{n^{-1}}\leq x_2\right)=\mathbf{\Psi}_1(x_2)$$

The dependence function of $\mathbf{F}$ *is the Clayton copula:* $\mathbf{C}\langle\mathbf{F}\rangle(u_1,u_2)=\left(u_1^{-\theta}+u_2^{-\theta}-1\right)^{-1/\theta}$. *We have:*

$$\begin{aligned}\lim_{u\to 0}\frac{1-\mathbf{C}\langle\mathbf{F}\rangle\left((1-u)^{t},(1-u)^{1-t}\right)}{u} &= \lim_{u\to 0}\frac{1-\left(1+\theta u+o(u)\right)^{-1/\theta}}{u}\\ &= \lim_{u\to 0}\frac{u+o(u)}{u}\\ &= 1\end{aligned}$$

We deduce that $\mathbf{C}\langle\mathbf{G}\rangle=\mathbf{C}^{\perp}$. *We obtain finally:*

$$\begin{aligned}\mathbf{G}(x_1,x_2) &= \lim_{n\to\infty}\Pr\left\{X_{n:n,1}-\ln n\leq x_1, n\left(X_{n:n,2}-1\right)\leq x_2\right\}\\ &= \mathbf{\Lambda}(x_1)\,\mathbf{\Psi}_1(x_2)\\ &= \exp\left(-e^{-x_1}\right)\exp(x_2)\end{aligned}$$

If we change the copula $\mathbf{C}\langle\mathbf{F}\rangle$, only the copula $\mathbf{C}\langle\mathbf{G}\rangle$ is modified. For instance, when $\mathbf{C}\langle\mathbf{F}\rangle$ is the Gaussian copula with parameter $\rho<1$, then $\mathbf{G}(x_1,x_2)=\exp\left(-e^{-x_1}\right)\exp(x_2)$. When the copula parameter ρ is equal to 1, we obtain $\mathbf{G}(x_1,x_2)=\min\left(\exp\left(-e^{-x_1}\right),\exp(x_2)\right)$. When $\mathbf{C}\langle\mathbf{F}\rangle$ is the Gumbel copula, the MEV distribution becomes $\mathbf{G}(x_1,x_2)=\exp\left(-\left(e^{-\theta x_1}+(-x_2)^{\theta}\right)^{1/\theta}\right)$.

16.3.3 Tail dependence of extreme values

We can show that the (upper) tail dependence of $\mathbf{C}\langle\mathbf{G}\rangle$ is equal to the (upper) tail dependence of $\mathbf{C}\langle\mathbf{F}\rangle$:

$$\lambda^{+}(\mathbf{C}\langle\mathbf{G}\rangle)=\lambda^{+}(\mathbf{C}\langle\mathbf{F}\rangle)$$

This implies that extreme values are independent if the copula function $\mathbf{C}\langle\mathbf{F}\rangle$ has no (upper) tail dependence.

16.4 Exercises

16.4.1 Uniform order statistics

We assume that $X_1,\ldots,X_n$ are independent uniform random variables.

1. Show that the density function of the order statistic $X_{i:n}$ is:

$$f_{i:n}(x)=\frac{\Gamma(i)\,\Gamma(n-i+1)}{\Gamma(n+1)}x^{i-1}(1-x)^{n-i}$$

2. Calculate the mean $\mathbb{E}[X_{i:n}]$.

3. Show that the variance is equal to:

$$\operatorname{var}(X_{i:n})=\frac{i(n-i+1)}{(n+1)^2(n+2)}$$

4. We consider 10 samples of 8 independent observations from the $\mathcal{U}_{[0,1]}$ probability distribution:

Sample	Observation							
	1	2	3	4	5	6	7	8
1	0.24	0.45	0.72	0.14	0.04	0.34	0.94	0.55
2	0.12	0.32	0.69	0.64	0.31	0.25	0.97	0.57
3	0.69	0.50	0.26	0.17	0.50	0.85	0.11	0.17
4	0.53	0.00	0.77	0.58	0.98	0.15	0.98	0.03
5	0.89	0.25	0.15	0.62	0.74	0.85	0.65	0.46
6	0.74	0.65	0.86	0.05	0.93	0.15	0.25	0.07
7	0.16	0.12	0.63	0.33	0.55	0.61	0.34	0.95
8	0.96	0.82	0.01	0.87	0.57	0.11	0.14	0.47
9	0.68	0.83	0.73	0.78	0.27	0.85	0.55	0.57
10	0.89	0.94	0.91	0.28	0.99	0.40	0.99	0.68

For each sample, find the order statistics. Calculate the empirical mean and standard deviation of $X_{i:8}$ for $i=1,\ldots,8$ and compare these values with the theoretical results.

5. We assume that n is odd, meaning that $n = 2k + 1$. We consider the median statistic $X_{k+1:n}$. Show that the density function of $X_{i:n}$ is right asymmetric if $i \leq k$, symmetric about .5 if $i = k+1$ and left asymmetric otherwise.

6. We now assume that the density function of $X_1, \ldots, X_n$ is symmetric. What becomes the results obtained in Question 5?

16.4.2 Order statistics and return period

1. Let X and $\mathbf{F}$ be the daily return of a portfolio and the associated probability distribution. We note $X_{n:n}$ the maximum of daily returns for a period of n trading days. Using the standard assumptions, define the cumulative probability distribution $\mathbf{F}_{n:n}$ of $X_{n:n}$ if we suppose that $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

2. How could we test the hypothesis $\mathcal{H}_0 : X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ using $\mathbf{F}_{n:n}$?

3. Define the notion of return period. What is the return period associated to the statistics $\mathbf{F}^{-1}(99\%)$, $\mathbf{F}_{1:1}^{-1}(99\%)$, $\mathbf{F}_{5:5}^{-1}(99\%)$ and $\mathbf{F}_{21:21}^{-1}(99\%)$?

4. We consider the random variable $X_{20:20}$. Find the confidence level α which ensures that the return period associated to the quantile $\mathbf{F}_{20:20}^{-1}(\alpha)$ is equivalent to the return period of the daily value-at-risk with a 99.9% confidence level.

16.4.3 Extreme order statistics of exponential random variables

1. We note $\boldsymbol{\tau} \sim \mathcal{E}(\lambda)$. Show that:

$$\Pr\{\boldsymbol{\tau} > t \mid \boldsymbol{\tau} > s\} = \Pr\{\boldsymbol{\tau} > t - s\}$$

with $t > s$. Comment on this result.

2. Let $\boldsymbol{\tau}_i$ be the random variable of distribution $\mathcal{E}(\lambda_i)$. Calculate the probability distribution of $\min(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ and $\max(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n)$ in the independent case. Show that:

$$\Pr\{\min(\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n) = \tau_i\} = \frac{\lambda_i}{\sum_{j=1}^{n} \lambda_j}$$

3. Same question if the random variables $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n$ are comonotone.

16.4.4 Construction of a stress scenario with the GEV distribution

1. We note a_n and b_n the normalization constraints and $\mathbf{G}$ the limit distribution of the Fisher-Tippet theorem.

(a) Find the limit distribution $\mathbf{G}$ when $X \sim \mathcal{E}(\lambda)$, $a_n = \lambda^{-1}$ and $b_n = \lambda^{-1} \ln n$.

(b) Same question when $X \sim \mathcal{U}_{[0,1]}$, $a_n = n^{-1}$ and $b_n = 1 - n^{-1}$.

(c) Same question when X is a Pareto distribution:

$$\mathbf{F}(x) = 1 - \left(\frac{\theta}{\theta + x} \right)^{\alpha},$$

$a_n = \theta \alpha^{-1} n^{1/\alpha}$ and $b_n = \theta n^{1/\alpha} - \theta$.

2. We denote by $\mathbf{G}$ the GEV probability distribution:

$$\mathbf{G}(x) = \exp \left\{ - \left[1 + \xi \left(\frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\}$$

What is the interest of this probability distribution? Write the log-likelihood function associated to the sample $\{x_1, \ldots, x_T\}$.

3. Show that for $\xi \to 0$, the distribution $\mathbf{G}$ tends toward the Gumbel distribution:

$$\mathbf{\Lambda}(x) = \exp \left(- \exp \left(- \left(\frac{x - \mu}{\sigma} \right) \right) \right)$$

4. We consider the minimum value of daily returns of a portfolio for a period of n trading days. We then estimate the GEV parameters associated to the sample of the opposite of the minimum values. We assume that ξ is equal to 1.

(a) Show that we can approximate the portfolio loss (in %) associated to the return period $\mathcal{T}$ with the following expression:

$$r(\mathcal{T}) \simeq - \left(\hat{\mu} + \left(\frac{\mathfrak{t}}{n} - 1 \right) \hat{\sigma} \right)$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates of GEV parameters.

(b) We set n equal to 21 trading days. We obtain the following results for two portfolios:

Portfolio	$\hat{\mu}$	$\hat{\sigma}$	ξ
#1	1%	3%	1
#2	10%	2%	1

Calculate the stress scenario for each portfolio when the return period is equal to one year. Comment on these results.

16.4.5 Extreme value theory in the bivariate case

1. What is an extreme value (EV) copula $\mathbf{C}$?

2. Show that $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$ are EV copulas. Why $\mathbf{C}^{-}$ can not be an EV copula?

3. We define the Gumbel-Houggaard copula as follows:

$$\mathbf{C}(u_1, u_2) = \exp\left(-\left[(-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right]^{1/\theta}\right)$$

with $\theta \geq 1$. Verify that it is an EV copula.

4. What is the definition of the upper tail dependence λ? What is its usefulness in multivariate extreme value theory?

5. Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$. If $g'(x_0) \neq 0$, L'Hospital's rule states that:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Deduce that the upper tail dependence λ of the Gumbel-Houggaard copula is $2 - 2^{1/\theta}$. What is the correlation of two extremes when $\theta = 1$?

6. We define the Marshall-Olkin copula as follows:

$$\mathbf{C}(u_1, u_2) = u_1^{1-\theta_1} u_2^{1-\theta_2} \min\left(u_1^{\theta_1}, u_2^{\theta_2}\right)$$

with $\{\theta_1, \theta_2\} \in [0, 1]^2$.

   (a) Verify that it is an EV copula.
   (b) Find the upper tail dependence λ of the Marshall-Olkin copula.
   (c) What is the correlation of two extremes when $\min(\theta_1, \theta_2) = 0$?
   (d) In which case are two extremes perfectly correlated?

16.4.6 Max-domain of attraction in the bivariate case

1. We consider the following distributions of probability:

Distribution		$\mathbf{F}(x)$
Exponential	$\mathcal{E}(\lambda)$	$1 - e^{-\lambda x}$
Uniform	$\mathcal{U}_{[0,1]}$	x
Pareto	$\mathcal{P}(\theta, \alpha)$	$1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}$

For each distribution, we give the normalization parameters a_n and b_n of the Fisher-Tippet theorem and the corresponding limit distribution distribution $\mathbf{G}(x)$:

Distribution	a_n	b_n	$\mathbf{G}(x)$
Exponential	λ^{-1}	$\lambda^{-1}\ln n$	$\mathbf{\Lambda}(x) = e^{-e^{-x}}$
Uniform	n^{-1}	$1 - n^{-1}$	$\mathbf{\Psi}_1(x-1) = e^{x-1}$
Pareto	$\theta\alpha^{-1}n^{1/\alpha}$	$\theta n^{1/\alpha} - \theta$	$\mathbf{\Phi}_\alpha\left(1+\frac{x}{\alpha}\right) = e^{-\left(1+\frac{x}{\alpha}\right)^{-\alpha}}$

We note $\mathbf{G}(x_1, x_2)$ the asymptotic distribution of the bivariate random vector $(X_{1,n:n}, X_{2,n:n})$ where $X_{1,i}$ (resp. $X_{2,i}$) are *iid* random variables.

(a) What is the expression of $\mathbf{G}(x_1, x_2)$ when $X_{1,i}$ and $X_{2,i}$ are independent, $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{U}_{[0,1]}$?

(b) Same question when $X_{1,i} \sim \mathcal{E}(\lambda)$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

(c) Same question when $X_{1,i} \sim \mathcal{U}_{[0,1]}$ and $X_{2,i} \sim \mathcal{P}(\theta, \alpha)$.

2. What becomes the previous results when the dependence function between $X_{1,i}$ and $X_{2,i}$ is the Normal copula with parameter $\rho < 1$?

3. Same question when the parameter of the Normal copula is equal to one.

4. Find the expression of $\mathbf{G}(x_1, x_2)$ when the dependence function is the Gumbel-Houggaard copula.

Chapter 17

Monte Carlo Simulation Methods

17.1 Random variate generation

17.1.1 Copula functions

17.1.1.1 The method of multivariate distributions

17.1.1.2 The method of conditional distributions

17.1.1.3 The method of empirical distributions

17.2 Stochastic process simulation

17.3 Monte Carlo methods

17.4 Exercises

17.4.1 Simulation of an Archimedean copula

We recall that an Archimedean copula has the following expression:

$$\mathbf{C}(u_1, u_2) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2))$$

where φ is the generator function.

1. What are the conditions on φ in order to verify that $\mathbf{C}$ is a copula?
2. We consider the following dependence functions: $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$. Which copulas are Archimedean? Give the corresponding generator.
3. We assume that $\varphi(u) = (-\ln u)^\theta$ with $\theta \geq 1$. Find the corresponding copula.
4. Calculate the conditional distribution $\mathbf{C}_{2|1}$ associated to the previous Archimedean copula. Deduce an algorithm to simulate it.

17.4.2 Simulation of the bivariate Normal copula

Let $X = (X_1, X_2)$ be a standard Gaussian vector with correlation ρ. We note $U_1 = \Phi(X_1)$ and $U_2 = \Phi(X_2)$.

1. We note Σ the matrix defined as follows:

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

   Calculate the Cholesky decomposition of Σ. Deduce an algorithm to simulate X.

2. Show that the copula of (X_1, X_2) is the same that the copula of the random vector (U_1, U_2).

3. Deduce an algorithm to simulate the Normal copula with parameter ρ.

4. Calculate the conditional distribution of X_2 knowing that $X_1 = x$. Then show that:

$$\Phi_2(x_1, x_2; \rho) = \int_{-\infty}^{x_1} \Phi\left(\frac{x_2 - \rho x}{\sqrt{1-\rho^2}}\right) \phi(x) \, \mathrm{d}x$$

5. Deduce an expression of the Normal copula.

6. Calculate the conditional copula function $\mathbf{C}_{2|1}$. Deduce an algorithm to simulate the Normal copula with parameter ρ.

7. Show that this algorithm is equivalent to the Cholesky algorithm found in Question 3.

Appendix A

Technical Appendix

A.1 Numerical analysis

A.1.1 Linear algebra

Following Horn and Johnson (1991), we recall some definitions about matrices:

- the square matrix A is symmetric if it is equal to its transpose $A^\top$;
- the square matrix A is hermitian if it is equal to its own conjugate transpose A^*, implying that we have $A_{i,j} = \operatorname{conj} A_{j,i}$;
- we say that A is an orthogonal matrix if we have $AA^\top = A^\top A = I$ and an unitary matrix if we have $A^* = A^{-1}$.

A.1.1.1 Eigendecomposition

The λ is an eigenvalue of the $n \times n$ matrix A if there exists a non-zero eigenvector v such that we have $Av = \lambda v$. Let V denote the matrix composed of the n eigenvectors. We have:

$$AV = V\Lambda$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of eigenvalues. We finally obtain the eigendecomposition of the matrix A:

$$A = V\Lambda V^{-1} \tag{A.1}$$

If A is an hermitian matrix, then the matrix V of eigenvectors is unitary. It follows that:

$$A = V\Lambda V^*$$

In particular, if A is a symmetric real matrix, we obtain[1]:

$$A = V\Lambda V^\top \tag{A.2}$$

[1] We have:

$$\begin{aligned} A^\top &= \left(V\Lambda V^{-1}\right)^\top \\ &= \left(V^{-1}\right)^\top \Lambda V^\top \end{aligned}$$

We deduce that $V^{-1} = V^\top$.

Remark 65 *A related decomposition is the singular value decomposition. Let A be a rectangular matrix with dimension $m \times n$. We have:*

$$A = U \Sigma V^* \tag{A.3}$$

where U is a $m \times m$ unitary matrix, Σ is a $m \times n$ diagonal matrix with elements $\sigma_i \geq 0$ and V is a $n \times n$. unitary matrix. σ_i are the singular values of A, u_i are the left singular vectors of A, and v_i are the left singular vectors of A.

A.1.1.2 Schur decomposition

The Schur decomposition of the $n \times n$ matrix A is equal to:

$$A = QTQ^* \tag{A.4}$$

where Q is a unitary matrix and T is an upper triangular matrix[2]. This decomposition is useful to calculate matrix functions.

Let us consider the matrix function in the space $\mathbb{M}$ of square matrices:

$$\begin{aligned} f: \quad & \mathbb{M} \longrightarrow \mathbb{M} \\ & A \longmapsto B = f(A) \end{aligned}$$

For instance, if $f(x) = \sqrt{x}$ and A is positive, we can define the matrix B such that:

$$BB^* = B^*B = A$$

B is called the square root of A and we note $B = A^{1/2}$. This matrix function generalizes the scalar-valued function to the set of matrices. Let us consider the following Taylor expansion:

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \ldots$$

We can show that if the series converge for $|x - x_0| < \alpha$, then the matrix $f(A)$ defined by the following expression:

$$f(A) = f(x_0) + (A - x_0 I) f'(x_0) + \frac{(A - x_0 I)^2}{2!} f''(x_0) + \ldots$$

converges to the matrix B if $|A - x_0 I| < \alpha$ and we note $B = f(A)$. In the case of the exponential function, we have:

$$f(x) = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We deduce that the exponential of the matrix A is equal to:

$$B = e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

[2] Q and T are also called the transformation matrix and the Schur form of A.

In a similar way, the logarithm of A is the matrix B such that $e^B = A$ and we note $B = \ln A$.

Let A and B be two $n \times n$ square matrices. Using the Taylor expansion, Golub and Van Loan (2013) show that $f\left(A^\top\right) = f(A)^\top$, $Af(A) = f(A)A$ and $f\left(B^{-1}AB\right) = B^{-1}f(A)B$. It follows that:

$$e^{A^\top} = \left(e^A\right)^\top$$

and:

$$e^{B^{-1}AB} = B^{-1}e^A B$$

If $AB = BA$, we can also prove that $Ae^B = e^B A$ and $e^{A+B} = e^A e^B = e^B e^A$.

Remark 66 *There are different ways to compute numerically $f(A)$. For transcendental functions, we have:*

$$f(A) = Qf(T)Q^*$$

where $A = QTQ^*$ is the Schur decomposition of A. Because T is an upper diagonal matrix, $f(T)$ is also a diagonal matrix whose elements can be calculated with Algorithm 9.1.1 of Golub and Van Loan (2013). This algorithm is reproduced below[3].

Algorithm 2 Schur-Parlett matrix function $f(A)$

Compute the Schur decomposition $A = QTQ^*$
Initialize F to the matrix $\mathbf{0}_{n\times n}$
for $i = 1 : n$ **do**
  $f_{i,i} \leftarrow f(t_{i,i})$
end for
for $p = 1 : n-1$ **do**
  for $i = 1 : n-p$ **do**
    $j \leftarrow i + p$
    $s \leftarrow t_{i,j}(f_{j,j} - f_{i,i})$
    for $k = i+1 : j-1$ **do**
      $s \leftarrow s + t_{i,k}f_{k,j} - f_{i,k}t_{k,j}$
    end for
    $f_{i,j} \leftarrow s/(t_{j,j} - t_{i,i})$
  end for
end for
$B \leftarrow QFQ^*$
return B

Source: Golub and Van Loan (2013), page 519.

[3] For the exponential matrix, we may prefer to use the Pade approximation method, which is described in Algorithm 9.3.1 (Scaling and Squaring) of Golub and Van Loan (2013).

A.1.2 Approximation methods

A.1.2.1 Semidefinite approximation

A.1.2.2 Numerical integration

A.1.2.3 Finite difference method

A.2 Statistics

A.2.1 Probability distributions

A.2.1.1 The normal distribution

Let C be a correlation matrix. We consider the standardized Gaussian random vector $X \sim \mathcal{N}(\mathbf{0}, C)$ of dimension n. We note $\phi_n(x; C)$ the associated density function defined as:

$$\phi_n(x; C) = (2\pi)^{-n/2} |C|^{-1/2} \exp\left(-\frac{1}{2} x^\top C^{-1} x\right)$$

We deduce that the expression of cumulative density function is:

$$\Phi_n(x; C) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_2} \phi_n(u; C) \, \mathrm{d}u$$

By construction, we have $\mathbb{E}[X] = \mathbf{0}$ and $\operatorname{cov}(x) = C$. In the bivariate case, we use the notations $\phi_2(x_1, x_2; \rho) = \phi_2(x; C)$ and $\Phi_2(x_1, x_2; \rho) = \Phi_2(x; C)$ where $\rho = C_{1,2}$ is the correlation between the components X_1 and X_2. In the univariate case, we also consider the alternative notations $\phi(x) = \phi_1(x; 1)$ and $\Phi(x) = \Phi_1(x; 1)$. The density function reduces then to:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} x^2\right)$$

Concerning the moments, we have $\mu(X) = 0$, $\sigma(X) = 1$, $\gamma_1(X) = 0$ and $\gamma_2(X) = 0$.

Adding a mean vector μ and a covariance matrix Σ is equivalent to apply the linear transformation to X:

$$Y = \mu + \sigma X$$

where $\sigma = \operatorname{diag}^{1/2}(\Sigma)$.

A.2.1.2 The Student's t distribution

Let $X \sim \mathcal{N}(\mathbf{0}, \Sigma)$ and $V \sim \chi_\nu^2 / \nu$ independent of X. We define the multivariate Student's t distribution as the one corresponding to the linear transformation:

$$Y = V^{-1/2} X$$

The corresponding density function is:

$$\mathbf{t}_n(y;\Sigma,\nu)=\frac{\Gamma\left(\frac{(\nu+n)}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)(\nu\pi)^{n/2}}\left|\Sigma\right|^{-1/2}\left(1+\frac{1}{\nu}y^\top\Sigma^{-1}y\right)^{-(\nu+n)/2}$$

We note $\mathbf{T}_n(y;\Sigma,\nu)$ the cumulative density function:

$$\mathbf{T}_n(y;\Sigma,\nu)=\int_{-\infty}^{y_1}\cdots\int_{-\infty}^{y_2}\mathbf{t}_n(u;\Sigma,\nu)\,\mathrm{d}u$$

The first two moments[4] of Y are $\mathbb{E}[Y]=\mathbf{0}$ and $\operatorname{cov}(Y)=\nu(\nu-2)^{-1}\Sigma$. Adding a mean μ is equivalent to consider the random vector $Z=\mu+Y$. We also verify that Y tends to the Gaussian random vector X when the number of degrees of freedom tends to ∞.

In the univariate case, the standardized density function becomes:

$$\mathbf{t}_1(y;\nu)=\frac{\Gamma\left(\frac{(\nu+1)}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\nu\pi}}\left(1+\frac{y^2}{\nu}\right)^{-(\nu+1)/2}$$

We also use the alternative notations $\mathbf{t}_\nu(y)=\mathbf{t}_1(y;\nu)$ and $\mathbf{T}_\nu(y)=\mathbf{T}_1(y;\nu)$. Concerning the moments[5], we obtain $\mu(Y)=0$, $\sigma^2(Y)=\nu/(\nu-2)$, $\gamma_1(Y)=0$ and $\gamma_2(Y)=6/(\nu-4)$.

A.2.1.3 The skew normal distribution

The seminal work of Azzalini (1985) has led to a rich development on skew distributions with numerous forms, parameterizations and extensions[6]. We adopt here the construction of Azzalini and Dalla Valle (1996).

The multivariate case Azzalini and Dalla Valle (1996) define the density function of the skew normal (or SN) distribution as follows:

$$f(x)=2\phi_n(x-\xi;\Omega)\,\Phi_1\left(\eta^\top\omega^{-1}(x-\xi)\right)$$

with $\omega=\operatorname{diag}^{1/2}(\Omega)$. We say that X follows a multivariate skew normal distribution with parameters ξ, Ω and η and we write $X\sim\mathcal{SN}(\xi,\Omega,\eta)$. We notice that the distribution of $X\sim\mathcal{SN}_n(\xi,\Omega,\mathbf{0})$ is the standard normal distribution $\mathcal{N}(\xi,\Omega)$. We verify the property $X=\xi+\omega Y$ where $Y\sim\mathcal{SN}(\mathbf{0},C,\eta)$ and $C=\omega^{-1}\Omega\omega^{-1}$ is the correlation matrix of Ω. Azzalini and Dalla Valle (1996) demonstrate that the first two moments are:

$$\begin{aligned}\mathbb{E}[X] &= \xi+\sqrt{\frac{2}{\pi}}\omega\delta\\ \operatorname{cov}(X) &= \omega\left(C-\frac{2}{\pi}\delta\delta^\top\right)\omega^\top\end{aligned}$$

[4]The second moment is not defined if $\nu\leq 2$.

[5]The skewness is not defined if $\nu\leq 3$ whereas the excess kurtosis is infinite if $2<\nu\leq 4$.

[6]See for instance Arellano-Valle and Genton (2005) and Lee and McLachlan (2013) for a review.

with $\delta = \left(1 + \eta^\top C \eta\right)^{-1/2} C\eta$.

Azzalini and Capitanio (1999) show that $Y \sim \mathcal{SN}(\mathbf{0}, C, \eta)$ has the following stochastic representation:

$$Y = \begin{cases} U & \text{if } U_0 > 0 \\ -U & \text{otherwise} \end{cases}$$

with:

$$\begin{pmatrix} U_0 \\ U \end{pmatrix} \sim \mathcal{N}\left(\mathbf{0}, C_+(\delta)\right), \qquad C_+(\delta) = \begin{pmatrix} 1 & \delta^\top \\ \delta & C \end{pmatrix}$$

and $\delta = \left(1 + \eta^\top C \eta\right)^{-1/2} C\eta$. We deduce that:

$$\begin{aligned} \Pr\{X \le x\} &= \Pr\left\{Y \le \omega^{-1}(x - \xi)\right\} \\ &= \Pr\left\{U \le \omega^{-1}(x - \xi) \mid U_0 > 0\right\} \\ &= \frac{\Pr\left\{U \le \omega^{-1}(x - \xi), U_0 > 0\right\}}{\Pr\{U_0 > 0\}} \\ &= 2\left(\Pr\left\{U \le \omega^{-1}(x - \xi)\right\} - \Pr\left\{U \le \omega^{-1}(x - \xi), U_0 \le 0\right\}\right) \\ &= 2\left(\Phi_n\left(\omega^{-1}(x - \xi); C\right) - \Phi_{n+1}\left(u_+; C_+(\delta)\right)\right) \\ &= 2\Phi_{n+1}\left(u_+; C_+(-\delta)\right) \end{aligned}$$

with $u_+ = \left(0, \omega^{-1}(x - \xi)\right)$. We can therefore use this representation to simulate the random vector $X \sim \mathcal{SN}(\xi, \Omega, \eta)$ and compute the cumulative distribution function.

Let A be a $m \times n$ matrix and $X \sim \mathcal{SN}(\xi, \Omega, \eta)$. Azzalini and Capitanio (1999) demonstrate that the linear transformation of a skew normal vector is still a skew normal vector:

$$AX \sim \mathcal{SN}(\xi_A, \Omega_A, \eta_A)$$

with:

$$\begin{aligned} \xi_A &= A\xi \\ \Omega_A &= A\Omega A^\top \\ \eta_A &= \frac{\omega_A \Omega_A^{-1} B^\top \eta}{\left(1 + \eta^\top \left(C - B\Omega_A^{-1} B^\top\right) \eta\right)^{1/2}} \end{aligned}$$

where $\omega = \operatorname{diag}^{1/2}(\Omega)$, $C = \omega^{-1}\Omega\omega$, $\omega_A = \operatorname{diag}^{1/2}(\Omega_A)$ and $B = \omega^{-1}\Omega A^\top$. This property also implies that the marginal distributions of a subset of X is still a skew normal distribution.

The univariate case When the dimension n is equal to 1, the density function of $X \sim \mathcal{SN}\left(\xi, \omega^2, \eta\right)$ becomes:

$$f(x) = \frac{2}{\omega}\phi\left(\frac{x - \xi}{\omega}\right)\Phi\left(\eta\left(\frac{x - \xi}{\omega}\right)\right)$$

Using the previous stochastic representation, we have:

$$\begin{aligned}
\Pr\{X \leq x\} &= 2\left(\Phi\left(\frac{x-\xi}{\omega}\right) - \Phi_2\left(0, \frac{x-\xi}{\omega}; \delta\right)\right) \\
&= 2\Phi_2\left(0, \frac{x-\xi}{\omega}; -\delta\right)
\end{aligned}$$

with:

$$\delta = \frac{\eta}{\sqrt{1+\eta^2}}$$

We note $m_0 = \delta\sqrt{2/\pi}$. The moments of the univariate SN distribution are:

$$\begin{aligned}
\mu(X) &= \xi + \omega m_0 \\
\sigma^2(X) &= \omega^2\left(1 - m_0^2\right) \\
\gamma_1(X) &= \left(\frac{4-\pi}{2}\right)\frac{m_0^3}{\left(1-m_0^2\right)^{3/2}} \\
\gamma_2(X) &= 2(\pi-3)\frac{m_0^4}{\left(1-m_0^2\right)^2}
\end{aligned}$$

A.2.1.4 The skew t distribution

The multivariate case Let $X \sim \mathcal{SN}(\mathbf{0}, \Omega, \eta)$ and $V \sim \chi^2_\nu / \nu$ independent of X. Following Azzalini and Capitanio (2003), the mixture transformation $Y = \xi + V^{-1/2}X$ has a skew t distribution and we write $Y \sim \mathcal{ST}(\xi, \Omega, \eta, \nu)$. The density function of Y is related to the multivariate t distribution as follows:

$$f(y) = 2t_n(y-\xi; \Omega, \nu)\,\mathbf{T}_1\left(\eta^\top\omega^{-1}(y-\xi)\sqrt{\frac{\nu+n}{Q+\nu}}; \nu+n\right)$$

with $Q = (y-\xi)^\top\Omega^{-1}(y-\xi)$. We notice that we have:

$$\begin{aligned}
\Pr\{Y \leq y\} &= \Pr\left\{V^{-1/2}X \leq \omega^{-1}(y-\xi)\right\} \\
&= \Pr\left\{V^{-1/2}U \leq \omega^{-1}(y-\xi) \mid U_0 > 0\right\} \\
&= 2\Pr\left\{V^{-1/2}\begin{pmatrix} -U_0 \\ U \end{pmatrix} \leq \begin{pmatrix} 0 \\ \omega^{-1}(y-\xi) \end{pmatrix}\right\} \\
&= 2\left(\mathbf{T}_n\left(\omega^{-1}(y-\xi); C, \nu\right) - \mathbf{T}_{n+1}(u_+; C_+(\delta), \nu)\right) \\
&= 2\mathbf{T}_{n+1}(u_+; C_+(-\delta), \nu)
\end{aligned}$$

with $u_+ = \left(0, \omega^{-1}(y-\xi)\right)$.

Like the multivariate skew normal distribution, the skew t distribution satisfies the closure property under linear transformation. Let A be a $m \times n$ matrix and $Y \sim \mathcal{SN}(\xi, \Omega, \eta)$. We have:

$$AY \sim \mathcal{SN}(\xi_A, \Omega_A, \eta_A, \nu_A)$$

with:

$$\begin{aligned}
\xi_A &= A\xi \\
\Omega_A &= A\Omega A^\top \\
\eta_A &= \frac{\omega_A \Omega_A^{-1} B^\top \eta}{\left(1 + \eta^\top \left(C - B\Omega_A^{-1} B^\top\right) \eta\right)^{1/2}} \\
\nu_A &= \nu
\end{aligned}$$

where $\omega = \operatorname{diag}^{1/2}(\Omega)$, $C = \omega^{-1}\Omega\omega$, $\omega_A = \operatorname{diag}^{1/2}(\Omega_A)$ and $B = \omega^{-1}\Omega A^\top$. This property also implies that the marginal distributions of a subset of Y is still a skew t distribution.

The univariate case The density function becomes:

$$f(y) = \frac{2}{\omega} t_1\left(\frac{y-\xi}{\omega}; \nu\right) \mathbf{T}_1\left(\eta\left(\frac{y-\xi}{\omega}\right)\sqrt{\frac{\nu+1}{Q+\nu}}; \nu+1\right)$$

with $Q = (y-\xi)^2/\omega^2$. To compute the cumulative density function, we use the following result:

$$\Pr\{Y \le y\} = 2\mathbf{T}_2\left(0, \frac{y-\xi}{\omega}; -\delta; \nu\right)$$

Let m_0 and v_0 be two scalars defined as follows[7]:

$$\begin{aligned}
m_0 &= \delta\sqrt{\frac{\nu}{\pi}} \exp\left(\ln\Gamma\left(\frac{\nu-1}{2}\right) - \ln\Gamma\left(\frac{\nu}{2}\right)\right) \\
v_0 &= \frac{\nu}{\nu-2} - \mu_0^2
\end{aligned}$$

As shown by Azzalini and Capitanio (2003), the moments of the univariate ST distribution are:

$$\begin{aligned}
\mu(Y) &= \xi + \omega m_0 \\
\sigma^2(Y) &= \omega^2 v_0 \\
\gamma_1(Y) &= m_0 v_0^{-3/2}\left(\frac{\nu\left(3-\delta^2\right)}{\nu-3} - \frac{3\nu}{\nu-2} + 2m_0^2\right) \\
\gamma_2(Y) &= m_0 v_0^{-2}\left(\frac{3\nu^2}{(\nu-2)(\nu-4)} - \frac{4m_0^2\nu\left(3-\delta^2\right)}{\nu-3} + \frac{6m_0^2\nu}{\nu-2} - 3m_0^4\right) - 3
\end{aligned}$$

[7] We remind that $\delta = \alpha \Big/ \sqrt{1+\alpha^2}$.

A.2.2 Special results

A.2.2.1 Affine transformation of random vectors

The univariate case Let X be a random variable with probability distribution $\mathbf{F}$. We consider the affine transformation $Y = a + bX$. If $b > 0$, the cumulative probability distribution $\mathbf{H}$ of Y is:

$$\begin{aligned} \mathbf{H}(y) &= \Pr\{Y \leq y\} \\ &= \Pr\left\{X \leq \frac{y-a}{b}\right\} \\ &= \mathbf{F}\left(\frac{y-a}{b}\right) \end{aligned}$$

and its density function is:

$$h(y) = \partial_y \mathbf{H}(y) = \frac{1}{b} f\left(\frac{y-a}{b}\right)$$

If $b < 0$, we obtain:

$$\begin{aligned} \mathbf{H}(y) &= \Pr\{Y \leq y\} \\ &= \Pr\left\{X \geq \frac{y-a}{b}\right\} \\ &= 1 - \mathbf{F}\left(\frac{y-a}{b}\right) \end{aligned}$$

and:

$$h(y) = \partial_y \mathbf{H}(y) = -\frac{1}{b} f\left(\frac{y-a}{b}\right)$$

The mean and the variance of Y are respectively $a + b\mu(X)$ and $b^2 \operatorname{var}(X)$. The centered moments are:

$$\mathbb{E}\left[(Y - \mu(Y))^r\right] = b^r \mathbb{E}\left[(X - \mu(X))^r\right]$$

We deduce that the excess kurtosis of Y is the same as for X whereas the skewness is equal to:

$$\gamma_1(Y) = \operatorname{sgn}(b)\, \gamma_1(X)$$

As an illustration, we consider the random variable $Y = \mu + \sigma X$ with $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ and $\sigma > 0$. We obtain:

$$\mathbf{H}(y) = \Phi\left(\frac{y-\mu}{\sigma}\right)$$

and

$$h(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp -\frac{1}{2}\left(\frac{y-\mu}{\sigma}\right)^2$$

We also deduce that

$$\mathbf{H}^{-1}(\alpha) = \mu + \sigma\Phi^{-1}(\alpha)$$

For the moments, we obtain $\mu(Y) = \mu$, $\sigma^2(Y) = \sigma^2$, $\gamma_1(Y) = 0$ and $\gamma_2(Y) = 0$.

The multivariate case Let X be a random vector of dimension n, A a $(m \times 1)$ vector and B a $(m \times n)$ matrix. We consider the affine transformation $Y = A + BX$. The moments verify $\mu(Y) = A + B\mu(X)$ and $\operatorname{cov}(Y) = B\operatorname{cov}(X)B^\top$. In the general case, it is not possible to find the distribution of Y. However, if $X \sim \mathcal{N}(\mu, \Sigma)$, Y is also a normal random vector with $Y \sim \mathcal{N}\left(A + B\mu, B\Sigma B^\top\right)$.

A.2.2.2 Relationship between density and quantile functions

Let $\mathbf{F}(x)$ be a cumulative density function. The density function is $f(x) = \partial_x \mathbf{F}(x)$. We note $\alpha = \mathbf{F}(x)$ and $x = \mathbf{F}^{-1}(\alpha)$. We have:

$$\frac{\partial\,\mathbf{F}^{-1}(\mathbf{F}(x))}{\partial\,x} = \frac{\partial\,\mathbf{F}^{-1}(\alpha)}{\partial\,\alpha}\left(\frac{\partial\,\mathbf{F}(x)}{\partial\,x}\right) = 1$$

We deduce that:

$$\frac{\partial\,\mathbf{F}^{-1}(\alpha)}{\partial\,\alpha} = \left(\frac{\partial\,\mathbf{F}(x)}{\partial\,x}\right)^{-1} = \frac{1}{f\left(\mathbf{F}^{-1}(\alpha)\right)}$$

and:

$$f(x) = \frac{1}{\partial_\alpha\,\mathbf{F}^{-1}(\mathbf{F}(x))}$$

For instance, we can use this result to compute the moments of the random variable X with the quantile function instead of the density function:

$$\mathbb{E}[X^r] = \int_{-\infty}^{\infty} x^r f(x)\,\mathrm{d}x = \int_0^1 \left(\mathbf{F}^{-1}(\alpha)\right)^r\,\mathrm{d}\alpha$$

A.2.2.3 Conditional expectation in the case of the Normal distribution

Let us consider a Gaussian random vector defined as follows:

$$\left(\begin{array}{c} Y \\ X \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_y \\ \mu_x \end{array}\right), \left(\begin{array}{cc} \Sigma_{yy} & \Sigma_{yx} \\ \Sigma_{xy} & \Sigma_{xx} \end{array}\right)\right)$$

The conditional distribution of Y given $X = x$ is a multivariate Normal distribution. We have:

$$\begin{array}{rcl} \mu_{y|x} & = & \mathbb{E}[Y \mid X = x] \\ & = & \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}(x - \mu_x) \end{array}$$

and:

$$\begin{aligned}\Sigma_{yy|x} &= \sigma^2 \left[Y \mid X = x\right] \\ &= \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}\end{aligned}$$

We deduce that:

$$Y = \mu_y + \Sigma_{yx}\Sigma_{xx}^{-1}\left(x - \mu_x\right) + u$$

where u is a centered Gaussian random variable with variance $\sigma^2 = \Sigma_{yy|x}$. It follows that:

$$Y = \underbrace{\left(\mu_y - \Sigma_{yx}\Sigma_{xx}^{-1}\mu_x\right)}_{\beta_0} + \underbrace{\Sigma_{yx}\Sigma_{xx}^{-1}}_{\beta^\top} x + u$$

We recognize the linear regression of Y on a constant and a set of exogenous variables X:

$$Y = \beta_0 + \beta^\top X + u$$

Moreover, we have:

$$\begin{aligned}R^2 &= 1 - \frac{\sigma^2}{\Sigma_{yy}} \\ &= \frac{\Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}}{\Sigma_{yy}}\end{aligned}$$

A.2.2.4 Calculation of a useful integral function in credit risk models

We consider the following integral:

$$I = \int_{-\infty}^{c} \Phi\left(a + bx\right)\phi\left(x\right)\,\mathrm{d}x$$

We have:

$$\begin{aligned}I &= \int_{-\infty}^{c}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a+bx}\exp\left(-\frac{1}{2}y^2\right)\mathrm{d}y\right)\phi\left(x\right)\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\infty}^{c}\int_{-\infty}^{a+bx}\exp\left(-\frac{y^2+x^2}{2}\right)\mathrm{d}y\,\mathrm{d}x\end{aligned}$$

By considering the change of variables $(x, z) = \varphi\left(x, y\right)$ such that $z = y - bx$, we obtain[8]:

$$I = \frac{1}{2\pi}\int_{-\infty}^{c}\int_{-\infty}^{a}\exp\left(-\frac{z^2 + 2bzx + b^2x^2 + x^2}{2}\right)\mathrm{d}z\,\mathrm{d}x$$

---

[8] We use the fact that the Jacobian of $\varphi\left(x, y\right)$ has the following expression:

$$J_\varphi = \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

and its determinant $\left|J_\varphi\right|$ is equal to 1.

If we consider the new change of variable $t = \left(1 + b^2\right)^{-1/2} z$ and use the notation $\delta = 1 + b^2$, we have:

$$\begin{aligned} I &= \frac{\sqrt{\delta}}{2\pi} \int_{-\infty}^{c} \int_{-\infty}^{\frac{a}{\sqrt{1+b^2}}} \exp\left(-\frac{\delta t^2 + 2b\sqrt{\delta} t x + \delta x^2}{2}\right) \mathrm{d}t \, \mathrm{d}x \\ &= \frac{\sqrt{\delta}}{2\pi} \int_{-\infty}^{c} \int_{-\infty}^{\frac{a}{\sqrt{1+b^2}}} \exp\left(-\frac{\delta}{2}\left(t^2 + \frac{2b}{\sqrt{\delta}} x t + x^2\right)\right) \mathrm{d}t \, \mathrm{d}x \end{aligned}$$

We recognize the expression of the cumulative bivariate Normal distribution[9], whose correlation parameter ρ is equal to $-b \Big/ \sqrt{\delta}$:

$$\int_{-\infty}^{c} \Phi\left(a + bx\right) \phi\left(x\right) \, \mathrm{d}x = \Phi_2\left(c, \frac{a}{\sqrt{1+b^2}}; \frac{-b}{\sqrt{1+b^2}}\right)$$

[9]We recall that $\Phi_2\left(x, y; \rho\right)$ is the cumulative distribution function of the bivariate Gaussian vector (X, Y) with correlation ρ on the space $[-\infty, x] \times [-\infty, y]$.

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